STUDY OF SOME PROBLEMS IN FUZZY TOPOLOGY

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By

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DEDICATED TO

MY RESPECTED PARENTS

AND

MY CHILDHOOD TEACHER & INSPIRER

MR. D. P. RAI

LIST OF PAPERS OF THE CANDIDATE

1. On fuzzy nearly compact regular open topology, (With Deb Ray, A.) *Advances in Fuzzy Mathematics*, 4(1)(2009), 59-68.

2. On fuzzy topological ring valued fuzzy continuous functions, (with Deb Ray, A.) *Applied.Mathematical Sciences*, 3(24)(2009), 1177-1188.

3. A note on fuzzy compact open topology, (with Deb Ray, A.) *To Appear.*

4. On pseudo δ-open fuzzy sets and pseudo fuzzy δ-continuous functions, (with Deb Ray, A.) *Communicated.*

5. On starplus nearly compact pseudo regular open fuzzy topology, (with Deb Ray, A.) *Communicated.*

6. Fuzzy pseudo nearly compact spaces and *ps-ro* continuous functions, (with Deb Ray, A.) *Communicated.*

7. Further on fuzzy pseudo near compactness and *ps-ro* continuous functions, (with Deb Ray, A.) *Communicated.*

8. More on Left fuzzy topological rings, (with Deb Ray, A. and Sahu, M.) *Communicated.*

9. A note on α-s-closedness of fuzzy topological spaces, *International Transactions in Mathematical Sciences and Computer*, 2(1)(2009), 29-33.

Contents

Chapter 1

Introduction and Preliminaries

1.1 Historical Background

The classical theory of sets was introduced by Cantor towards the end of nineteenth century. This is still regarded as the foundation of mathematics in general. Based on this theory, vast amount of significant researches have been carried out by several distinguished mathematicians in different fields of Mathematics. To mention a few, subjects like Topology, Algebra, Functional Analysis, Measure Theory were immensely developed since the time of Cantor to the present day.

Even then, applications of this set theory had their own limitations. It is with this point of view as the background, Zadeh in 1965 enunciated a new branch of study known as fuzzy mathematics. He defined in his preliminary work [94], the concept of fuzzy sets as a function from an ordinary set X into the closed interval $[0, 1]$. In this contemporary period, Goguen [38] replaced the closed interval $[0, 1]$ by a complete distributive lattice and named it L-fuzzy set. Soon thereafter in 1971, Brown [7] modified the definition of Goguen by defining a fuzzy set as a function from a set to a Boolean lattice. He applied his definition to some of the results based on Zadeh's fuzzy sets and found them consistent in his own framework.

In this period, the theory of fuzzy sets had been developed considerably by several workers. Fuzzy set theory is applied in various branches of Mathematics, such as, Topology, Algebra, Graph Theory, Logic and so on. Apart from these, it has applications in subjects like Operation Research, Mathematical Programming and various diversified fields of applications as Pattern Recognition, Information Processing, Signal Processing, Production Management, Robotics, Automations, etc. Lucid presentation of the subject including some of its applications are given in the book of Zimmerman [95].

In 1968, C.L. Chang [10] became the pioneer to present the idea of fuzzy topology. Fundamentally, he replaced the classical notion of open sets by a notion called fuzzy open sets. However, later in 1976, Lowen [53] found that many of the well known results in general topology cannot be obtained if one follows Chang's definition of fuzzy topology. So, he redefined fuzzy topology by including constant fuzzy sets in it. This implies that in Chang's definition of fuzzy topology one more axiom was included. It should be mentioned that those who follow Chang's definition of fuzzy topology, call Lowen's definition of it as "fully stratified" fuzzy topology. In this thesis, we have also followed the same nomenclature.

To study in fuzzy topology, analogous local topological concepts, such as, local compactness, local connectedness, 1^{st} countability and continuity at a point, convergence of nets and filters to a point, the well known concept of fuzzy point was introduced by Wong in 1974 [93]. According to Wong's definition, a fuzzy point on a space X is a particular type of fuzzy set having value zero at all but one point of X, while the value at the later point x (say) of X is α (with $0 < \alpha < 1$). This was later modified by Pu and Liu [72] in 1980, to include $\alpha = 1$. It is only when $\alpha = 1$ is taken, that crisp singletons follow as a particular case of fuzzy points. In this thesis, we have adopted the above modified definition as given by Pu and Liu.

To Pu and Liu [72] is attributed the inception of the concepts of q-concidence and q-neighbourhoods, which in succeeding years led with great strides the study of fuzzy topology.

Among them, many investigators whose works involve the concept of fuzzy points, the names of Wong [92], Kotze [50], De Mitri and Pascali [17], Kandid and El-Etriby [47] need to be mentioned.

In 1976, Lowen [53] introduced two functors ω and i which allow one to see more clearly the fuzzy topological spaces and topological spaces. The functor ω carries over a topological space to a fuzzy topological space; whereas the functor i_α, for each $\alpha \in [0, 1)$, carries over a fuzzy topological space to a topological space. The topology $i_\alpha(\tau)$, where τ stands for a fuzzy topology, has been named by Kohli and Prasannan [49], strong α-level topology.

Pioneering researches on various separation axioms were done by Hutton [42], Hutton and Reilly [43], Rodabough [77] and many others, of whom the names of Ganguly and Saha [33], Sinha [83], [82] should be cited.

In the following sections, the gradual developments of the topics relevant to the thesis are enumerated in details.

1.2 Operators and functions on fuzzy topological spaces

In the last three or four decades, various mathematicians initiated a variety of closure-like operators in general topology. Significant among them are semi-closures [14], δ-closure [87], θ-closure [87], α-

closure [70], pre-closure [60] and δ-preclosure [76] operators. These in turn resulted in the introduction of their corresponding open-like sets. Several authors in this period applied these concepts in their studies of separation axioms, different compact-like covering properties, continuous-like functions, connectedness and so on.

In this contemporary period, generalizations of the above concepts have been made in fuzzy topological space by different authors. Among them, Azad [3] introduced what is known as fuzzy semi-closure operator. Following his concepts, fuzzy semi-separation axioms, semi-continuous and semi-closed mappings, fuzzy semi-connectedness and fuzzy S-closedness were developed by several authors and to name a few are Ganguly and Saha [32], Ghosh [37], Mukherjee and Ghosh [62] and Das [16].

The δ-closure operator as enunciated by Velicko [87] in general topology was extended in fuzzy settings by Ganguly and Saha [33], and was successfully applied in their study of fuzzy δ-continuity and fuzzy δ-connectedness. In topology, the δ-closure operator plays an important role to characterize near compactness. Later, these results were reviewed or generalized in fuzzy framework by Hayder [41], Mukherjee and Ghosh [63], [65], among others.

Apart from the operators described above, there are other significant operators on fuzzy topological spaces, of which α-closure operators for L-fuzzy topological spaces, as introduced by Rodabaugh [77] is one and fuzzy θ-closure operator, introduced by Mukherjee and Sinha [69] is one more among such operators.

A new pair of operators which we name as fuzzy ps-closure and fuzzy ps-interior has been used in our work in this thesis.

Different types of continuous-like functions in fuzzy topology were studied by many mathematicians. Chang [10] introduced fuzzy continuous maps between two fuzzy topological spaces, whereas Wong [92] initiated fuzzy open and closed maps. Mashhour and Ghanim [61] made a further generalization of fuzzy continuity in fuzzy closure spaces.

T. Noiri formulated various types of functions between topological spaces. His introduction of δ-continuity opened vast areas for further researches in general topology. The corresponding analogue in fuzzy version was fruitfully studied by Saha [78], Ganguly and Saha [33], El-Monsef et al. [28], Mukherjee and Ghosh [64] and others.

Apart from fuzzy δ-continuity, there are other forms of fuzzy continuity and their mutual relations.

In this thesis two distinct types of fuzzy continuous-like maps are defined and successfully applied in our investigations of certain compact-like fuzzy covering properties.

1.3 Fuzzy compactness, fuzzy near compactness and starplus compactness

Fuzzy compactness has been defined differently by different researchers. None of such definitions could singly construct many of the well known properties in fuzzy topology, in analogy to their corresponding properties in general topology.

C.L Chang [10] in 1968, showed that the fuzzy compactness is preserved under fuzzy continuous functions. In 1973, J.A. Goguen [39] pointed out a deficiency in Chang's definition of compactness, by showing that Tychonoff theorem does not hold for infinite product.

Later, C.K Wong [91], [92] introduced the notion of countable compactness, sequential compactness, semi-compactness and local compactness in fuzzy framework.

In the process of his investigations, Lowen [53] gave a successful definition of fuzzy compactness.

The concept of N-closed spaces was introduced by Carnahan [8] in general topology. Now, by replacing the topology of subspace by the topology of whole space is formed, what is known as nearly compact space. Singal and Mathur [81] are its profounders.

Hayder in 1987, defined near compactness in fuzzy topology and then after Mukherjee and Ghosh [63] made considerable studies of it. Fuzzy locally nearly compact space was introduced and characterized by Bakier [4]

In 2001 Kohli and Prasannan [49] introduced the notion of Starplus compactness by extending Lowen's notion of strong fuzzy cmpactness [53] to an arbitrary set. Earlier, Lowen had shown that for each $\alpha \in [0, 1)$, a functor i_α carries over a fuzzy topological space to a topological space. Kohli and Prasannan utilized this functor in defining Starplus compact fuzzy sets.

1.4 Fuzzy function spaces

Y. W. Peng [75], G. Jagar [45],[44], Kohli and Prasannan [49], [48] are among the early investigators on fuzzy function spaces. The introduction of fuzzy topologies on a given function space was first made by Peng [75]. He defined pointwise convergent topology and a version of compact open topology on a family of functions from a fuzzy topological space to another. Later, Alderton [1] considered such a problem from categorical point of view and made use of the theory of Cartesian closedness of monotopological categories to function spaces. Subsequently, Dang and Behera [15] induced a topology in lieu of a fuzzy topology, on a family of functions from a fuzzy topological space to another.

Later, Gunther Jagar [44] made significant contributions to the study of compactness in the category of fuzzy convergence spaces as defined by Lowen [53], Lowen and Wuyts [59]. The notions of splitting and conjoining structures on fuzzy subsets have been utilized by them in their works. As a special case, they studied a notion of fuzzy compact open topology.

Important developments of fuzzy topologies on function spaces were made by Kohli and Prasannan [48], [49]. In the first of these two papers, they introduced and studied three different fuzzy topologies on function spaces, which are analogues of the topology of pointwise convergence, compact open topology and topology of joint-continuity. Interrelations among them, analogous to their counter part in general topology, were also obtained. In their second paper, these authors extended Lowen's [53] notion of strong fuzzy compactness to an arbitrary fuzzy set; thereby they introduced the notion of starplus compact fuzzy set. They have shown that the category of starplus compact fuzzy topological space is productive and that starplus compactness is a good extension of the notion of compactness. Besides, they introduced the notion of starplus compact open fuzzy topology on a function space. Its interrelations with fuzzy topology of point-wise convergence and the fuzzy topology of joint fuzzy continuity were studied.

1.5 Left fuzzy topological ring

As is very well known, topological ring is the study of the topological behavior of a space, in presence of its ring structure. The properties of topological ring were extensively investigated by several renowned mathematicians, namely, Arnautov, Ursul, Warner and others. Lucid review of these works is given in the book "Topological rings satisfying compactness conditions" by Mihail Ursul [86] and also in the book "Introduction to the theory of topological rings and modules" by Arnautov, Glavatsky and Mikhalev [2].

Deb Ray [18] has recently framed a new form of fuzzy topological ring in the light of topological rings referred above. This author has named it a left fuzzy topological ring and comments that analogously, a right fuzzy topological ring can be studied. The basic aim of this paper is to characterize the fundamental system of fuzzy neighbourhoods of the fuzzy point 0_α $(0 < \alpha \leq 1)$ and to establish that any fuzzy neighbourhood of a fuzzy point x_α is of the form $x_\alpha + V$, where V is a fuzzy neighourhood of x_α.

In this section, many more properties of the above left fuzzy topological ring are obtained. Further, fuzzy continuous functions, having values in left fuzzy topological ring have been studied.

1.6 Basic definitions and results

The previously known definitions and results, which we have adopted in this thesis are enumerated as follows.

Definition 1.6.1 [94] Let X be a non empty set and I denote the closed interval $[0, 1]$. A fuzzy set μ on X is a function from X into I. Let I^X denote the collection of all fuzzy sets on X. The support of a fuzzy set μ, denoted by $supp(\mu)$, is the crisp set $\{x \in X : \mu(x) > 0\}$. A fuzzy set with a singleton as its support is called a fuzzy point, denoted by x_α, where $0 < \alpha \leq 1$ and defined as

$$x_\alpha(z) = \begin{cases} \alpha, & for\ z = x \\ 0, & otherwise. \end{cases}$$

The fuzzy sets on X taking the constant values 0 and 1 respectively are denoted by 0_X and 1_X or simply by 0 and 1 respectively. A fuzzy set A on X is called non-null if $A \neq 0$. If a fuzzy set A takes only the values 0 and 1, then A is called a crisp subset of X, i.e., A becomes the characteristic function of an ordinary subset of X.

Definition 1.6.2 [94] Let f be a function from a set X into another set Y and A and B be fuzzy sets on X and Y respectively. Then $f(A)$ and $f^{-1}(B)$ are respective fuzzy sets on Y and X, given by,

$$f(A)(y) = \begin{cases} sup\{A(x) : x \in f^{-1}(y)\}, & if\ f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

and $f^{-1}(B)(x) = B(f(x))$

Definition 1.6.3 [94] Two fuzzy sets A and B on a set X are said to be equal, i.e., $A = B$, if $A(x) = B(x), \forall\ x \in X$. A fuzzy point x_α is said to belong or is contained in a fuzzy set A if $\alpha \leq A(x)$. For two fuzzy sets A and B on X, A is said to be subset of B, written as $A \leq B$ if $A(x) \leq B(x), \forall\ x \in X$. Clearly, $A = B$ iff $A \leq B$ and $B \leq A$.

Definition 1.6.4 [94] The complement of a fuzzy set A on X, denoted by $1 - A$ or A^c is given by $(1 - A)(x) = 1 - A(x), \forall\ x \in X$.

Definition 1.6.5 [94] Let $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy sets on X, where Λ denotes an indexing set. The union $\cup\{A_\alpha : \alpha \in \Lambda\}$ and the intersection $\cap\{A_\alpha : \alpha \in \Lambda\}$ (also denoted by $\vee\{A_\alpha : \alpha \in \Lambda\}$ and $\wedge\{A_\alpha : \alpha \in \Lambda\}$, respectively) are given by

$(\vee\mathcal{A})(x) = sup\{A_\alpha(x) : \alpha \in \Lambda\}, \forall\ x \in X$

$(\wedge\mathcal{A})(x) = inf\{A_\alpha(x) : \alpha \in \Lambda\}, \forall\ x \in X$

De Morgan's laws were also established in [94]:

Theorem 1.6.1 [94] For any family $\{A_\alpha : \alpha \in \Lambda\}$ of fuzzy sets on X, the following hold:

(i) $1 - \vee\{A_\alpha : \alpha \in \Lambda\} = \wedge\{1 - A_\alpha : \alpha \in \Lambda\}$

(ii) $1 - \wedge\{A_\alpha : \alpha \in \Lambda\} = \vee\{1 - A_\alpha : \alpha \in \Lambda\}$.

Definition 1.6.6 [46] Let X and Y be non empty sets and A, B be fuzzy sets on X and Y respectively. Then the product fuzzy set $A \times B$ on $X \times Y$ is defined by $(A \times B)(x, y) = inf\{A(x), B(y) : (x, y) \in X \times Y\}$

Theorem 1.6.2 [3], [10], [73] Let f be a function from a set X into a set Y. Then the following hold:

(i) $f^{-1}(1 - B) = 1 - f^{-1}(B)$, for any fuzzy set B on Y.

(ii) $B_1 \leq B_2 \Rightarrow f^{-1}(B_1) \leq f^{-1}(B_2)$, for any fuzzy sets B_1 and B_2 on Y.

(iii) $A_1 \leq A_2 \Rightarrow f(A_1) \leq f(A_2)$, for any fuzzy sets A_1 and A_2 on X.

(iv) $ff^{-1}(B) \leq B$, for any fuzzy set B on Y.

(v) $f^{-1}f(A) \geq A$, for any fuzzy set A on X.

(vi) $f^{-1}(B)^c = (f^{-1}(B))^c$, for any fuzzy set B on Y.

(vii) For any family $\{A_\alpha : \alpha \in \Lambda\}$ of fuzzy sets on X,

1. $f(\vee\{A_\alpha : \alpha \in \Lambda\}) = \vee\{f(A_\alpha) : \alpha \in \Lambda\}$.

2. $f(\wedge\{A_\alpha : \alpha \in \Lambda\}) \leq \wedge\{f(A_\alpha) : \alpha \in \Lambda\}$.

(viii) For any family $\{B_\alpha : \alpha \in \Lambda\}$ of fuzzy sets on Y,

1. $f^{-1}(\vee\{B_\alpha : \alpha \in \Lambda\}) = \vee\{f^{-1}(B_\alpha) : \alpha \in \Lambda\}$.

2. $f^{-1}(\wedge\{B_\alpha : \alpha \in \Lambda\}) = \wedge\{f^{-1}(B_\alpha) : \alpha \in \Lambda\}$.

Definition 1.6.7 [10] A collection $\tau \subseteq I^X$ is called a fuzzy topology on X if the following conditions are satisfied:

(i) $0, 1 \in \tau$

(ii) $\forall\ \mu_1, \mu_2, ..., \mu_n \in \tau \Rightarrow \wedge_{i=1}^{n} \mu_i \in \tau$

(iii) $\mu_\alpha \in \tau$, $\forall\ \alpha \in \Lambda$ (where Λ is an index set) $\Rightarrow \vee\mu_\alpha \in \tau$.

Then (X, τ) is called a fuzzy topological space (fts, for short). The members of τ are called fuzzy open sets on X and their complements are called fuzzy closed sets on X.

The definition of fuzzy topology generalizes ordinary (classical) set topology. In what follow, we always mean a fuzzy topological space (X, τ) by fts.

Definition 1.6.8 [53] A collection $\tau \subseteq I^X$ is called a laminated [49] or stratified [48] or fully stratified [51] fuzzy topology on X if the following conditions are satisfied:

(i) $\forall\ c \in I, \overline{c} \in \tau$ where $\overline{c}(x) = c, \forall\ x \in X$.

(ii) $\forall\ \mu_1, \mu_2, ..., \mu_n \in \tau \Rightarrow \wedge_{i=1}^{n} \mu_i \in \tau$

(iii) $\mu_\alpha \in \tau$, $\forall\ \alpha \in \Lambda$ (where Λ is an index set) $\Rightarrow \vee\mu_\alpha \in \tau$.

Definition 1.6.9 [92] Let (X, τ) be a fts. A subfamily $\mathcal{B}$ of τ is called a base for τ if each member of τ can be expressed as a union of some members of $\mathcal{B}$. A subfamily $\mathcal{F}$ of τ is called a subbase for τ if the collection of all finite intersections of members of $\mathcal{F}$ forms a base for τ. The members of $\mathcal{B}$ are called basic fuzzy open sets and that of $\mathcal{F}$ are called subbasic fuzzy open sets.

Definition 1.6.10 [10] For any fuzzy set A on a fts (X, τ), the fuzzy closure of A, denoted by clA or $\bar{A}$, and the fuzzy interior, denoted by $intA$ or A^0, are defined respectively as follows:

$clA = inf\{B \in I^X : A \leq B, (1 - B) \in \tau\}$

$intA = sup\{B \in I^X : B \leq A, B \in \tau\}$.

It is clear that clA is the smallest fuzzy closed set containing A and $intA$ is the largest fuzzy open set contained in A.

Theorem 1.6.3 [3] For any fuzzy set A on a fts (X, τ),

(i) $1 - intA = cl(1 - A)$

(ii) $1 - clA = int(1 - A)$

(iii) A is fuzzy open (closed) iff $A = intA$ (respectively, $A = clA$)

Definition 1.6.11 [3] For any family $\{A_\alpha : \alpha \in \Lambda\}$ of fuzzy sets on a fts (X, τ),

(i) $\vee\{clA_\alpha : \alpha \in \Lambda\} \leq cl(\vee\{A_\alpha : \alpha \in \Lambda\})$, the equality holds if Λ is finite.

(ii) $\vee\{intA_\alpha : \alpha \in \Lambda\} \leq int(\vee\{A_\alpha : \alpha \in \Lambda\})$.

Definition 1.6.12 [72] In a fts (X, τ), a fuzzy set A is said to be a neighborhood (*nbd.* for short) of a fuzzy point x_α if there is a fuzzy open set B such that $x_\alpha \in B \leq A$. In addition, if A is fuzzy open, the *nbd.* is called fuzzy open *nbd.*

Definition 1.6.13 [72] In a fts (X, τ), a fuzzy set A is said to be a quasi neighborhood or simply *q-nbd.* of a fuzzy point x_α if there is a fuzzy open set B such that $x_\alpha qB \leq A$. In addition, if A is fuzzy open, the *q-nbd.* is called fuzzy open *q-nbd.*

Definition 1.6.14 [72] In a fts (X, τ), a fuzzy set A is said to be quasi coincident (q-coincident, for short) with another fuzzy set B (written as AqB) if there exists $x \in X$ such that $A(x) + B(x) > 1$. If A and B are not q-coincident, we write $A \not{q} B$.

A fuzzy point x_α is q-coincident with a fuzzy set B, denoted by $x_\alpha qB$, if $\alpha + B(x) > 1$.

Theorem 1.6.4 [72] A fuzzy set is fuzzy open iff it is a fuzzy *nbd.* of every point contained in it.

Theorem 1.6.5 [72] In a fts (X, τ), a family $\mathcal{B}$ of τ is a base for τ iff for each fuzzy point x_α in (X, τ) and for each fuzzy open *q-nbd.* U of x_α, there exists a member $B \in \mathcal{B}$ such that $x_\alpha qB \leq A$.

Definition 1.6.15 [72] In a fts (X, τ), a fuzzy point x_α is called a fuzzy cluster point of a fuzzy set A if every *q-nbd.* (or equivalently, every fuzzy open *q-nbd.*) of x_α is q-coincident with A.

Theorem 1.6.6 [72] For a fuzzy point x_α and a fuzzy set A on a fts (X, τ), $x_\alpha \in clA$ iff x_α is a fuzzy cluster point of A.

Definition 1.6.16 [3] A fuzzy set A on a fts (X, τ) is called fuzzy regular open in X if $int(clA) = A$. A is called fuzzy regular closed in X if $cl(intA) = A$.

Theorem 1.6.7 [3] A fuzzy set A on a fts (X, τ) is fuzzy regular open iff $(1 - A)$ is fuzzy regular closed in X.

Definition 1.6.17 [32] A fuzzy point x_α is said to be a fuzzy δ-cluster point of a fuzzy set A on a fts (X, τ), if every fuzzy regular open *q-nbd.* U of x_α is q-coincident with A. The union of fuzzy δ-cluster points of A is called the fuzzy δ-closure of A and is denoted by δ-clA. A fuzzy set A is called fuzzy δ-closed if $A = \delta\text{-}clA$ and the complement of such fuzzy set is called fuzzy δ-open. Another operator δ-interior of a fuzzy set A, denoted by δ-$intA$ is defined as $\delta\text{-}intA = 1 - \delta\text{-}cl(1 - A)$. It can be proved that A is fuzzy δ-open iff $A = \delta\text{-}intA$.

Remark 1.6.1 [3] A fuzzy δ-open set is the union of some fuzzy regular open sets. The complement of a fuzzy δ-open set is fuzzy δ-closed set. Hence, every fuzzy regular open set is fuzzy δ-open.

Definition 1.6.18 Let (X, τ) and (Y, σ) be two fuzzy topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i) fuzzy continuous [10], if $f^{-1}(\mu)$ is fuzzy open on X, for all fuzzy open set μ on Y.

(ii) fuzzy open (fuzzy closed) [93], if for each fuzzy open (fuzzy closed) set μ on X , $f(\mu)$ is fuzzy open (respectively fuzzy closed) on Y.

(iii) fuzzy homeomorphism [10], if f is bijective and both f and f^{-1} are fuzzy continuous.

A property $\mathcal{P}$ of a fts (X, τ) is called a fuzzy topological property if it remains invariant under fuzzy homeomorphism.

Theorem 1.6.8 [73] Let (X, τ) and (Y, σ) be two fuzzy topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function, then the following are equivalent:

(i) f is fuzzy continuous.

(ii) $f^{-1}(\mu)$ is fuzzy closed on X, for all fuzzy closed set μ on Y.

(iii) For each member V of a subbase $\mathcal{B}$ for σ, $f^{-1}(V)$ is τ open.

(iv) For each fuzzy point x_β on X and any fuzzy *nbd.* V of $(f(x))_\beta$ on Y, there exists fuzzy *nbd.* U of x_β on X such that $f(U) \leq V$.

(v) For each x_β on X and any fuzzy *q-nbd.* V of $(f(x))_\beta$ on Y, there exists fuzzy *q-nbd.* U of x_β on X such that $f(U) \leq V$.

(vi) For any fuzzy set A on X, $f(clA) \leq clf(A)$.

(vii) For any fuzzy set B on Y, $cl(f^{-1}(B)) \leq f^{-1}(clB)$.

Theorem 1.6.9 [73] Let f be a function from a fts X to another fts Y. Then the following hold:

(i) If $A_1, A_2 \in I^X$ and $A_1 \wedge A_2 \neq 0_X$, then $f(A_1 \wedge A_2) \neq 0_Y$.

(ii) If $A_1, A_2 \in I^X$, $f(A_1 \wedge A_2) \leq f(A_1) \wedge f(A_2)$.

(iii) If $A \in I^X$ and $B \in I^Y$, then $f(A) \leq B \Rightarrow A \leq f^{-1}(B)$.

(iv) If $\dot{A}, B \in I^X$ such that AqB, then $f(A)qf(B)$.

(v) If $A, B \in I^Y$ such that $A \not{q} B$, then $f^{-1}(A) \not{q} f^{-1}(B)$.

Definition 1.6.19 [92] Let (X, τ) and (Y, σ) be two fts. The fuzzy product space $X \times Y$ of X and Y is a fts which is endowed with the product fuzzy topology ρ generated by the family $\{(p_1^{-1}(A), p_2^{-1}(B)) : A \in \tau, B \in \sigma\}$ as a base, where p_1 and p_2 are the usual projection mappings of $X \times Y$ onto X and Y respectively.

According to [74], the definition of fuzzy T_2 space is

Definition 1.6.20 A fts (X, τ) is called fuzzy T_2 if for any two fuzzy points x_α and y_β with $x \neq y$, there exists *q-nbd.* B and C of x_α and y_β respectively, such that $B \wedge C = 0$

According to [49], the definition of fuzzy Hausdorff space is

Definition 1.6.21 A fts (X, τ) is called fuzzy Hausdorff if for every pair of fuzzy points x_α, y_α on X with distinct supports there exist fuzzy open sets U and V on X such that $x_\alpha \in U$, $y_\alpha \in V$ and $U \wedge V = 0$

Definition 1.6.22 [33] A fts (X, τ) is called fuzzy GS-T_2 if for any two distinct fuzzy points x_α and y_β,

(i) when $x \neq y$, x_α and y_β have fuzzy open *nbds*. which are not q-coincident,

(ii) when $x = y$ and $\alpha < \beta$(say), x_α has a fuzzy open *nbd.* U and y_β has a fuzzy open *q-nbd.* V such that $U \not{q} V$.

Definition 1.6.23 [67] A fts (X, τ) is called fuzzy regular if each fuzzy point x_α on X and each fuzzy open *q-nbd.* U of x_α there exists a fuzzy open *q-nbd.* V such that $\overline{V} \leq U$,where $\overline{V}$ stands for the closure of V.

Definition 1.6.24 [67] A fts (X, τ) is called fuzzy almost regular if for each fuzzy point x_α and any fuzzy regular open set A with $x_\alpha q A$, there exists a fuzzy regular open set B such that $x_\alpha q B$ and $\overline{B} \leq A$.

Definition 1.6.25 [53] A fuzzy set A on a fts (X, τ) is called fuzzy compact if for every family $\mathcal{U}$ of fuzzy open sets on X such that $A \leq sup\{U : U \in \mathcal{U}\}$ and for every $\epsilon > 0$, there exist, a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $A \leq sup\{U : U \in \mathcal{U}'\} + \epsilon$. Extending this notion to X, the definition of fuzzy compact space is obtained.

Similarly, fuzzy nearly compact set is defined as follows:

Definition 1.6.26 A fuzzy set A is called fuzzy nearly compact if for every family $\mathcal{U}$ of fuzzy open sets on X such that $A \leq sup\{U : U \in \mathcal{U}\}$ and for every $\epsilon > 0$, there exist, a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $A \leq sup\{\overline{U}^0 : U \in \mathcal{U}'\} + \epsilon$, where $\overline{U}^0$ denotes $int(clU)$.

Definition 1.6.27 [53], [54] For each $i \in \Lambda$, if $f_i : X \to (Y_i, \tau_i)$ are the functions from a set X into fts (Y_i, τ_i), then the smallest fuzzy topology on X for which the functions $f_i, i \in \Lambda$ are fuzzy continuous is called initial fuzzy topology on X generated by the collection of functions $\{f_i : i \in \Lambda\}$. For a fuzzy set μ on X, the set $\mu^\alpha = \{x \in X : \mu(x) > \alpha\}$ is called the strong α-level set of X. For a topological space (X, τ), $w(\tau)$ denotes the collection of all lower semi continuous functions from X into I, i.e., $w(\tau) = \{\mu : \mu^{-1}(a, 1] \in \tau\}, \forall\ a \in [0, 1]$. For a topolog τ on X, $w(\tau)$ is a fully stratified fuzzy topology on X. In a fuzzy topological space (X, τ), for each $\alpha \in I_1 = [0, 1)$, the collection $i_\alpha(\tau) = \{\mu^{-1}(\alpha, 1] : \mu \in \tau\}$ is a topology on X and is called strong α-level topology.

We have also used the following definitions and results from general topology in our work.

Definition 1.6.28 [87] A set S in a topological space in (X, τ) is called regular open if $S = int(clS)$. Complement of a regular open set is regular closed. A δ-open set in X is the union of some regular open sets in X . Every regular open set is δ-open set.

Definition 1.6.29 [81] A set S is called nearly compact if every regular open cover of S has a finite subcover.

Theorem 1.6.10 [81] A topological space (X, τ) is nearly compact iff every family $\mathcal{F}$ of regular closed sets with $\bigcap\{f : f \in \mathcal{F}\} = \Phi$, there exist a finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $\bigcap\{f : f \in \mathcal{F}_0\} = \Phi$.

Theorem 1.6.11 [81] (i) Every regular closed set in nearly compact space is nearly compact.

(ii) In a Hausdorff space, every nearly compact set is δ-closed.

Definition 1.6.30 [71] Let X and Y be two topological spaces. A function $f : X \to Y$ is δ-continuous if $f^{-1}(V)$ is δ-open in X for each δ-open set V in Y.

Definition 1.6.31 [31] Let X, Y be two topological spaces. For any nearly compact set C in X and regular open set U in Y, $T(C, U) = \{f \in Y^X : f(C) \subset U\}$. Then the collection $\{T(C, U)\}$ forms a subbase for some topology on Y^X called nearly compact regular open topology.

We have denoted this topology by N_R topology in our work.

Definition 1.6.32 [31] Let X, Y be two topological spaces and $Z \subset Y^X$. A topology on Z is said to be jointly δ-continuous or δ-admissible if the evaluation mapping $P : Z \times X \to Y$ given by $P(f, x) = f(x)$, where $f \in Z$, $x \in X$ is δ-continuous and $Z \times X$ is endowed with the product topology.

Definition 1.6.33 [79] A category $\mathcal{C}$ consists of three items:

(a) a class of objects, to be denoted by letters A, B, C,..., etc.,

(b) for each pair (A, B) of objects in $\mathcal{C}$, a set $Mor(A, B)$, called the set of morphisms or arrows $f : A \to B$, and

(c) a product $Mor(A, B) \times Mor(B, C) \to Mor(A, C)$ for each triple (A, B, C) of objects in $\mathcal{C}$, called composition and taking the pair (f, g) to $g \circ f \in Mor(A, C)$.

These are required to satisfy the following two axioms:

(i) For any three morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have $h \circ (g \circ f) = (h \circ g) \circ f$, i.e., the composition of morphisms is associative.

(ii) for each object A in $\mathcal{C}$, there exist a morphism $I_A : A \to A$, called identity morphism, which has the property that for all $f : A \to B$ and all $g : C \to A$, we have $f \circ I_A = f$, $I_A \circ g = g$.

A subcategory $\mathcal{S}$ of $\mathcal{C}$ is a collection of some of the objects and some of the arrows of $\mathcal{C}$ such that $\mathcal{S}$ itself is a category.

Definition 1.6.34 [79] A covariant functor F from a catrgory $\mathcal{C}$ to a category $\mathcal{D}$ is a pair of functions (both defined by the same letter F) which maps objects of $\mathcal{C}$ to the objects of $\mathcal{D}$ and, for any pair (A, B) of objects of $\mathcal{C}$, it maps the set $Mor(A, B)$ to the set $Mor(F(A), F(B))$ and is required to satisfy the folowing two conditions:

(i) $F(I_A) = I_{F(A)}$, for every $A \in \mathcal{C}$

(ii) $F(g \circ f) = F(g) \circ F(f) : F(A) \to F(C)$ for any two morphisms $f : A \to B$ and $g : B \to C$ in the category $\mathcal{C}$.

A contravariant functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is similarly defined. The only difference is that the contravariant functor maps a morphism $f : A \to B$ to a morphism $F(f) : F(B) \to F(A)$ in the opposite direction to that of f. Thus, the two conditions in this case will be

(i) $F(I_A) = I_{F(A)}$ for every $A \in \mathcal{C}$

(ii) $F(g \circ f) = F(f) \circ F(g) : F(C) \to F(A)$

A functor $T : \mathcal{C} \to \mathcal{D}$ is full when to every pair c, c' of objects of $\mathcal{C}$ and to every morphism $g : Tc \to Tc'$ of $\mathcal{D}$, there is a morphism $f : c \to c'$ of $\mathcal{C}$ such that $g = Tf$.

Proposition 1.6.1 [79] If $\mathcal{S}$ is a subcategory of a category $\mathcal{C}$ and $i : \mathcal{S} \to \mathcal{C}$ sends each object and each arrow of $\mathcal{S}$ to itself in $\mathcal{C}$, then i is a functor, called inclusion functor.

Definition 1.6.35 [79] A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is called full subcategory if the inclusion functor from $\mathcal{S}$ to $\mathcal{C}$ is a full functor.

1.7 A brief survey of this thesis

As follows, contents of the various Chapters of this thesis are reviewed in short.

In chapter 2, two fuzzy topologies on function spaces have been studied, one known as fuzzy compact open topology, as given by Gunther Jagar and the other defined by us as fuzzy nearly compact regular open topology.

Under both the topologies, we find that a collection of functions $\mathcal{F}$ from a fts X to another fts Y become fuzzy GS-T_2, if the range space Y is fuzzy GS-T_2. Moreover, the fuzzy regularity (fuzzy almost regularity) of the range space Y induces somewhat regularity (somewhat almost regularity) on the space of functions with fuzzy compact open topology (respectively, fuzzy nearly compact regular open topology).

The concepts of jointly fuzzy continuous on fuzzy compacta and jointly fuzzy δ-continuous on fuzzy near compacta have been introduced and studied.

In Chapter 3, by suitably defining pseudo fuzzy δ-continuous functions, we find that such functions correspond to the well known δ-continuous functions in general topology, under the functorial correspondence i_α, for each $\alpha \in [0, 1)$ and some of its characterizations are obtained.

By introducing a compact-like notion of fuzzy sets, which we call starplus near compactness, we see that it generalizes the existing notion of starplus compactness, and some of its properties are studied.

In Chapter 4, with the help of pseudo regular open fuzzy sets and starplus nearly compact fuzzy sets, discussed in Chapter 3, we construct a fuzzy topology on function spaces. Also, defining pseudo admissible fuzzy topology we have investigated the interrelation between this and the fuzzy topology stated above.

In Chapter 5, the notion of pseudo near compactness is introduced and it has been studied via fuzzy nets, fuzzy filterbase and so on.

Besides, two operators, named by us, fuzzy ps-closure and fuzzy ps-interior are introduced and studied.

Further, a new type of continuous like functions, which we have called pseudo fuzzy ro-continuous functions, has been introduced and its characterizations are obtained. We have also shown that this type of functions preserve pseudo near compactness of a fts.

In Chapter 6, Left fuzzy topological ring (left ftr), as introduced by Deb Ray, has been discussed in general and certain properties of the same from categorical point of view are interpreted.

We find that the collection of all left ftr-valued fuzzy continuous functions on a fuzzy topological space form a ring. The interplay between its ring structure and its topological and fuzzy topological behaviour are also observed.

Chapter 2

Fuzzy topologies on Function spaces

2.1 Introduction

Function spaces play an important role in the study of various branches of mathematics, such as, functional analysis, topology, differential equations, differential geometry, and complex analysis among others. Considerable amount of work has been done since long past, by introducing different topologies on a given collection of functions. But not much researches have been done so far, by applying fuzzy topologies on function spaces. Some fruitful attempts in this direction which need be mentioned, were by Kohli and Prasannan [48], [49], Gunther Jagar [45] and Peng [75].

Section (2.2) begins with the definition of fuzzy compact open topology on function spaces, as given by Gunther Jagar in 1999 [45]. This author has slightly changed the definition of fuzzy compact open topology as due to Peng, by replacing Wang's [88] definition of N-compactness by Lowen's definition [53] of compactness. In [48], Kohli and Prasannan, following the definition of compactness due to Wang [88], made some studies on fuzzy compact open topology on function spaces and named it N-compact open topology. In this paper, the authors also left a problem open that if the range space is fuzzy regular, whether the space of all fuzzy continuous functions too is fuzzy regular or not. In this section, we have shown that a collection of functions $\mathcal{F}$ (from a fts X to another fts Y) endowed with fuzzy compact open topology becomes fuzzy T_2, when the range space Y is fuzzy T_2. Moreover, the fuzzy regularity of the range space induces somewhat regularity on the space of functions with fuzzy compact open topology.

Besides, we have introduced a new fuzzy topology called fuzzy F_{NR} topology on functions between two fts. The NR topology was introduced and studied in detail by Ganguly and Dutta [31] for functions between two topological spaces. Analogous to the definition of fuzzy compact open topology, given by Gunther Jagar in [45], we have introduced fuzzy nearly compact regular open (F_{NR}) topology and studied the function space under this fuzzy topology, imposing conditions on the respective range and domain fts. It is observed that the fuzzy GS-T_2-ness (fuzzy almost regularity) of the range fts Y induces fuzzy GS-T_2-ness (respectively, fuzzy somewhat almost regularity) in $\mathcal{F}$. We have also obtained that the evaluation map on $(\mathcal{F}, F_{NR})$ is fuzzy δ-continuous, if the range space is almost regular.

In Section (2.3), the notions of jointly fuzzy continuous on fuzzy compacta and jointly fuzzy δ-continuous on fuzzy near compacta are initiated, in connection with the fuzzy compact open topology and F_{NR} topology respectively. A special class of functions with fuzzy compact open topology is shown to be jointly fuzzy continuous on fuzzy compacta. The conditions for the F_{NR} topology to be jointly fuzzy δ-continuous are also formulated.

2.2 Fuzzy compact open and nearly compact regular open topology

In this section, we investigate fuzzy T_2-ness of function spaces under fuzzy compact open topology, as defined by Jagar and also by our own definition of fuzzy nearly compact regular open topology. Several forms of fuzzy T_2-ness are available in the literature, two of such being fuzzy T_2-ness [74] and GS-T_2-ness [33]. We find that

GS-T_2-ness of the codomain space is responsible for GS-T_2-ness of fuzzy compact open topology, Δ_{co} as well as fuzzy nearly compact regular open topology, F_{NR}. Moreover, we see that fuzzy T_2-ness of the codomain space also induces fuzzy T_2-ness on $(\mathcal{F}, \Delta_{co})$ and $(\mathcal{F}, F_{NR})$.

The last part of this section is mainly involved in finding out the behaviour of the function spaces, when the codomain space is fuzzy regular or fuzzy almost regular, under both the fuzzy topologies mentioned above.

Definition 2.2.1 [45] Let (X, τ) and (Y, σ) be two topological spaces and $\mathcal{F}$ be a nonempty collection of functions from X to Y. For each fuzzy compact set K on X and each fuzzy open set G on Y, a fuzzy set K_G on $\mathcal{F}$ is given by $K_G(g) =^{inf}_{x \in supp(K)} G(g(x))$. The collection of all such K_G forms a subbase for some fuzzy topology on $\mathcal{F}$, called fuzzy compact open topology and it is denoted by Δ_{co}.

Replacing fuzzy compact set K by fuzzy nearly compact set N on X and fuzzy open set G by fuzzy regular open set R on Y, we get a fuzzy set N_R on $\mathcal{F}$, given by $N_R(g) =^{inf}_{x \in supp(N)} R(g(x))$. The collection of all such fuzzy sets N_R, forms a subbase for some fuzzy topology on $\mathcal{F}$. We call this, fuzzy nearly compact regular open topology and denote it by F_{NR}.

Remark 2.2.1 Every crisp fuzzy point x_1 on a fts is fuzzy compact.

Proof. If x_1 is a crisp fuzzy point and $\mathcal{U}$ is any collection of fuzzy open sets on X with $x_1 \leq sup\{U : U \in \mathcal{U}\}$ then $1 \leq sup\{U : U \in \mathcal{U}\}(x)$ and for $y \neq x$, $0 \leq sup\{U : U \in \mathcal{U}\}(y)$. By definition of supremum, for any $\epsilon > 0$, $\exists\ U_k \in \mathcal{U}$ such that $U_k(x) > 1 - \epsilon$. Again for $y \neq x, U_k(y) \geq 0$, so that $(U_k + \epsilon)(y) > 0$. Hence, $\{U_k\}$ forms a finite subcollection of $\mathcal{U}$ such that $x_1 < U_k + \epsilon$. This proves that x_1 is fuzzy compact.

Remark 2.2.2 As x_1 is fuzzy compact, it is fuzzy nearly compact.

Remark 2.2.3 Here, in particular if we take $N = x_1$, we get $(x_1)_G(f) = Gf(x)$. Then the fuzzy topology generated by all $(x_1)_G$ is called fuzzy point regular open topology, denoted by F_{PR}.

Lemma 2.2.1 In a fts X,

(1) if x_α and y_β are any two fuzzy points with $x \neq y$ and A_1, B_1 are fuzzy open sets with $x_\alpha \in A_1$, $y_\beta \in B_1$ and $A_1 \not{q} B_1$ then there exist fuzzy regular open sets A and B such that $x_\alpha \in A$, $y_\beta \in B$ and $A \not{q} B$.

(2) if x_α and x_β $(\alpha < \beta)$ are any two fuzzy points and A_2, B_2 are fuzzy open sets with $x_\alpha \in A_2$, $x_\beta q B_2$ and $A_2 \not{q} B_2$ then there exist fuzzy regular open sets A and B such that $x_\alpha \in A$, $x_\beta q B$ and $A \not{q} B$.

Proof. **(1)** Consider $A = \overline{A_1}^0$ and $B = \overline{B_1}^0$. $x_\alpha \in A_1 \Rightarrow \alpha \leq A_1(x)$.

So, $\alpha \leq \overline{A_1}^0(x)$. Hence, $x_\alpha \in \overline{A_1}^0 = A$. Similarly, $y_\beta \in B_1$

$\Rightarrow y_\beta \in \overline{B_1}^0 = B$. $A_1 \not{q} B_1 \Rightarrow \forall\, x \in X,\ A_1(x) + B_1(x) \leq 1$.

$\Rightarrow A_1(x) \leq 1 - B_1(x)$.

$\Rightarrow \overline{A_1}(x) \leq (1 - B_1)(x)$.

$\Rightarrow \overline{A_1}^0(x) \leq \overline{A_1}(x) \leq (1 - B_1)(x)$.

$\Rightarrow \overline{A_1}^0(x) + B_1(x) \leq 1$.

$\Rightarrow B_1(x) \leq 1 - \overline{A_1}^0(x), \forall\, x \in X$

As $\overline{A_1}^0$ is a fuzzy open set, by similar argument as above,

$\overline{B_1}^0(x) \leq \overline{B_1}(x) \leq 1 - \overline{A_1}^0(x), \forall\, x \in X$.

Hence, $\overline{A_1}^0(x) + \overline{B_1}^0(x) \leq 1, \forall\, x \in X \Rightarrow \overline{A_1}^0 \not{q} \overline{B_1}^0 \Rightarrow A \not{q} B$. Hence, the result.

(2) x_α and x_β $(\alpha < \beta)$ are any two fuzzy points and A_2, B_2 are fuzzy open sets with $x_\alpha \in A_2$, $x_\beta q B_2$ and $A_2 \not{q} B_2$. $\alpha \leq A_2$, $\beta + B_2(x) > 1$ and $A_2(z) + B_2(z) \leq 1, \forall\, z \in X$. Choose $A = \overline{A_2}^0$ and $B = \overline{B_2}^0$. Then $\alpha \leq A_2(x) \leq \overline{A_2}^0(x) = A(x)$. So, $x_\alpha \in A$.

$\beta + B(x) = \beta + \overline{B_2}^0(x) \geq \beta + B_2(x) > 1$. So, $x_\beta q B$ and as in (1), $A \not{q} B$.

Theorem 2.2.1 Let (X, τ) and (Y, σ) be two fts, $\mathcal{F} \subseteq Y^X$, endowed with F_{NR} topology is fuzzy GS-T_2 when (Y, σ) is so.

Proof. Let f_λ and g_μ be two fuzzy points on $\mathcal{F}$.

Case (i): Suppose, $f \neq g$. Then $f(x) \neq g(x)$ for some $x \in X$. Now,

$(f(x))_\lambda$ and $(g(x))_\mu$ be two fuzzy points on Y with $f(x) \neq g(x)$ and Y is fuzzy GS-T_2, there exist fuzzy open sets U_1, V_1 with $(f(x))_\lambda \in U_1$, $(g(x))_\mu \in V_1$ and $U_1 \not{q} V_1$. By Lemma (2.2.1(1)), there exist fuzzy regular open sets U and V such that $(f(x))_\lambda \in U$, $(g(x))_\mu \in V$ and $U \not{q} V$. Which gives, $\lambda \leq U(f(x))$, $\mu \leq V(g(x))$ and $U \not{q} V$.

$\Rightarrow \lambda \leq (x_1)_U(f)$, $\mu \leq (x_1)_V(g)$ and $U \not{q} V$.

$\Rightarrow f_\lambda \in (x_1)_U$, $g_\mu \in (x_1)_V$ and $U \not{q} V$.

To show $U \not{q} V \Rightarrow (x_1)_U \not{q} (x_1)_V$.

$U \not{q} V \Rightarrow \forall\, z \in X, U(z) + V(z) \leq 1$. Now,

$\forall\, h \in \mathcal{F}, (x_1)_U(h) + (x_1)_V(h) = U(h(x)) + V(h(x)) \leq 1, \forall\, x \in X$.

Hence, $(x_1)_U \not{q} (x_1)_V$.

Case (ii): Suppose $f = g$. without loss of generality we assume $\lambda < \mu$. Then $f(x) = g(x), \forall\, x \in X$. $(f(x))_\lambda$ and $(g(x))_\mu$ with $\lambda < \mu$ are fuzzy points on Y with $f(x) = g(x)$. By fuzzy GS-T_2 ness of Y, $\exists$ fuzzy open sets A_2, B_2 on Y such that $(f(x))_\lambda \in A_2$, $(g(x))_\mu q B_2$ and $A_2 \not{q} B_2$. By Lemma (2.2.1(2)), there exist fuzzy regular open sets A and B on Y such that $(f(x))_\lambda \in A$, $(g(x))_\mu q B$ and $A \not{q} B$

$\Rightarrow \lambda \leq A(f(x))$, $\mu + B(g(x)) > 1$ and $A(z) + B(z) \leq 1, \forall\, z \in X$.

$\Rightarrow f_\lambda \in (x_1)_A$, $g_\mu(g) + (x_1)_B(g) > 1$ and $A(z) + B(z) \leq 1, \forall\, z \in X$.

Thus, $f_\lambda \in (x_1)_A$, $g_\mu q (x_1)_B$ and $A(z) + B(z) \leq 1, \forall\, z \in X$. Now, $\forall\, \psi \in \mathcal{F}$, $(x_1)_A(\psi) + (x_1)_B(\psi) = A(\psi(x)) + B(\psi(x)) \leq 1$.

So, $(x_1)_A \not q\ (x_1)_B$. Hence, $\mathcal{F}$ endowed with F_{NR} topology is fuzzy GS-T_2.

As F_{PR} is a special case of F_{NR} topology, we have,

Corollary 2.2.1 Let (X, τ) and (Y, σ) be two fts, $\mathcal{F} \subseteq Y^X$, endowed with F_{PR} topology is fuzzy GS-T_2 when (Y, σ) is fuzzy GS-T_2.

We append here, an analogous result for fuzzy compact open topology. As the method of proving this result is similar to that of Theorem (2.2.1), the proof is not given.

Theorem 2.2.2 Let (X, τ) and (Y, σ) be two fts, $\mathcal{F} \subseteq Y^X$, endowed with fuzzy compact open topology Δ_{co}. Then $(\mathcal{F}, \Delta_{co})$ is fuzzy GS-T_2 if (Y, σ) is so.

We also find that fuzzy T_2-ness on the codomain space induces fuzzy T_2-ness on functions with fuzzy compact open topology.

Theorem 2.2.3 Let (X, τ) and (Y, σ) be two fuzzy topological spaces and $\mathcal{F}$ be a nonempty collection of functions from X to Y, endowed with the fuzzy compact open topology Δ_{co}. Then $(\mathcal{F}, \Delta_{co})$ is fuzzy T_2 when (Y, σ) is fuzzy T_2.

Proof. Let f_λ and g_μ be two fuzzy points in $\mathcal{F}$ having supp$(f_\lambda) \neq$ supp(g_μ); i.e., $f \neq g$. Hence, there exists $x \in X$ such that $f(x) \neq g(x)$. Now, let us consider two fuzzy points $f(x)_\lambda$ and $g(x)_\mu$ on Y.

Clearly, $supp(f(x)_\lambda) \neq supp(g(x)_\mu)$. Hence, by fuzzy T_2-ness of Y, there exist *q-nbd.s* B and C of $f(x)_\lambda$ and $g(x)_\mu$ respectively such that $B \wedge C = 0$. As B is a *q-nbd.* of $f(x)_\lambda$, there exist a fuzzy open set A in Y such that $f(x)_\lambda\ qA$ and $A \leq B$. i.e., $f(x)_\lambda(f(x)) + A(f(x)) > 1$ and putting $K = x_1$ we get $K_A \leq K_B$ where K is fuzzy compact in X. i.e., $\lambda + A(f(x)) > 1$ and $K_A \leq K_B$.

i.e., $f_\lambda + inf[A(f(x) : x \in \{x\}] > 1$ and $K_A \leq K_B$.

i.e., $f_\lambda(f) + K_A(f) > 1$ and $K_A \leq K_B$.

i.e., $f_\lambda q K_A$ and $K_A \leq K_B$.

Hence, K_B is a *q-nbd.* of f_λ. Similarly, we can prove that K_C is a *q-nbd.* of g_μ. Now,

$(K_B \wedge K_C)(h)$

$= inf(K_B(h), K_C(h))$

$= inf[inf\{B(h(x)) : x \in supp(K)\}, inf\{C(h(x)) : x \in supp(K)\}]$

$= inf[B(h(x)), C(h(x))]$

$= (B \wedge C)(h(x))$

$= 0$

Corollary 2.2.2 Let (X, τ) and (Y, σ) be two fts, $\mathcal{F} \subseteq Y^X$, endowed with F_{PR} topology is fuzzy T_2 when (Y, σ) is fuzzy T_2.

Proof. Follows from the above Theorem, as F_{PR} is a special case of fuzzy compact open topology.

Lemma 2.2.2 Let (X,τ) and (Y,σ) be two fts, $\mathcal{F} \subset Y^X$ and $N = x_1$ be a crisp fuzzy point on X.

(i) If $\mathcal{F}$ is endowed with fuzzy nearly compact regular open topology, F_{NR} and F is fuzzy regular open on Y then $N_{\overline{F}} \geq \overline{N_F} \geq N_F$.

(ii) If $\mathcal{F}$ is endowed with fuzzy compact open topology, Δ_{co} and F is fuzzy open on Y then $N_{\overline{F}} \geq \overline{N_F} \geq N_F$

Proof. **(i)** Suppose, $N = x_1$, for some $x \in X$. Then,

$(1 - (x_1)_F)(g)$

$= 1 - (x_1)_F(g)$

$= 1 - F(g(x))$

$= inf\{(1 - F)(g(x)) : x \in supp(x_1)\}$

$= N_{1-F}(g)$. Hence, $1 - N_F = N_{1-F}$. Since $(x_1)_F$ is a member of F_{NR}, $1-(x_1)_F$ is fuzzy closed in $\mathcal{F}$. So, N_{1-F} is closed in $\mathcal{F}$. In particular if V is a fuzzy regular open set, $\overline{V}$ is fuzzy regular closed in Y. Hence, $N_{\overline{V}} = N_{1-(1-\overline{V})} = 1 - N_{1-\overline{V}}$. In other words, $N_{\overline{V}}$ is a fuzzy closed set on $\mathcal{F}$ such that $N_V \leq N_{\overline{V}}$, as $V(f(x)) \leq \overline{V}(f(x))$. Consequently, $\overline{N_V} \leq N_{\overline{V}}$, as $\overline{N_V}$ is the smallest fuzzy closed set containing N_V.

(ii) Similar to (i).

Theorem 2.2.4 Let $\mathcal{F}$ be a collection of functions from a fts (X,τ) to a fts (Y,σ). Consider the evaluation map $e_x : \mathcal{F} \to Y$ defined by $e_x(f) = f(x)$ for each $x \in X$.

(i) If $\mathcal{F}$ is endowed with F_{NR} topology and Y is fuzzy almost regular, then each e_x is fuzzy δ-continuous.

(ii) If $\mathcal{F}$ is endowed with Δ_{co} topology and Y is fuzzy regular, then each e_x is fuzzy continuous.

Proof. (i) Let f_λ be a fuzzy point on $\mathcal{F}$. As $e_x(f_\lambda)$ is a fuzzy set on Y, for any $y \in Y$,we have $e_x(f_\lambda) = (f(x))_\lambda$. Let $(f(x))_\lambda$ be a fuzzy point on Y and U be a fuzzy regular open *q-nbd.* of $(f(x))_\lambda$. As Y is fuzzy almost regular, there exist fuzzy regular open set V such that $(f(x))_\lambda qV$ and $\overline{V} \leq U$. Hence, $\lambda + V(f(x)) > 1$ and $V \leq \overline{V} \leq U$
$\Rightarrow 1 - \lambda < V(f(x)) \leq \overline{V}(f(x)) \leq U(f(x))$.
So, $(1 - f_\lambda)(f) < (x_1)_V(f) \leq (x_1)_{\overline{V}}(f) \leq (x_1)_U(f)$.
Using Lemma (2.2.2), we have $\overline{(x_1)_V}^0 \leq \overline{(x_1)_V} \leq (x_1)_{\overline{V}} \leq (x_1)_U$.
Hence, $f_\lambda q\overline{(x_1)_V}^0$. Now, for each $y \in Y$,
$e_x[\overline{(x_1)_V}^0](y)$
$= \sup_{e_x(g)=y}[\overline{(x_1)_V}^0(g)]$
$\leq \sup_{e_x(g)=y}[(x_1)_{\overline{V}}(g)]$
$= \sup_{e_x(g)=y}[\overline{V}(g)(x)]$
$\leq U(y)$. Hence, $e_x[\overline{(x_1)_V}^0] \leq U$. So, $\overline{(x_1)_V}^0$ is fuzzy regular *q-nbd.* of f_λ, as desired.

(ii) Similar to (i).

Theorem 2.2.5 (i) If K_μ is a fuzzy open set on $\mathcal{F}$ endowed with F_{NR}

topology, then $K_{\overline{\mu}}$ is a fuzzy δ-closed set on $\mathcal{F}$ and $K_\mu \le \overline{K_\mu} \le \delta$-$cl(K_\mu) \le K_{\overline{\mu}}$.

(ii) If K_μ is a fuzzy open set on $\mathcal{F}$ endowed with Δ_{co} topology, then $K_{\overline{\mu}}$ is a fuzzy closed set on $\mathcal{F}$ and $K_\mu \le \overline{K_\mu} \le K_{\overline{\mu}}$.

Proof. (i) Since μ is a fuzzy regular open set on Y, $\overline{\mu}$ is fuzzy regular closed and hence by fuzzy δ-continuity of $e_x : \mathcal{F} \to Y$ given by $e_x(f) = f(x), \forall\ x \in X$, $e_x^{-1}(\overline{\mu})$ is δ-closed in $\mathcal{F}$. Now, $\forall f \in \mathcal{F}$,

$K_{\overline{\mu}}(f)$

$= inf\{\overline{\mu}(f(x)) : x \in supp(K)\}$

$= inf\{e_x^{-1}(\overline{\mu})(f) : x \in supp(K)\}$.

So, $K_{\overline{\mu}} = inf\{e_x^{-1}(\overline{\mu}) : x \in supp(K)\}$. Hence, $K_{\overline{\mu}}$ is fuzzy δ-closed set on $\mathcal{F}$. Again, $\forall\ x \in X$, $e_x^{-1}(\mu) \le e_x^{-1}(\overline{\mu})$. Hence, $K_\mu \le \overline{K_\mu} \le \delta$-$cl(K_\mu) \le K_{\overline{\mu}}$.

(ii) Since μ is a fuzzy open set on Y, $\overline{\mu}$ is fuzzy closed and hence by fuzzy continuity of $e_x : \mathcal{F} \to Y$ given by $e_x(f) = f(x), \forall\ x \in X$, $e_x^{-1}(\overline{\mu})$ is fuzzy closed in $\mathcal{F}$. Now, $\forall\ f \in \mathcal{F}$,

$K_{\overline{\mu}}(f)$

$= inf\{\overline{\mu}(f(x)) : x \in supp(K)\}$

$= inf\{e_x^{-1}(\overline{\mu})(f) : x \in supp(K)\}$.

So, $K_{\overline{\mu}} = inf\{e_x^{-1}(\overline{\mu}) : x \in supp(K)\}$.

Hence, $K_{\overline{\mu}}$ is fuzzy closed set on $\mathcal{F}$. Again, $\forall\ x \in X$, $e_x^{-1}(\mu) \le e_x^{-1}(\overline{\mu})$.

Consequently, $K_\mu \leq \overline{K_\mu} \leq K_{\overline{\mu}}$.

Definition 2.2.2 A fts X is said to be fuzzy somewhat regular if for each fuzzy point x_α $(0 < \alpha < 1)$ and any fuzzy open set A with $x_\alpha q A$, there exists a fuzzy open set B and γ with $0 < \alpha < \gamma < 1$ such that $x_\gamma q B$ and $\overline{B} \leq A$.

Definition 2.2.3 A fts X is said to be fuzzy somewhat almost regular if for each fuzzy point x_α $(0 < \alpha < 1)$ and any fuzzy regular open set A with $x_\alpha q A$, there exists a fuzzy regular open set B and γ with $0 < \alpha < \gamma < 1$ such that $x_\gamma q B$ and $\bar{B} \leq A$.

Remark 2.2.4 It is clear that a fuzzy regular space is fuzzy somewhat regular and a fuzzy almost regular space is fuzzy somewhat almost regular.

Finally, we observe in the following two results that fuzzy regularity (fuzzy almost regularity) of the range space induces fuzzy somewhat regularity (respectively, fuzzy somewhat almost regularity) on the function space, equipped with fuzzy compact open topology (respectively, fuzzy nearly compact regular open topology).

Theorem 2.2.6 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F} \subset Y^X$ be endowed with Δ_{co} topology. Then $\mathcal{F}$ is fuzzy somewhat regular if Y is fuzzy regular.

Proof. Let (Y, σ) be fuzzy regular. Let g_λ be any fuzzy point on $\mathcal{F}$ and K_G be a subbasic fuzzy open set on $\mathcal{F}$ such that $g_\lambda q K_G$, where K is fuzzy compact on X and G is fuzzy open on Y. So, $g_\lambda(g) + K_G(g) > 1$. i.e., $1 - \lambda < \inf\{G(g(x)) : x \in supp(K)\}$. Hence, for all $x \in supp(K)$ and $(g(x))_\lambda$ on Y, we get $(g(x))_\lambda q G$. By fuzzy regularity of Y there exist a fuzzy open set F on Y, such that $(g(x))_\lambda q F$ and $\overline{F} \leq G$. i.e., $1 - \lambda < F(g(x)) \leq \overline{F}(g(x)) \leq G(g(x))$
So, $1 - \lambda \leq K_F(g) \leq K_{\overline{F}}(g) \leq K_G(g)$. Choose any β such that $0 < \lambda < \beta < 1$. Then $1-\beta < 1-\gamma$ and $1-g_\beta(g) \leq K_F(g) \leq K_{\overline{F}}(g) \leq K_G(g)$. Hence, using Theorem (2.2.5) $1-g_\beta(g) \leq K_F(g) \leq \overline{K_F}(g) \leq K_{\overline{F}}(g) \leq K_G(g)$. So, $g_\beta q K_F$ and $\overline{K_F} \leq K_G$. Hence, $(\mathcal{F}, \Delta_{co})$ is fuzzy somewhat regular.

Theorem 2.2.7 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F} \subset Y^X$ be endowed with F_{NR} topology. Then $\mathcal{F}$ is fuzzy somewhat almost regular if Y is fuzzy almost regular.

Proof. Let (Y, σ) be fuzzy almost regular. Let g_λ be any fuzzy point on $\mathcal{F}$ and N_R be a subbasic fuzzy open set on $\mathcal{F}$ such that $g_\lambda q N_R$, where N is fuzzy nearly compact on X and R is fuzzy regular open in Y. So, $g_\lambda(g) + N_R(g) > 1$.
i.e., $1 - \lambda < \inf\{R(g(x)) : x \in supp(N)\}$. Hence, $(g(x))_\lambda q R$, $\forall\ x \in supp(N)$. By fuzzy almost regularity of Y there exist a fuzzy regular

open set F in Y such that $(g(x))_\lambda qF$ and $\bar{F} \leq R$. i.e., $1-\lambda < F(g(x)) \leq \overline{F}(g(x)) \leq R(g(x))$

$\Rightarrow 1-\lambda \leq N_F(g) \leq N_{\overline{F}}(g) \leq N_R(g)$. Choose any β such that $0 < \lambda < \beta < 1$. Then $1-\beta < 1-\gamma$ and $1-g_\beta(g) \leq N_F(g) \leq N_{\overline{F}}(g) \leq N_R(g)$. Hence, using Theorem (2.2.5) $1-g_\beta(g) \leq N_F(g) \leq \overline{N_F}^0(g) \leq \overline{N_F}(g) \leq N_{\overline{F}}(g) \leq N_R(g)$. So, $g_\beta q\overline{N_F}^0$ and $\overline{\overline{N_F}^0} \leq N_R$. Hence, $(\mathcal{F}, F_{NR})$ is fuzzy somewhat almost regular.

2.3 Jointly fuzzy continuous and δ-continuous fuzzy topologies

The notion of joint continuity and joint δ-continuity have significant roles in general topology. In this section, defining the analogues of such topologies in fuzzy setting, we observe their interrelations with fuzzy compact open topology and fuzzy nearly compact regular open topology. We have also found a family of functions on which Δ_{co} (F_{NR}) becomes jointly fuzzy continuous on fuzzy compacta (respectively, jointly fuzzy δ-continuous on fuzzy near compacta).

Definition 2.3.1 Let (X,τ) and (Y,σ) be two fuzzy topological spaces and $\mathcal{F}$ be a nonempty collection of functions from X to Y. A fuzzy topology on $\mathcal{F}$ is said to be

(i) jointly fuzzy continuous (jointly fuzzy continuous on fuzzy compacta) if a function $P : \mathcal{F} \times X \to Y$ given by $P(f, x) = f(x)$ is fuzzy continuous (respectively, $P \mid_{\mathcal{F} \times supp(K)}$ is fuzzy continuous for each fuzzy compact set K on X).

(ii) jointly fuzzy δ-continuous (jointly fuzzy δ-continuous on fuzzy near compacta) if a function $P : \mathcal{F} \times X \to Y$ given by $P(f, x) = f(x)$ is fuzzy δ-continuous (respectively, $P \mid_{\mathcal{F} \times supp(N)}$ is fuzzy δ-continuous for each fuzzy nearly compact set N on X).

Definition 2.3.2 Let (X, τ) and (Y, σ) be two fuzzy topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i) fuzzy continuous on a fuzzy compact set K on X if for any fuzzy open set μ, $f \mid_{supp(K)}^{-1} (\mu)$ is fuzzy open on $supp(K)$. (ii) fuzzy δ-continuous on a fuzzy nearly compact set N on X if for any fuzzy regular open set μ, $f \mid_{supp(N)}^{-1} (\mu)$ is fuzzy regular open on $supp(N)$.

The following two results are immediate from the definitions and hence the proofs are omitted.

Theorem 2.3.1 (i) Each fuzzy topology on $\mathcal{F}$ which is jointly fuzzy continuous on fuzzy compacta is larger than the fuzzy compact open topology on $\mathcal{F}$.

(ii) Each fuzzy topology on $\mathcal{F}$ which is jointly fuzzy δ-continuous on fuzzy near compacta is larger than the fuzzy nearly compact regular open topology on $\mathcal{F}$.

We observe that each member of the collection of functions $\mathcal{F}$ is necessarily fuzzy continuous (fuzzy δ-continuous) if $\mathcal{F}$ is endowed with any jointly fuzzy continuous topology (respectively, jointly fuzzy δ-continuous topology). Since the methods of developments for both the cases are similar, we prove only one case in the following theorem and state the other as its subsequent theorem.

Theorem 2.3.2 If $\mathcal{F}$ is endowed with the fuzzy topology which is jointly fuzzy δ-continuous then each $f \in \mathcal{F}$ is fuzzy δ-continuous.

Proof. Let Δ be a fuzzy topology on $\mathcal{F}$ which is jointly fuzzy δ-continuous. Then P is fuzzy δ-continuous. Let x_β be any fuzzy point on X and V any fuzzy regular open *nbd.* of $(f(x))_\beta$ on Y. Now, $(f_1 \times x_\beta)(h,t)$

$= f_1(h) \wedge x_\beta(t)$

$$= \begin{cases} \beta, & if h = f \text{ and } t = x \\ 0, & otherwise. \end{cases}$$

Hence, $(f_1 \times x_\beta) = (f,x)_\beta$. So, $P(f_1 \times x_\beta) = P((f,x)_\beta) = (f(x))_\beta$. Using the fuzzy δ continuity of P, there exist fuzzy regular open *nbd.s* U_1 of f_1 in $\mathcal{F}$ and U_2 of x_β on X such that $P(U_1 \times U_2) \leq V$. Now, $P(U_1 \times U_2)(y)$

$= \sup\{(U_1 \times U_2)(h,t) : (h,t) \in \ P^{-1}(y)\}$, *where* $(h,t) \in \mathcal{F} \times X$.

Also, $P(U_1 \times U_2)(y)$

$\geq \sup\{(f_1 \times U_2)(h,t) : P(h,t) = y\}$

$= \sup\{(f_1 \times U_2)(h,t) : h(t) = y\}$

$= \sup\{(f_1(h) \wedge U_2(t) : h(t) = y\}$

$= \sup\{U_2(t) : f(t) = y\}$

$= f(U_2)(y)$

Hence, $f(U_2) \leq P(U_1 \times U_2) \leq V$, as desired.

Theorem 2.3.3 If $\mathcal{F}$ is endowed with the fuzzy topology which is jointly fuzzy continuous then each $f \in \mathcal{F}$ is fuzzy continuous.

Definition 2.3.3 A family $\mathcal{F}$ of functions from a fts X to a fts Y is said to be

(i) fuzzy equicontinuous on fuzzy compacta, if for each fuzzy compact set K on X and any fuzzy open set V on Y with $(f(x))_\alpha qV$, for some $f \in \mathcal{F}, x \in supp(K)$, then there exist a fuzzy open set U on $supp(K)$ such that $\forall\ h \in \mathcal{F}$, $h(U) < V$ and $t_\alpha qU, \forall\ t \in supp(K)$.

(ii) fuzzy δ-equicontinuous on fuzzy near compacta, if for each fuzzy nearly compact set N on X and any fuzzy regular open set V on Y with $(f(x))_\alpha qV$, for some $f \in \mathcal{F}, x \in supp(N)$, then there exist a fuzzy regular open set U on $supp(N)$ such that $\forall\ h \in \mathcal{F}$, $h(U) < V$ and $t_\alpha qU, \forall\ t \in supp(N)$.

Theorem 2.3.4 Let (X,τ) and (Y,σ) be two fts. If $\mathcal{F}$ is fuzzy δ-

equicontinuous on fuzzy near compacta then $\mathcal{F}$ endowed with F_{NR} topology is jointly fuzzy δ-continuous on fuzzy near compacta.

Proof. Let $(f,x)_\alpha$ be a fuzzy point on $\mathcal{F} \times supp(N)$ and V be any fuzzy regular open set on Y with $(P(f,x))_\alpha qV$ i.e., $(f(x))_\alpha qV$, $x \in supp(N)$. Since $\mathcal{F}$ is fuzzy δ-equicontinuous, there exist a fuzzy regular open set U on $supp(N)$ such that $\forall\ h \in \mathcal{F},\ h(U) < V$ and $t_\alpha qU, \forall\ t \in supp(N)$. Now, it is easy to see that $h(U) < V \Rightarrow U(z) < V(h(z)), \forall\ z \in X$. since $(N_{\overline{V}})^0$ and U are respectively fuzzy regular open in $\mathcal{F}$ and $supp(N)$, we need to show that $(f,x)_\alpha q\ ((N_{\overline{V}})^0 \times U)$ and $P((N_{\overline{V}})^0 \times U) \leq V$. Now,

$((N_{\overline{V}})^0 \times U)(f,x) + \alpha$

$\geq (N_V(f) \wedge U(x)) + \alpha$

$= [\inf_{t \in supp(N)} V(f(t)) \wedge U(x)] + \alpha$

$> [\inf_{t \in supp(N)} U(t) \wedge U(x)] + \alpha$

$\geq (1-\alpha) + \alpha.$

Hence, $(f,x)_\alpha q\ ((N_{\overline{V}})^0 \times U)$.

Again, $P((N_{\overline{V}})^0 \times U)(y)$

$= \sup_{P(h,t)=y} [((N_{\overline{V}})^0 \times U)(h,t)]$

$\leq \sup_{P(h,t)=y} [(N_{\overline{V}} \times U)(h,t)]$

$= \sup_{h(t)=y} [\inf_{s \in supp(N)} \overline{V}(h(s)) \wedge U(t)]$

$< \sup_{h(t)=y} [\inf_{s \in supp(N)} \overline{V}(h(s)) \wedge V(h(t))]$

$\leq^{sup}_{h(t)=y} [V(h(t))]$

$= V(y)$

Hence, $P((N_{\overline{V}})^0 \times U) \leq V$.

In a similar way it can be shown that:

Theorem 2.3.5 Let (X, τ) and (Y, σ) be two fts. If $\mathcal{F}$ is fuzzy equicontinuous on fuzzy compacta then $\mathcal{F}$ endowed with Δ_{co} is jointly fuzzy continuous on fuzzy compacta.

Chapter 3

Pseudo fuzzy δ-continuous functions and Starplus near compactness

3.1 Introduction

As we have already mentioned in Chapter 1, Lowen [53] introduced two functors ω and i between the category of all fuzzy topological spaces and fuzzy continuous functions and the category of all topological spaces and continuous functions. It was also observed in the same work that for each α with $\alpha \in [0, 1)$, the functor i_α associates a topology $i_\alpha(\tau)$ with the fuzzy topology τ.

It is a natural question, whether the fuzzy topologies on a family of functions studied in Chapter 2, correspond to the known compact open topology and NR topology in the general setting, via the functor i_α, for each $\alpha \in [0,1)$. In the process of their investigations, Kohli and Prasannan [49], introduced another form of fuzzy topology on function spaces, which they have called starplus compact open fuzzy topology and have shown that such fuzzy topology is the one that corresponds to the usual compact open topology.

In carrying out our research along the same line, we first notice that fuzzy regular open sets do not behave as fuzzy open sets under the above functorial correspondence.

In Section (3.2), we have observed that a strong α-level set $i_\alpha(\mu) = \mu^\alpha$ need not be regular open, if μ is fuzzy regular open. On the other hand, even the regular openness of μ^α, for all $\alpha \in [0,1)$ may fail to imply the fuzzy regularity of μ. This observation leads to the definition of new type of fuzzy sets termed as pseudo regular open fuzzy sets. In a similar manner, pseudo regular closed, pseudo δ-open, pseudo δ-closed fuzzy sets are also introduced. It is quite interesting to notice that a pseudo regular closed fuzzy is not a complement of pseudo regular open fuzzy set, and a pseudo δ-closed fuzzy set is not a complement of pseudo δ-open fuzzy set. Also, surprisingly, a pseudo δ-open fuzzy set may not be expressible as a union of pseudo regular open fuzzy sets, though a union of pseudo regular open fuzzy sets is indeed a pseudo δ-open fuzzy set. This fact gave birth to two new fuzzy topologies, that we have called *ps-δ* and *ps-ro* fuzzy topologies on X. We have also discussed their interrelations with the original fuzzy topology τ, in the same section.

In Section (3.3), we have introduced a class of functions, called pseudo fuzzy δ-continuous functions whose functorial counterpart, via the functor i_α for each $\alpha \in [0, 1)$ is precisely the family of all δ-continuous functions in general topology.

In Section (3.4), we have defined a new form of compact-like fuzzy sets and have called them starplus nearly compact fuzzy sets. We have discussed some fundamental properties of the same and also a couple of necessary conditions for starplus nearly compact fts, which play important role in determining when (X, τ) is not starplus nearly compact.

Through out this thesis, we denote $[0, 1)$ by I_1.

3.2 New fuzzy topologies from old

We begin this section with an example showing that on a fts (X, τ), if μ is fuzzy regular open, then μ^α need not be regular open in the corresponding topological space $(X, i_\alpha(\tau)), \alpha \in I_1$ and also μ^α may be regular open in $(X, i_\alpha(\tau)), \alpha \in I_1$, inspite of μ being not fuzzy regular open in the fts (X, τ).

Example 3.2.1 Let X be a set with at least two elements. Fix an element $y \in X$. Clearly, $\tau = \{0, 1, A\}$ is a fuzzy topology on X, where A is defined as $A(x) = \begin{cases} 0.5, & \text{for } x = y \\ 0.3, & otherwise. \end{cases}$

The fuzzy closed sets on (X, τ) are $0, 1$ and $1 - A$ where

$(1 - A)(x) = \begin{cases} 0.5, & \text{for } x = y \\ 0.7, & otherwise. \end{cases}$

Clearly, $A \leq 1 - A$ and hence $int(clA) = A$. i.e., A is fuzzy regular open in (X, τ). Now, in the corresponding topological space $(X, i_\alpha(\tau))$, $\alpha \in I_1$, the open sets are Φ, X and A^α where $A^\alpha =$

$$\begin{cases} X, & \text{for } \alpha < 0.3 \\ \{y\}, & \text{for } 0.3 \leq \alpha < 0.5 \\ \Phi, & \text{for } \alpha \geq 0.5. \end{cases}$$

For, $0.3 \leq \alpha < 0.5$, the closed sets on $(X, i_\alpha(\tau))$ are Φ, X and $X - \{y\}$. It is clear that $int(clA^\alpha) = X$. Hence, A^α is not regular open in $(X, i_\alpha(\tau))$ for $0.3 \leq \alpha < 0.5$.

Example 3.2.2 Let $X = \{x, y, z\}$. Define fuzzy sets μ, γ and η as follows: $\mu(a) = 0.4$, $\gamma(a) = 0.55$ and $\eta(a) = 0.6$, $\forall\ a \in X$. If $\tau = \{0, 1, \mu, \gamma, \eta\}$ then (X, τ) is a fts. The closed fuzzy sets are $(1 - \mu) = \eta$, $(1 - \gamma)(a) = 0.45$, $\forall\ a \in X$ and $(1 - \eta) = \mu$. Here,

$cl(\gamma) = \eta$ and $int(cl\gamma) = \eta$. Hence, γ is not fuzzy regular open set. But, $\gamma^\alpha = \{x : \gamma(x) > \alpha\}$. For $\alpha \geq 0.55$, $\gamma^\alpha = \Phi$, which is regular open. For $\alpha < 0.55$, $\gamma^\alpha = \{x, y, z\} = X$, which is also regular open. Hence for all $\alpha \in I_1, \gamma^\alpha$ is regular open in $(X, i_\alpha(\tau))$.

In view of these examples we define the following:

Definition 3.2.1 A fuzzy open set (fuzzy closed set) μ on a fts (X, τ) is said to be pseudo regular open (respectively, pseudo regular closed) fuzzy set if the strong α-level set μ^α is regular open (respectively, regular closed) in $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1$.

The following example establishes that pseudo regular closed and pseudo regular open fuzzy sets are not complements of each other.

Example 3.2.3 Let $X = \{x, y, z, w\}, \tau = \{0, 1, \mu\}$ where μ is defined as $\mu(x) = 0.1, \mu(y) = 0.2, \mu(z) = 0.3, \mu(w) = 0.4$.
Clearly, (X, τ) is a fts. If $\alpha = 0.3$, $i_\alpha(\tau) = \{\Phi, X, \mu^\alpha\}$ and $\mu^\alpha = \{x \in X : \alpha < \mu(x) \leq 1\} = \{w\}$. Closed sets on $(X, i_\alpha(\tau))$ are Φ, X and $\{x, y, z\}$. Here we shall show that μ is not pseudo regular open but its complement $(1 - \mu)$ is pseudo regular closed fuzzy set on (X, τ). As the smallest closed set containing μ^α is X, $cl(\mu^\alpha) = X$. So, $int(cl\mu^\alpha) = X \neq \mu^\alpha$. Hence, μ^α is not regular open in $(X, i_\alpha(\tau))$. This shows that μ is not pseudo regular open

in (X,τ). Here, $(1-\mu)(x) = 0.9, (1-\mu)(y) = 0.8, (1-\mu)(z) = 0.7, (1-\mu)(w) = 0.6$. $(1-\mu)^\alpha = \{x \in X : \alpha < (1-\mu)(x) \leq 1\}$. For, $\alpha \geq 0.6$, $i_\alpha(\tau) = \{\Phi, X\}$. So, $cl[int(1-\mu)^\alpha] = X$. Also, for $\alpha < 0.6, (1-\mu)^\alpha = X, cl[int(1-\mu)^\alpha] = X$, whatever be $i_\alpha(\tau)$. Hence, in any case $(1-\mu)^\alpha$ is regular closed in $(X, i_\alpha(\tau))$. This shows, $(1-\mu)$ is pseudo regular closed fuzzy set on (X,τ).

Definition 3.2.2 A fuzzy set μ on a fts (X,τ) is said to be pseudo δ-open (respectively, pseudo δ-closed) fuzzy set if the strong α-level set μ^α is δ-open (respectively, δ-closed) in $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1$.

Theorem 3.2.1 The collection of all pseudo δ-open fuzzy sets on a fts (X,τ) forms a fuzzy topology on X.

Proof. Straightforward.

Definition 3.2.3 The fuzzy topology as obtained in the above theorem is called pseudo δ-fuzzy topology (in short *ps*-δ fuzzy topology) on X. The complements of the members of *ps*-δ fuzzy topology are known as *ps*-δ closed fuzzy sets.

Theorem 3.2.2 In a fts (X,τ), union of pseudo regular open fuzzy sets is pseudo δ-open.

Proof. Let $\mu = \vee\{\mu_i : i \in \Lambda\}$, where μ_i is pseudo regular open fuzzy set on a fts (X,τ), for each $i \in \Lambda$. Here, μ_i^α is regular open in

$(X, i_\alpha(\tau)), \forall\ i \in \Lambda$. As, $(\vee \mu_i)^\alpha = \cup \mu_i^\alpha$ is δ-open in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$, μ is pseudo δ-open fuzzy set on (X, τ).

The following example shows that the converse of Theorem (3.2.2) is not true in general. i.e., Any pseudo δ-open fuzzy set on a fts (X, τ) need not be expressible as union of pseudo regular open fuzzy sets.

Example 3.2.4 Let $X = \{x, y, z\}$ and the fuzzy topology τ, generated by μ , γ , η where $\mu(x) = 0.4, \mu(y) = 0.4, \mu(z) = 0.5$, $\gamma(x) = 0.4$,$\gamma(y) = 0.6$, $\gamma(z) = 0.4$ and $\eta(x) = 0.5$, $\eta(y) = 0.5$, $\eta(z) = 0.6$. Consider $i_\alpha(\tau)$, for each α as follows:

Case 1: For $\alpha < 0.4$, $\mu^\alpha = \gamma^\alpha = \eta^\alpha = X$ and hence $i_\alpha(\tau) = \{X, \Phi\}$. Consequently, $\mu^\alpha, \gamma^\alpha$ and η^α are all regular open.

Case 2: For $\alpha \geq 0.6$, $\mu^\alpha = \gamma^\alpha = \eta^\alpha = \Phi$ and hence $i_\alpha(\tau) = \{X, \Phi\}$. Consequently, $\mu^\alpha, \gamma^\alpha$ and η^α are all regular open.

Case 3: For $0.4 \leq \alpha < 0.5$, $\mu^\alpha = \{z\}$, $\gamma^\alpha = \{y\}$, $\eta^\alpha = X$ and hence $i_\alpha(\tau) = \{X, \Phi, \{y\}, \{z\}, \{y, z\}\}$. We observe that $int(cl\mu^\alpha) = \mu^\alpha$, $int(cl\gamma^\alpha) = \gamma^\alpha$ and $int(cl\eta^\alpha) = \eta^\alpha$, proving all of them to be regular open. But $(\mu \vee \gamma)^\alpha = \{y, z\}$ is not regular open as $int(cl\{y, z\}) = X \neq \{y, z\}$.

Case 4: For $0.5 \leq \alpha < 0.6$, $\mu^\alpha = \Phi$, $\gamma^\alpha = \{y\}$, $\eta^\alpha = \{z\}$ and hence $i_\alpha(\tau) = \{X, \Phi, \{y\}, \{z\}, \{y, z\}\}$. In this case too all μ^α,γ^α and η^α are regular open but $(\gamma \vee \eta)^\alpha = \{y, z\}$ is not so.

Now, we consider a fuzzy set K on X as follows: $K(x) = 0.4$, $K(y) = 0.6$ and $K(z) = 0.6$. Clearly,

$$K^\alpha = \begin{cases} X, & \text{for } \alpha < 0.4 \\ \{y, z\}, & \text{for } 0.4 \leq \alpha < 0.6 \\ \Phi, & \text{for } \alpha \geq 0.6. \end{cases}$$

Hence, $K^\alpha = \begin{cases} X, & \text{for } \alpha < 0.4 \\ \mu^\alpha \cup \gamma^\alpha, & \text{for } 0.4 \leq \alpha < 0.5 \\ \gamma^\alpha \cup \eta^\alpha, & \text{for } 0.5 \leq \alpha < 0.6 \\ \Phi, & \text{for } \alpha \geq 0.6. \end{cases}$

and so, K^α is δ-open in $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1$. Therefore, K is pseudo δ-open fuzzy set in (X, τ). It can be shown easily that K is neither a pseudo regular open fuzzy set in (X, τ) nor is expressible as union of pseudo regular open fuzzy sets on (X, τ).

It is clear from the above example that $A^\alpha = B^\alpha$, for some $\alpha \in I_1$ does not imply $A = B$. However we have the following Theorem:

Theorem 3.2.3 If A and B are two fuzzy sets on a fts (X, τ) such that $A^\alpha = B^\alpha$, $\forall\ \alpha \in I_1$ then $A = B$.

Proof. For $\alpha = 0, A^0 = B^0 \Rightarrow A(x) > 0$ iff $B(x) > 0$. Hence, $A(x) = 0$ iff $B(x) = 0$. Suppose, $y \in Y$ is such that $A(y) > 0, B(y) > 0$ and

$A(y) \neq B(y)$. Let $A(y) = \alpha_1$ and $B(y) = \alpha_2$. Without any loss of generality, let us take $\alpha_1 > \alpha_2$. Since $A(y) = \alpha_1 > \alpha_2$, $y \in A^{\alpha_2}$, but $y \notin B^{\alpha_2}$ i.e., $A^{\alpha_2} \neq B^{\alpha_2}$, which is a contradiction. Hence, $\forall\ y \in Y$, $A(y) = B(y)$ i.e., $A = B$.

Theorem 3.2.4 If $\{\mu_i\}$ be a collection of all pseudo regular open fuzzy sets on a *fts* (X, τ), then

(i) $0, 1 \in \{\mu_i\}$.

(ii) $\forall\ \mu_1, \mu_2 \in \{\mu_i\} \Rightarrow \mu_1 \wedge \mu_2 \in \{\mu_i\}$.

Proof. As 0 and 1 are pseudo regular open fuzzy sets on a *fts* (X, τ), $0, 1 \in \{\mu_i\}$. Let $\mu_1, \mu_2 \in \{\mu_i\}$. Then $\mu_1^\alpha, \mu_2^\alpha$ are regular open in $(X, i_\alpha(\tau))$. Now, $\mu_1^\alpha \cap \mu_2^\alpha$ is regular open in $(X, i_\alpha(\tau))$. i.e., $(\mu_1 \wedge \mu_2)^\alpha = \mu_1^\alpha \cap \mu_2^\alpha$ is regular open in $(X, i_\alpha(\tau))$. Which shows $\mu_1 \wedge \mu_2$ is pseudo regular open fuzzy set on (X, τ).

Remark 3.2.1 In view of Theorem (3.2.4) the collection of all pseudo regular open fuzzy sets on (X, τ) generates a fuzzy topology, which we call pseudo regular open fuzzy topology (in short, *ps-ro* fuzzy topology) on X. The members of this topology are termed as *ps-ro* open fuzzy sets and their complements as *ps-ro* closed fuzzy sets.

Theorem 3.2.5 In a *fts* (X, τ), *ps-ro* fuzzy topology is coarser than τ.

Proof. Straightforward.

Theorem 3.2.6 In a fts (X, τ), *ps-ro* fuzzy topology is coarser than *ps-δ* fuzzy topology.

Proof. Let $\mu \in$ *ps-ro* fuzzy topology. So, $\mu = \bigvee_i \gamma_i$ where γ_i's are pseudo regular open fuzzy sets on (X, τ). Hence, $\mu^\alpha = (\bigvee_i \gamma_i)^\alpha = \bigcup_i \gamma_i^\alpha$ is δ-open in $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1$. This shows that $\mu \in$ *ps-δ* fuzzy topology on X.

Remark 3.2.2 In view of Example (3.2.4), in general *ps-ro* fuzzy topology is strictly coarser than *ps-δ* fuzzy topology in a fts (X, τ).

3.3 Pseudo fuzzy δ-continuous functions

As proposed in the introduction of this Chapter, defining pseudo fuzzy δ-continuous functions, we establish that such functions correspond to the well known δ-continuous functions in general topology, under the functorial correspondence i_α, for each $\alpha \in [0, 1)$. Moreover, we characterize such pseudo fuzzy δ-continuous functions in terms of *ps-δ* closed fuzzy sets as well as fuzzy points.

Definition 3.3.1 A function f from a fts X to a fts Y is pseudo fuzzy δ-continuous if $f^{-1}(U)$ is pseudo δ-open fuzzy set on X, for each pseudo δ-open fuzzy set U on Y.

Theorem 3.3.1 If a function $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous then $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$, is δ-continuous for each $\alpha \in I_1$, where $(X, \tau), (Y, \sigma)$ are *fts.*

Proof. Let v be a δ-open set in $i_\alpha(\sigma)$. As every δ-open set is open, $v \in i_\alpha(\sigma)$ and so there exist, $\mu \in \sigma$ such that $v = \mu^\alpha$. Now, $f^{-1}(\mu^\alpha)$

$= \{x \in X : f(x) \in \mu^\alpha\}$

$= \{x \in X : \mu(f(x)) > \alpha\}$

$= \{x \in X : (\mu f)(x) > \alpha\}$

$= \{x \in X : (f^{-1}(\mu))(x) > \alpha\}$

$= \{x \in X : x \in (f^{-1}(\mu))^\alpha\}$

$= (f^{-1}(\mu))^\alpha.$

Consider a fuzzy set ζ on Y given by

$$\zeta(z) = \begin{cases} 1 & if\ \mu(z) > \alpha \\ \alpha & otherwise. \end{cases}$$

Then $\zeta^\beta = \begin{cases} \mu^\alpha & if\ \beta \geq \alpha \\ Y & \beta < \alpha. \end{cases}$

Consequently, ζ^β is δ-open for all $\beta \in I_1$. Hence, ζ is a pseudo δ-open on Y. Since, f is pseudo fuzzy δ-continuous, $f^{-1}(\zeta)$ is pseudo δ-open fuzzy set on X. Now, $(f^{-1}(\zeta))^\alpha = f^{-1}(\zeta^\alpha) = f^{-1}(\mu^\alpha) = f^{-1}(v)$. Hence, $f^{-1}(v)$ is δ-open set whenever v is so. This proves that f is δ-continuous, for each $\alpha \in I_1$.

Theorem 3.3.2 A function $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$, is δ-continuous for each $\alpha \in I_1$, where $(X, \tau), (Y, \sigma)$ are fts then $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous.

Proof. Let μ be any fuzzy pseudo δ-open set in (Y, σ). μ^α is δ-open in $(Y, i_\alpha(\sigma))$. By the δ-continuity of $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$, $f^{-1}(\mu^\alpha) = (f^{-1}(\mu))^\alpha$ is δ-open in $(X, i_\alpha(\tau))$. Hence, $f^{-1}(\mu)$ is pseudo δ-open fuzzy set on (X, τ), proving f to be pseudo fuzzy δ-continuous.

A necessary condition for pseudo fuzzy δ-continuous functions follows next.

Theorem 3.3.3 If a function $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous then $f^{-1}(\mu)$ is pseudo δ-closed fuzzy set on (X, τ), for all pseudo δ-closed fuzzy set μ in (Y, σ).

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be pseudo fuzzy δ-continuous.
So, $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous for each $\alpha \in I_1$. Now, μ is pseudo δ-closed fuzzy set on (Y, σ). Hence, $\forall\ \alpha \in I_1$, μ^α is δ-closed in $(Y, i_\alpha(\sigma))$, that is $(Y - \mu^\alpha)$ is δ-open in $(Y, i_\alpha(\sigma))$. Now,

$f^{-1}(Y - \mu^\alpha)$

$= \{x \in X : f(x) \notin \mu^\alpha\}$

$= \{x \in X : \mu(f(x)) \leq \alpha\}$

$= \{x \in X : (\mu f)(x)) \leq \alpha\}$

$= X - \{x \in X : (\mu f)(x)) > \alpha\}$

$= X - \{x \in X : (f^{-1}(\mu))(x) > \alpha\}$

$= X - (f^{-1}(\mu))^{\alpha}$. As, $f^{-1}(Y - \mu^{\alpha})$ is δ-open, $(f^{-1}(\mu))^{\alpha}$ is δ-closed in $(X, i_{\alpha}(\tau))$, $\forall\ \alpha \in I_1$. Hence, $f^{-1}(\mu)$ is pseudo δ-closed fuzzy set on (Y, σ).

As a pseudo δ-closed set need not be the complement of a pseudo δ-open set, the converse of the Theorem (3.3.3) may not hold true. However, the following theorem characterizes pseudo fuzzy δ-continuous functions in terms of *ps*-δ closed fuzzy sets.

Theorem 3.3.4 A function $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous iff $f^{-1}(\mu)$ is *ps*-δ closed fuzzy set on a fts (X, τ), where μ is *ps*-δ closed fuzzy set on a fts (Y, σ).

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be pseudo fuzzy δ-continuous.
Then $f : (X, i_{\alpha}(\tau)) \to (Y, i_{\alpha}(\sigma))$ is δ-continuous, for each $\alpha \in I_1$.
Let μ be *ps*-δ closed fuzzy set on fts (Y, σ). Then $1 - \mu$ is pseudo δ-open fuzzy set on (Y, σ). So $(1 - \mu)^{\alpha}$ is δ-open and $f^{-1}((1 - \mu)^{\alpha}) = (f^{-1}(1 - \mu))^{\alpha}$ is δ-open fuzzy set on $(X, i_{\alpha}(\tau)), \forall\ \alpha \in I_1$. This shows that $f^{-1}(1 - \mu)$ is pseudo δ-open fuzzy set on (X, τ). Now,
$(1 - f^{-1}(1 - \mu))(x)$
$= 1 - f^{-1}(1 - \mu)(x)$
$= 1 - (1 - \mu)(f(x))$
$= \mu(f(x))$

$= f^{-1}(\mu)(x)$.

Hence, $f^{-1}(\mu)$ is ps-δ closed fuzzy set on (X,τ).

Conversely, Let μ be any pseudo δ-open and so, $(1-\mu)$ is ps-δ closed fuzzy set on (Y,σ). As, $f^{-1}(1-\mu)$ is ps-δ closed, $1-f^{-1}(1-\mu)$ is pseudo δ-open fuzzy set on $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. Again, $f^{-1}(\mu) = 1-f^{-1}(1-\mu), f^{-1}(\mu)$ is pseudo δ-open fuzzy set on (X,τ). Hence, f is pseudo fuzzy δ-continuous.

We see in the following theorem that pseudo fuzzy δ-continuous functions may also be characterized by means of fuzzy points.

Theorem 3.3.5 A function $f:(X,\tau)\to(Y,\sigma)$ is pseudo fuzzy δ-continuous iff for any pseudo δ-open fuzzy set μ on Y with $(f(x))_\alpha q\mu$ there exists a pseudo δ-open fuzzy set ν on X with $x_\alpha q\nu$ and $f(\nu)\leq \mu$.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be pseudo fuzzy δ-continuous and μ be any pseudo δ-open fuzzy set on Y with $(f(x))_\alpha q\mu$. Then $\mu(f(x))+\alpha>1$. i.e., $(f^{-1}(\mu))(x)+\alpha>1$. So, $x_\alpha q f^{-1}(\mu)$. Since f pseudo fuzzy δ-continuous, $f^{-1}(\mu)$ is pseudo δ-open in X. Now, $f(f^{-1}(\mu))\leq\mu$ is always true, which proves the result.

Conversely, let the condition hold. We shall prove $f:(X, i_\alpha(\tau))\to(Y, i_\alpha(\sigma))$ is δ-continuous, $\forall\ \alpha\in I_1$, which is sufficent to prove $f:(X,\tau)\to(Y,\sigma)$ is pseudo fuzzy δ-continuous. Let μ^α be δ-open in Y

with $f(x) \in \mu^\alpha$. i.e., $\mu f(x) > \alpha$. Let us consider a fuzzy set ζ on Y by $\zeta(z) = \begin{cases} 1, & \text{if } \mu(t) > \alpha \\ \alpha, & \text{otherwise.} \end{cases}$

For $\beta > \alpha$, $y \in \zeta^\beta \Rightarrow \zeta(y) > \beta$

$\Rightarrow \zeta(y) > \alpha \Rightarrow \zeta(y) = 1 \Rightarrow \mu(y) > \alpha$. So, $(\zeta)^\beta \subseteq (\mu)^\alpha$. Similarly, we have $(\mu)^\alpha \subseteq (\zeta)^\beta$. So, $(\mu)^\alpha = (\zeta)^\beta$. For $\alpha > \beta, \forall\, y \in Y$, by definition of ζ, $\zeta(y) > \beta \Rightarrow y \in \zeta^\beta$. i.e, $Y \subseteq \zeta^\beta$. So, $Y = \zeta^\beta$. Also, for $\alpha = \beta$, $(\mu)^\alpha = (\zeta)^\beta$. So, $\zeta^\beta = \begin{cases} \mu^\alpha, & \text{if } \beta \geq \alpha \\ Y, & \text{if } \beta < \alpha. \end{cases}$

Hence, ζ^β is δ-open, $\forall\ \beta \in I_1$. Thus, ζ is pseudo δ-open fuzzy set on Y. As $\mu f(x) > \alpha$, $\zeta(f(x)) = 1 > \alpha \Rightarrow \zeta(f(x)) + (1 - \alpha) > 1 \Rightarrow (f(x))_{1-\alpha} q \zeta$. By the given condition there exist a pseudo δ-open fuzzy set ν on X with $x_{1-\alpha} q \nu$ and $f(\nu) \leq \zeta$. i.e., with $1 - \alpha + \nu(x) > 1 \Rightarrow \nu(x) > \alpha \Rightarrow x \in \nu^\alpha$ and $(f(\nu))^\alpha \subseteq \zeta^\alpha$, as $f(\nu) \leq \zeta \Rightarrow (f(\nu))^\alpha \subseteq \zeta^\alpha$. Hence, $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous, $\forall\ \alpha \in I_1$ and so, $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous.

Combining all the results proved earlier, we get:

Theorem 3.3.6 For a function f from a fts (X, τ) to another fts (Y, σ), the following are equivalent:

(a) f is pseudo fuzzy δ-continuous.

(b) $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous for each $\alpha \in I_1$.

(c) $f^{-1}(\mu)$ is *ps*-δ closed fuzzy set on (X, τ), where μ is *ps*-δ closed fuzzy set on (Y, σ).

(d) for any pseudo δ-open fuzzy set μ on Y with $(f(x))_\alpha q\mu$ there exists a pseudo δ-open fuzzy set ν on X with $x_\alpha q\nu$ and $f(\nu) \leq \mu$.

3.4 Starplus nearly compact fuzzy sets

In this section, we define compact-like fuzzy sets called starplus nearly compact fuzzy sets, and observe that it generalizes the existing notion of starplus compact fuzzy sets [49]. Some properties of starplus nearly compact fuzzy sets are discussed here, along with a couple of necessary conditions for a fuzzy set to be starplus nearly compact.

Definition 3.4.1 A fuzzy set μ on a fts (X, τ) is said to be starplus nearly compactif μ^α is nearly compact on $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. A fts (X, τ) is said to be starplus nearly compact fts if $(X, i_\alpha(\tau))$ is nearly compact, $\forall\ \alpha \in I_1$.

It is clear from the definition that starplus compact [49] implies starplus nearly compact.

Theorem 3.4.1 The pseudo fuzzy δ-continuous image of a starplus nearly compact fuzzy set is also so.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be pseudo fuzzy δ-continuous and μ, a starplus nearly compact fuzzy set on X. By Theorem (3.3.6), $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous for each $\alpha \in I_1$. As μ^α is nearly compact on $(X, i_\alpha(\tau))$, $f(\mu^\alpha)$ is nearly compact on $(Y, i_\alpha(\sigma))$. Again since $(f(\mu))^\alpha = f(\mu^\alpha), f(\mu)$ is starplus nearly compact fuzzy set in X.

Similarly, we have

Theorem 3.4.2 Every pseudo regular closed fuzzy set is starplus nearly compact on a starplus nearly compact fts.

Proof. Let (X, τ) be starplus nearly compact fts and μ be a pseudo regular closed fuzzy set on (X, τ). Hence, μ^α is regular closed in the nearly compact topological space $(X,\ i_\alpha(\tau))$ for each $\alpha \in I_1$. As every regular closed set on nearly compact space is nearly compact, μ^α is nearly compact in $(X,\ i_\alpha(\tau))$. Hence, μ is starplus nearly compact on (X, τ).

Theorem 3.4.3 Every pseudo δ-closed fuzzy set is starplus nearly compact fuzzy set on starplus nearly compact fts.

Theorem 3.4.4 The union of a finite number of starplus nearly compact fuzzy sets is starplus nearly compact.

Proof. Let μ, γ be two starplus nearly compact fuzzy sets on a fts

(X, τ). For all $\alpha \in I_1$, μ^α, γ^α and hence $\mu^\alpha \cup \gamma^\alpha$ is nearly compact on $(X, i_\alpha(\tau))$. As $(\mu \vee \gamma)^\alpha = \mu^\alpha \cup \gamma^\alpha$, $(\mu \vee \gamma)^\alpha$ is nearly compact on $(X, i_\alpha(\tau))$. Hence, $(\mu \vee \gamma)$ is starplus nearly compact fuzzy set on (X, τ).

Theorem 3.4.5 If μ is starplus nearly compact fuzzy set on a fts (X, τ), for any pseudo regular closed fuzzy set ϑ in $(X, \tau), \mu \wedge \vartheta$ is starplus nearly compact.

Proof. Considering that every regular closed set on nearly compact space is nearly compact and $(\mu \wedge \vartheta)^\alpha = \mu^\alpha \cap \vartheta^\alpha$, the theorem follows.

Theorem 3.4.6 Every starplus nearly compact fuzzy set on Hausdorff fts is pseudo δ-closed fuzzy set.

Proof. Let μ be starplus nearly compact fuzzy set on Hausdorff fts (X, τ). Then, $(X, i_\alpha(\tau))$ is Hausdorff and μ^α is nearly compact in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. So, μ^α is δ-closed in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. Hence, μ is pseudo δ-closed in (X, τ).

Theorem 3.4.7 A fuzzy set μ on a fts (X, τ) with a finite support is starplus nearly compact.

Proof. Straightforward and hence omitted.

Theorem 3.4.8 A fuzzy set μ on a fts (X, τ) with a finite fuzzy topology is starplus nearly compact.

Proof. Straightforward and hence omitted.

Lemma 3.4.1 [54] Let (X,τ) be a topological space and let $(X,w(\tau))$ be the fully stratified fts. Then for each $\alpha \in I_1, i_\alpha(w(\tau)) = \tau$.

Theorem 3.4.9 Let (X,τ) be a topological space and let $(X,w(\tau))$ be the corresponding fully stratified fts. Then for any starplus nearly compact fuzzy set μ in fts $(X,w(\tau))$, $supp(\mu)$ is nearly compact in (X,τ).

Proof. As μ is starplus nearly compact fuzzy set on fts $(X,w(\tau)), \mu^\alpha$ is nearly compact in $i_\alpha(w(\tau)), \forall\ \alpha \in I_1$. By Lemma (3.4.1), μ^α is nearly compact in $\tau, \forall\ \alpha \in I_1$. In particular, $supp(\mu) = \mu^0$ is nearly compact in τ.

We conclude this section with a couple of necessary conditions for starplus nearly compact fts, which play important role in determining when (X,τ) is not starplus nearly compact.

Theorem 3.4.10 If a fts (X,τ) is starplus nearly compact then
(i) every collection of pseudo regular open fuzzy sets $\{\mu_i\}$ with $\vee\mu_i = 1$, implies, there exist a finite subcollection $\{\mu_i : i = 1,2,..,n\}$ such that, $\bigvee_{i=1}^{n} \mu_i = 1$.
(ii) every family $\mathcal{F}$ of pseudo regular closed fuzzy sets with $\wedge\{\mu_i : \mu_i \in \mathcal{F}\} = 0$ implies for each $\alpha \in I_1 - \{0\}$ there exist a finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $\wedge\{\mu_i : \mu_i \in \mathcal{F}_0\} \leq \alpha$.

Proof. (i) Let (X,τ) be a starplus nearly compact fts and $\{\mu_i\}$ be a collection of pseudo regular open fuzzy sets with $\vee\mu_i = 1$. $(X, i_\alpha(\tau))$ is nearly compact $\forall\ \alpha \in I_1$. Now, $X = 1^\alpha = (\vee\mu_i)^\alpha = \cup\mu_i^\alpha$, which shows that $\{\mu_i^\alpha\}$ is a regular open cover of X. Since X is nearly compact, $\{\mu_i^\alpha\}$ has a finite subcover. i.e., there exist $i = 1, 2, .., n$ such that $X = \bigcup_{i=1}^n {\mu_i}^\alpha = (\bigvee_{i=1}^n \mu_i)^\alpha$. Hence, for each $\forall\ \alpha \in I_1$, $1^\alpha = (\bigvee_{i=1}^n \mu_i)^\alpha \Rightarrow 1 = \bigvee_{i=1}^n \mu_i$.

(ii) Let (X, τ) be a starplus nearly compact fts and $\mathcal{F}$ be a family of pseudo regular closed fuzzy sets with $\wedge\{\mu_i : \mu_i \in \mathcal{F}\} = 0$.

Hence, $\{{\mu_i}^\alpha\}$ is a collection of regular closed sets in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1 - \{0\}$. We claim that $\cap{\mu_i}^\alpha = \Phi$. If not, let $x \in \cap{\mu_i}^\alpha$

$\Rightarrow \forall\ i,\ x \in {\mu_i}^\alpha$

$\Rightarrow \mu_i(x) > \alpha$

$\Rightarrow \alpha < \mu_i(x) < 1 \Rightarrow \wedge\{\mu_i\} \neq 0$, which is a contradiction. Hence, $\cap{\mu_i}^\alpha = \Phi$. As (X, τ) is starplus nearly compact fts, $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1 - \{0\}$ is nearly compact. So, for each $\alpha \in I_1 - \{0\}$ there is a finite sub family $\mathcal{F}_0 = \{{\mu_i}^\alpha : i = 1, 2, ..., n\}$ of $\mathcal{F}$ such that $\cap\{{\mu_i}^\alpha : {\mu_i}^\alpha \in \mathcal{F}_0\} = \Phi$. We claim that $\wedge\{\mu_i : i = 1, 2, ..., n\} \leq \alpha$. If possible let for $i = 1, 2, ..., n$, $inf\{\mu_i(x)\} > \alpha$. So, $\mu_i(x) > \alpha$, for all $i = 1, 2, ..., n$. $\Rightarrow \cap{\mu_i}^\alpha \neq \Phi$, which is a contradiction. Hence, $\wedge\{\mu_i : i = 1, 2, ..., n\} \leq \alpha$

Chapter 4

Starplus nearly compact pseudo regular open fuzzy topology on function spaces

4.1 Introduction

With the help of pseudo regular open fuzzy sets and starplus nearly compact fuzzy sets studied in Chapter 3, we have constructed in this chapter, a new type of fuzzy topology on function spaces. We have termed it as Starplus nearly compact pseudo regular open fuzzy topology (τ_{*NC}) and found it finer than F_{NR}. On the other hand, if we start with the NR-topology [31], we are able to establish that $w(N_R)$ is coincident with τ_{*NC}.

In the following section, defining pseudo δ-admissibility, we have shown that any pseudo δ-admissible fuzzy topology is finer than starplus nearly compact pseudo regular open fuzzy topology. We have also obtained the result that the concept of pseudo δ-admissibility ensures δ-admissibility of the corresponding strong α-level topology.

4.2 Starplus nearly compact pseudo regular open fuzzy topology

Definition 4.2.1 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. For each starplus nearly compact fuzzy set K on X and each pseudo regular open fuzzy set μ on Y, a fuzzy set K_μ on $\mathcal{F}$ is given by

$$K_\mu(g) = \inf_{x \in supp(K)} \mu(g(x))$$

The collection of all such K_μ forms a subbase for some fuzzy topology on $\mathcal{F}$, called starplus nearly compact pseudo regular open fuzzy topology and it is denoted by τ_{*NC}.

Definition 4.2.2 [48] Let (X, τ) and (Y, σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. For each $x \in X$, define a map $e_x : \mathcal{F} \to Y$ by $e_x(g) = g(x)$. The map e_x is called the evaluation map at the point x. The initial fuzzy topology τ_p generated by the collection of maps $\{e_x : x \in X\}$ is called the pointwise fuzzy topology on $\mathcal{F}$.

Theorem 4.2.1 [48] If Y is a fuzzy Hausdorff fts, then the fts $(\mathcal{F}, \tau_p)$ is fuzzy Hausdorff.

Remark 4.2.1 $\forall\ x \in X$, $\forall\ g \in \mathcal{F}$ and any fuzzy set μ on Y, $({e_x}^{-1}(\mu))(g) = \mu(e_x(g)) = \mu(g(x))$.

So, $K_\mu(g)$

$= inf\{\mu(g(x)) : x \in supp(K)\}$

$= inf\{({e_x}^{-1}(\mu))(g) : x \in supp(K)\}$

$= (inf\{{e_x}^{-1}(\mu) : x \in supp(K)\})(g)$

Hence, K_μ

$= inf\{e_x^{-1}(\mu) : x \in supp(K)\}$

$= \bigwedge_{x \in supp(K)} e_x^{-1}(\mu)$.

Theorem 4.2.2 The starplus nearly compact pseudo regular open fuzzy topology τ_{*NC} is finer than the pointwise fuzzy topology τ_p on $\mathcal{F}$.

Proof. As every fuzzy set with finite support is starplus compact, it is starplus nearly compact, and so the theorem follows.

Theorem 4.2.3 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F}$ be endowed with starplus nearly compact pseudo regular open fuzzy topology. Then $(\mathcal{F}, \tau_{*NC})$ is fuzzy Hausdorff when (Y, σ) is fuzzy Hausdorff.

Proof. By Theorem (4.2.2) and (4.2.1), the theorem follows.

Remark 4.2.2 If $\mathcal{F}$ is a collection of functions from X to Y, then we shall denote the set $\{f \in \mathcal{F} : f(T) \subset U\}$, by $[T, U]$, where $T \subset X$ and $U \subset Y$.

Theorem 4.2.4 If K_μ is a subbasic open fuzzy set on τ_{*NC}, then $K_\mu^\alpha = [supp(K), \mu^\alpha]$.

Proof. In view of Remark (4.2.1),

K_μ^α

$= (\bigwedge_{x \in supp(K)} e_x^{-1}(\mu))^\alpha$

$= \{f \in \mathcal{F} : \bigwedge_{x \in supp(K)} e_x^{-1}(\mu)(f) > \alpha\}$

$= \{f \in \mathcal{F} : \bigwedge_{x \in supp(K)} \mu(f(x)) > \alpha\}$

$= \{f \in \mathcal{F} : f(supp(K)) \subseteq \mu^\alpha\}$

$= [supp(K), \mu^\alpha]$

Remark 4.2.3 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. Let us denote by N_R^α the ordinary nearly compact regular open topology [31] when X is endowed with 0-level topology $i_0(\tau)$ and Y with $i_\alpha(\sigma), \forall\ \alpha \in I_1$.

Theorem 4.2.5 The strong α-level topology $i_\alpha(\tau_{*NC})$ on $\mathcal{F}$, where $\alpha \in I_1$, is coarser than N_R^α topology on $\mathcal{F}$.

Proof. Let $\beta = \bigwedge_{i=1}^n K_{\mu_i}^i$ be a basic fuzzy open set on τ_{*NC}. The strong α-level set β^α is given by

β^α

$= (\bigwedge_{i=1}^{n} K_{\mu_i}^i)^\alpha$

$= \bigcap_{i=1}^{n} (K_{\mu_i}^i)^\alpha$

$= \bigcap_{i=1}^{n} [supp(K^i), \mu_i^\alpha]$.

As each K^i is starplus nearly compact, $supp(K^i)$ is nearly compact in $i_0(\tau)$. Again $\forall\ \alpha \in I_1$, μ_i^α being regular open in $i_\alpha(\sigma)$, it follows that β^α is a basic open set in N_R^α. This proves the theorem.

We shall take the same example as discussed in [49] to show that in general, two topologies $i_\alpha(\tau_{*NC})$ and N_R^α on $\mathcal{F}$ are not same.

Example 4.2.1 Let X be an infinite set and τ be the fuzzy topology on X generated by the collection
$\{(\frac{1}{2}\chi_U) \vee \chi_{X-U} : U \in X \text{ and } (X - U) \text{ is finite}\}$. Then

$$i_\alpha(\tau) = \begin{cases} \text{the discrete topology on } X, & for\ \alpha \geq \frac{1}{2} \\ \text{the indiscrete topology on } X, & for\ \alpha < \frac{1}{2} \end{cases}$$

For an infinite subset T of X, $\chi_T = K$(say), is a fuzzy set on X. Now, $supp(K) = T$ is compact and hence nearly compact in $i_0(\tau)$ but K^α is not nearly compact in $i_\alpha(\tau)$, for $\alpha \geq \frac{1}{2}$. Hence, K^α is not starplus nearly compact on (X, τ) and hence the fuzzy set $K_\mu \notin \tau_{*NC}$, for any pseudo regular open fuzzy set on Y. In fact, $K_\mu^\alpha = [supp(K), \mu^\alpha] = [T, \mu^\alpha] \in N_R^\alpha$. Hence, $i_\alpha(\tau_{*NC}) \neq N_R^\alpha$.

Theorem 4.2.6 Let (X,T) and (Y,U) be topological spaces and let (X,T_f) and (Y,U_f) denote corresponding characteristic fts, respectively. Then for each $\alpha \in I_1$, $i_\alpha(\tau_{*NC}) = N_R$, where N_R is the ordinary nearly compact regular open topology on $\mathcal{F}$.

Proof. $\forall\ \alpha \in I_1$, $i_\alpha(T_f) = T$ and $i_\alpha(U_f) = U, N_R^\alpha = N_R$. By Theorem (4.2.5) $i_\alpha(\tau_{*NC}) \subseteq N_R$. Now, let $[K,V] \in N_R$, where K is nearly compact in X and V is regular open in Y. Then the fuzzy set $S = \chi_K$ is starplus nearly compact in X and $\mu = \chi_V$ is pseudo regular open fuzzy set on Y, and $S_\mu^\alpha = [K,V]$, for each $\alpha \in I_1$. So, $[K,V] \in i_\alpha(\tau_{*NC})$. Hence, for each $\alpha \in I_1$, $i_\alpha(\tau_{*NC}) = N_R$.

Theorem 4.2.7 Let (X,T_X) and (Y,T_Y) be topological spaces and let N_R denote nearly compact regular open topology on $\mathcal{F}$. Then $\tau_{*NC} = w(N_R)$, where X and Y are endowed with the fuzzy topologies $w(T_X)$ and $w(T_Y)$, respectively.

Proof. Let N_R be the nearly compact regular open topology on $\mathcal{F}$, where X and Y are endowed with the fuzzy topologies $w(T_X)$ and $w(T_Y)$, respectively. Let $K_\mu \in \tau_{*NC}$, where K is starplus nearly compact in $w(T_X)$ and μ is pseudo regular open fuzzy set on $w(T_Y)$. By Theorem (4.2.4), $K_\mu^\alpha = [supp(K), \mu^\alpha]$. Using Theorem (3.4.9), $supp(K)$ is nearly compact in T_X. As μ^α is regular open in T_Y, $[supp(K), \mu^\alpha]$ is a subbasic open set in N_R and so K_μ is lower semi-continuous. Hence, $K_\mu \in w(N_R)$. Consequently, $\tau_{*NC} \subseteq w(N_R)$.

Conversely, let $v \in w(N_R)$. Then $\forall\ \alpha \in I_1, v^{-1}(\alpha, 1] \in N_R$. We will show that v is a τ_{*NC} neighborhood of each of its points. Let $g_\lambda \in v$. Then for $\alpha < \lambda$, $g \in v^{-1}(\alpha, 1]$. Since $v^{-1}(\alpha, 1] \in N_R$, there exist nearly compact sets $K_1, K_2, ...K_n$ in X and regular open sets $U_1, U_2, ...U_n$ in Y such that $g \in \cap_{i=1}^{n}[K_i, U_i] \subset v^{-1}(\alpha, 1]$. Now, the fuzzy set $S^i = \chi_{K_i}$ is starplus nearly compact in $w(T_X)$ with support K_i. Since for each $i = 1, 2..., n$, $U_i \in T_Y$ and $g(K_i) \subset U_i$, then the fuzzy set $\mu_i = (\chi_{U_i} \wedge \lambda) \in w(T_Y)$ such that $\mu_i(g(x)) = \lambda$, for each $x \in K_i$ and $\mu_i^\alpha = U_i$ for $\alpha < \lambda$. Hence, $g \in \cap_{i=1}^{n}[supp(S^i), \mu_i^\alpha] \subset v^{-1}(\alpha, 1]$. Now, using Theorem (4.2.5), $g_\lambda \in \wedge_{i=1}^{n} S_{\mu_i}^i \leq v$. Hence, v is a τ_{*NC} neighborhood of each of its points.

4.3 Pseudo δ-admissible fuzzy topology

Definition 4.3.1 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. A fuzzy topology T on $\mathcal{F}$ is said to be pseudo δ-admissible (pseudo δ-admissible on starplus near compacta) if a function $P : \mathcal{F} \times X \to Y$ given by $P(f, x) = f(x)$ is pseudo fuzzy δ-continuous (respectively, $P\ |_{\mathcal{F}\times supp(K)}$ is pseudo fuzzy δ-continuous for each starplus nearly compact set K on X), where $\mathcal{F} \times X$ is endowed with the product fuzzy topology.

Theorem 4.3.1 If T is a pseudo δ-admissible fuzzy topology on $\mathcal{F}$, then for each $\alpha \in I_1$, the strong α-level topology $i_\alpha(T)$ is jointly δ-continuous.

Proof. Let T be a pseudo δ-admissible fuzzy topology on $\mathcal{F}$. So, $P : \mathcal{F} \times X \to Y$ given by $P(f, x) = f(x)$ is pseudo fuzzy δ-continuous. Hence, $P : (\mathcal{F}, i_\alpha(T)) \times (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma)$ is δ-continuous, for all $\alpha \in I_1$. This shows that $i_\alpha(T)$ is jointly δ-continuous for all $\alpha \in I_1$.

Definition 4.3.2 Let μ be a fuzzy set on a fts (X, τ). The subspace fuzzy topology on μ is given by $\{v \mid_{supp(\mu)}: v \in \tau\}$ and is denoted by τ_μ. The pair $(supp(\mu), \tau_\mu)$ is called the subspace fts of μ.

Definition 4.3.3 Let (X, τ) and (Y, σ) be two fts. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be pseudo fuzzy δ-continuous on a fuzzy set μ on X, if $f \mid_{supp(\mu)}$ is pseudo fuzzy δ-continuous, where $supp(\mu)$ is endowed with the subspace fuzzy topology τ_μ.

Theorem 4.3.2 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F} \subset Y^X$. Then every fuzzy topology on $\mathcal{F}$ which is pseudo δ-admissible on starplus near compacta is finer than the starplus nearly compact pseudo regular open fuzzy topology τ_{*NC} on $\mathcal{F}$.

Proof. Let $(\mathcal{F}, T)$ be pseudo δ-admissible on starplus near compacta. Let K_μ be any subbasic fuzzy open set on τ_{*NC} where K is starplus near compact on X and μ is pseudo regular open fuzzy set on Y. The function $P\mid_{\mathcal{F}\times supp(K)}: \mathcal{F}\times supp(K)\to Y$ given by $P(f,x)=f(x)$ is pseudo fuzzy δ-continuous. So, $P\mid^{-1}_{\mathcal{F}\times supp(K)}(\mu)$ is pseudo δ-open fuzzy set on $\mathcal{F}\times supp(K)$. For simplicity of notation , instead of $P\mid_{\mathcal{F}\times supp(K)}$ we shall use the symbol P only. Let f_α be any fuzzy point in K_μ. i.e., $K_\mu(f)\geq\alpha\Rightarrow inf\{\mu(f(x)): x\in supp(K)\}\geq\alpha$. We now prove, $f_\alpha\times\chi_{supp(K)}\leq P^{-1}(\mu)$. Now,

$(f_\alpha\times\chi_{supp(K)})(g,t)$

$=f_\alpha(g)\wedge\chi_{supp(K)}(t)$

$$=\begin{cases}\alpha, & iff=g\ and\ t\in supp(K)\\[2ex] 0, & otherwise.\end{cases}$$

Again, $P^{-1}(\mu)(g,t)=\mu(P(g,t))=\mu(g(t))\geq\alpha$.

Hence, $f_\alpha\times\chi_{supp(K)}\leq P^{-1}(\mu)$.

Consider the first projection $\Pi_1:(\mathcal{F},i_\alpha(T))\times(X,i_\alpha(\tau))\to(\mathcal{F},i_\alpha(T))$.

Now, $(\Pi_1(P^{-1}(\mu)))^\alpha$

$=\{f:\Pi_1(P^{-1}(\mu))(f)>\alpha\}$

$=\{f:^{sup}_{\Pi_1(g,t)=f}[P^{-1}(\mu)(g,t)>\alpha]\}$

$=\{f:^{sup}_{g=f}P^{-1}(\mu)(g,t)>\alpha\}$

$=\{f:P^{-1}(\mu)(f,t)>\alpha\}$

$=\{\Pi_1(f,t):(f,t)\in(P^{-1}(\mu))^\alpha\}$

$=\Pi_1(P^{-1}(\mu))^\alpha$

So, $(\Pi_1(P^{-1}(\mu)))^\alpha=\Pi_1(P^{-1}(\mu))^\alpha$. As $P^{-1}(\mu)$ is pseudo δ-open fuzzy set, $(P^{-1}(\mu))^\alpha$ is δ-open. Π_1 being projection mapping, $(\Pi_1(P^{-1}(\mu)))^\alpha = \Pi_1(P^{-1}(\mu))^\alpha$ is δ-open set. So, $\Pi_1(P^{-1}(\mu))$ is pseudo δ-open fuzzy set. As, $f_\alpha \in K_\mu$ we have,

$K_\mu(f) \geq \alpha$

$\Rightarrow inf\{\mu(f(x)) : x \in supp(K)\} \geq \alpha$. So, $\mu(f(s)) \geq \alpha, \forall\ s \in supp(K)$. $\Pi_1(P^{-1}(\mu))(g,s)$

$=^{sup}_{\Pi_1(g,s)=f} [P^{-1}(\mu)(g,s)]$

$= P^{-1}(\mu)(f,s)$

$= \mu(f(s))$

$\geq \alpha$. i.e., $f_\alpha \leq \Pi_1(P^{-1}(\mu))$.

Now, $\Pi_1(P^{-1}(\mu)) \times \chi_{supp(K)})(g,t)$

$= \Pi_1(P^{-1}(\mu))(g) \wedge \chi_{supp(K)}(t)$

$=^{sup}_{\Pi_1(f,t)=g} [P^{-1}(\mu)(g,t)] \wedge \chi_{supp(K)}(t)$

$= P^{-1}(\mu)(g,t) \wedge \chi_{supp(K)}(t)$

$$= \begin{cases} \mu(g(t)), & if\ t \in supp(K) \\ 0, & otherwise. \end{cases}$$

$\leq P^{-1}(\mu)(g,t)$.

Hence, $\Pi_1(P^{-1}(\mu)) \times \chi_{supp(K)} \leq P^{-1}(\mu)$.

Now, $\Pi_1(P^{-1}(\mu))(g)$

$=^{sup}_{\Pi_1(h,s)=g} [P^{-1}(\mu)(h,s)\ : s \in supp(K)]$

$= P^{-1}(\mu)(g,s)$

$= \mu(g(s)), \forall\ s \in supp(K)$

$=^{inf}_{t \in supp(K)}\ \mu(g(t))$

$=\ K_\mu(g).$

So, for any fuzzy point g_λ on $\Pi_1(P^{-1}(\mu)) \in T$, $g_\lambda \in K_\mu$, which proves the theorem.

We now state a lemma to prove the final result of this section. It is to mention here that this lemma has already been established by us in Theorem(3.3.5).

Lemma 4.3.1 A function $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous iff for any pseudo δ-open fuzzy set μ on Y with $(f(x))_\alpha q\mu$ there exists a pseudo δ-open fuzzy set ν on X with $x_\alpha q\nu$ and $f(\nu) \leq \mu$.

Theorem 4.3.3 In a fully stratified Hausdorff fts (X, τ), if each member of $\mathcal{F} \subset Y^X$ is pseudo fuzzy δ-continuous on every starplus nearly compact fuzzy set of X, then the starplus nearly compact pseudo regular open fuzzy topology τ_{*NC} on $\mathcal{F}$ is pseudo δ-admissible on starplus near compacta.

Proof. Let (X, τ) be a fully stratified Hausdorff fts. Let $f \in \mathcal{F}$ and $(f(x))_\alpha q\mu$, for any pseudo δ-open fuzzy set on Y. Since K is starplus nearly compact in $(supp(K), \tau_K)$, where τ_K is the subspace fuzzy topology on $supp(K)$, K^α is nearly compact on $(supp(K), i_\alpha(\tau_K)), \forall\ \alpha \in$

I_1. Again $f:(supp(K),\tau_K)\to(Y,\sigma)$ is pseudo fuzzy δ-continuous, $f:(supp(K),i_\alpha(\tau_K))\to(Y,i_\alpha(\sigma))$ is δ-continuous, $\forall\ \alpha\in I_1$. (X,τ) being Hausdorff fts, $(supp(K),\tau_K)$ is Hausdorff $\forall\ \alpha\in I_1$. Hence, there exist a nearly compact *nbd.* M_α of x in $(supp(K),i_\alpha(\tau_K))$ and $f(M_\alpha)\subset\mu^\alpha$, as μ^α is open in $i_\alpha(\tau_K), \forall\ \alpha\in I_1$. Now, we choose $\beta>1-\lambda$. Let $K^*=(\chi_{M_\beta}\wedge\beta)$. As,

$$
\begin{aligned}
&(K^*)^\alpha\\
&=(\chi_{M_\beta}\wedge\beta)^\alpha\\
&=\{x:(\chi_{M_\beta}\wedge\beta)(x)>\alpha\}\\
&=\{x:\beta>\alpha\ and\ x\in M_\beta\}\\
&=\begin{cases} M_\beta, & if\ \beta>\alpha\\ \\ \Phi, & if\ \beta\le\alpha\end{cases}
\end{aligned}
$$

K^* is starplus nearly compact in the subspace fts $(supp(K),\tau_K)$ such that $f(K^*)\le\mu$. Now,

$$
\begin{aligned}
&(K^*_\mu\times\chi_{supp(K)})(f,x)\\
&=K^*_\mu(f)\wedge\chi_{supp(K)}(x)\\
&=\begin{cases} K^*_\mu(f), & if\ x\in supp(K)\\ \\ 0, & otherwise\end{cases}\\
&=inf\{\mu f(z):z\in supp(K^*)\}, x\in supp(K)\\
&>1-\lambda,\ as\ (f(x))_\lambda q\mu\ and\ z\in supp(K^*)\Rightarrow\mu f(z)>\beta.
\end{aligned}
$$

Hence, $(K^*_\mu\times\chi_{supp(K)})\mid_{(\mathcal{F}\times supp(K))}$ is a *q-nbd.* of the fuzzy point $(f,x)_\lambda$

on $(\mathcal{F}\times supp(K))$. Also, it can be seen that $(K_\mu^*\times\chi_{supp(K)})\,|_{(\mathcal{F}\times supp(K))}\leq P^{-1}(\mu)$. Hence the theorem.

Chapter 5

Pseudo nearly compact fuzzy sets and *ps*-*ro* continuous functions

5.1 Introduction

By accepting the necessary condition obtained in Theorem(3.4.10) (i) in Chapter 3, for a fuzzy set to be starplus nearly compact as a basic definition of a compact-like covering property, we introduce the notion of pseudo near compactness. The pseudo near compactness has been studied via fuzzy nets, fuzzy filterbase and properties of *ps*-*ro* closed fuzzy sets.

We have further introduced two operators, called by us fuzzy *ps*-closure and fuzzy *ps*-interior. In section (5.3), we have formed a new type of fuzzy continuous-like function, named by us, pseudo fuzzy *ro*-continuous function. This we have studied with the aid of the preceding two operators. Apart from these operators, we have also studied them through other concepts like fuzzy points, *ps-ro* closed fuzzy sets and so on. Besides, we are able to show that this type of functions preserve pseudo near compactness of a fts.

5.2 ps-closure and ps-interior operators

Definition 5.2.1 The union of all *ps-ro* open fuzzy sets, each contained in a fuzzy set A on a fts X is called fuzzy ps-interior of A and is denoted by *ps-int*(A). So, *ps-int*$(A) = \vee\{B : B \leq A,\ B$ is *ps-ro* open fuzzy set on $X\}$

Definition 5.2.2 The intersection of all *ps-ro* closed fuzzy sets on a fts X, each containing a fuzzy set A on X is called fuzzy ps-closure of A and is denoted by *ps-cl*(A). So, *ps-cl*$(A) = \wedge\{B : A \leq B,\ B$ is *ps-ro* closed fuzzy set on $X\}$

Some properties of *ps-cl* and *ps-int* operators are furnished below. Since the proofs are straightforward, we only state the properties without proof.

Theorem 5.2.1 For any fuzzy set A on a fts (X, τ), the following hold:

(i) *ps-cl*(A) is the smallest *ps-ro* closed fuzzy set containing A.

(ii) $ps\text{-}cl(A) \leq ps\text{-}cl(B)$ if $A \leq B$.

(iii) $ps\text{-}cl(A) = A$ iff A is *ps-ro* closed.

(iv) $ps\text{-}cl(ps\text{-}cl(A)) = ps\text{-}cl(A)$.

(v) $ps\text{-}cl(A \vee B) = ps\text{-}cl(A) \vee ps\text{-}cl(B)$

Theorem 5.2.2 For any fuzzy set A on a fts (X, τ), the following hold:

(i) $ps\text{-}int(A)$ is the largest *ps-ro* open fuzzy set contained in A.

(ii) $ps\text{-}int(A) \leq ps\text{-}int(B)$ if $A \leq B$.

(iii) $ps\text{-}int(A) = A$ iff A is *ps-ro* open.

(iv) $ps\text{-}int(ps\text{-}int(A)) = ps\text{-}int(A)$.

(v) $ps\text{-}int(A \wedge B) = ps\text{-}int(A) \wedge ps\text{-}int(B)$

Definition 5.2.3 In a fts (X, τ), a fuzzy set A is said to be a

(i) *ps-ro nbd.* of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha \in B \leq A$. In addition, if A is *ps-ro* open fuzzy set, the *ps-ro nbd.* is called *ps-ro* open *nbd.*

(ii) *ps-ro* quasi neighborhood or simply *ps-ro q-nbd.* of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha q B \leq A$. In addition, if A is *ps-ro* open, the *ps-ro q-nbd.* is called *ps-ro* open *q-nbd.*.

Definition 5.2.4 A fuzzy point x_α where $0 < \alpha \leq 1$ is called fuzzy *ps*-cluster point of a fuzzy set A if for every *ps-ro* open *q-nbd.* U of

x_α, UqA.

Theorem 5.2.3 In a fts (X, τ), for any fuzzy set A, *ps-cl*(A) is the union of all fuzzy ps-cluster points of A.

Proof. Let B = *ps-cl*(A). Let x_α be any fuzzy point such that $x_\alpha \leq B$ and if possible let there be *ps-ro* open *q-nbd.* U of x_α such that $U \not{q} A$. Then there exist a *ps-ro* open fuzzy set V on X such that $x_\alpha q V \leq U$. Consequently, $V \not{q} A$. So that $A \leq 1 - V$. As $1 - V$ is *ps-ro* closed fuzzy set, $B \leq 1 - V$. As $x_\alpha \not\leq 1 - V$, we have $x_\alpha > B$, a contradiction. Conversely, suppose $x_\alpha \not\leq B$. Then there exists a *ps-ro* closed fuzzy set F containing A such that $x_\alpha > F$. So, $x_\alpha q(1 - F)$ and $A \not{q}(1 - F)$. Further, $(1 - F)$ is *ps-ro* open fuzzy set, so that x_α is not ps-cluster point of A.

Theorem 5.2.4 For a fuzzy set A in a fts X, *ps-int*$(1 - A) = 1 -$*ps-cl*(A).

Proof. *ps-int*$(1 - A) = \vee\{B : B \leq (1 - A),\ B$ is *ps-ro* open on $X\}$

So, $1 -$ *ps-int*$(1 - A)$

$= 1 - \vee\{B : A \leq (1 - B),\ (1 - B)$ is *ps-ro* closed on $X\}$

$= \wedge\{(1 - B) : A \leq (1 - B),\ (1 - B)$ is *ps-ro* closed on $X\}$

$=$ *ps-cl*(A). Hence, *ps-int*$(1 - A) = 1 -$ *ps-cl*(A).

Remark 5.2.1 It may be observed that a fuzzy set A on a fts (X, τ) is *ps-ro* open iff A is *ps-ro nbd.* of each of the fuzzy points contained in A.

We conclude this section with some definitions that we require in the next section.

Definition 5.2.5 Let $\{S_n : n \in D\}$ be a fuzzy net on a fts X. i.e., for each member n of a directed set $(D, \leq)$, S_n be a fuzzy set on X. A fuzzy point x_α on X is said to be a fuzzy ps-cluster point of the fuzzy net if for every $n \in D$ and every *ps-ro* open *q-nbd.* V of x_α, there exists $m \in D$, with $n \leq m$ such that $S_m q V$.

Definition 5.2.6 Let x_α be a fuzzy point on a fts X. A fuzzy net $\{S_n : n \in (D, \geq)\}$ on X is said to ps-converge to x_α, written as $S_n \xrightarrow{ps} x_\alpha$ if for each *ps-ro* open *q-nbd.* W of x_α, there exists $m \in D$ such that $S_n q W$ for all $n \geq m$,$(n \in D)$.

Definition 5.2.7 A collection $\mathcal{B}$ of fuzzy sets on a fts (X, τ) is said to form a fuzzy filter base in X if for every finite subcollection $\{B_1, B_2, ..., B_n\}$ of $\mathcal{B}$, $\bigwedge_{i=1}^{n} B_i \neq 0$. If in addition, the members of $\mathcal{B}$ are *ps-ro* open (closed) fuzzy sets then $\mathcal{B}$ is called a *ps-ro* open fuzzy (respectively, *ps-ro* closed fuzzy) filter base in X. If every member of a fuzzy filterbase $\mathcal{B}$ on X is contained in some fuzzy set A in X, then $\mathcal{B}$ is called a fuzzy filterbase in A.

Definition 5.2.8 A fuzzy filterbase $\mathcal{B}$ on a fts (X, τ) is said to have a fuzzy ps-cluster point in a fuzzy set A if there exist a fuzzy point x_α in A such that $x_\alpha \leq \bigwedge\{ps\text{-}cl(U) : U \in \mathcal{B}\}$.

Definition 5.2.9 Let x_α be a fuzzy point on a fts X. A fuzzy filterbase $\mathcal{B}$ is said

(i) to ps-adhere at x_α written as $x_\alpha \leq ps\text{-}ad.\mathcal{B}$ if for each ps-ro open q-$nbd.$ U of x_α and each $B \in \mathcal{B}$, BqU.

(ii) to ps-converge to x_α, written as $\mathcal{B} \xrightarrow{ps} x_\alpha$ if for each ps-ro open q-$nbd.$ U of x_α, there coresponds some $B \in \mathcal{B}$ such that $B \leq U$.

5.3 Pseudo near compactness

Definition 5.3.1 Let A be a fuzzy set on a fts X. A collection $\mathcal{U}$ of fuzzy sets on X is called a cover of A if $sup\{U : U \in \mathcal{U}\} \geq A$. If in addition, the members of $\mathcal{U}$ are ps-ro open fuzzy sets on X, then $\mathcal{U}$ is called a ps-ro open cover of A. In particular, if $A = 1_X$, we get the definition of ps-ro open cover of the fts X. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in a fts is said to have a finite ps-ro open subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\vee\{U : U \in \mathcal{U}_0\} \geq A$.

Definition 5.3.2 A fuzzy set A on a fts (X, τ) is called fuzzy pseudo nearly compact set if every covering of A by ps-ro open fuzzy sets has a finite subcover. Clearly, for $A = X$, the fts (X, τ) becomes fuzzy pseudo nearly compact fts.

Remark 5.3.1 It is easy to observe, as pseudo regular open fuzzy sets form a base for *ps-ro* open fuzzy topology, replacing *ps-ro* open cover by pseudo regular open cover, we may obtain pseudo near compactness.

Theorem 5.3.1 A fts (X, τ) is fuzzy pseudo nearly compact iff every $\{B_\alpha : \alpha \in \Lambda\}$ of *ps-ro* closed fuzzy sets on X with $\wedge_{\alpha\in\Lambda} B_\alpha = 0$, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha\in\Lambda_0} B_\alpha = 0$.

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a *ps-ro* open cover of X. Now, $\wedge_{\alpha\in\Lambda}(1 - U_\alpha) = (1 - \vee_{\alpha\in\Lambda} U_\alpha) = 0$. As $\{1 - U_\alpha : \alpha \in \Lambda\}$ is a collection of *ps-ro* closed fuzzy sets on X, by given condition, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha\in\Lambda_0}(1 - U_\alpha) = 0 \Rightarrow 1 - \vee_{\alpha\in\Lambda_0} U_\alpha = 0$. i.e., $1 = \vee_{\alpha\in\Lambda_0} U_\alpha$. So, X is fuzzy pseudo nearly compact.

Conversely, Let $\{B_\alpha : \alpha \in \Lambda\}$ be a family of *ps-ro* closed fuzzy sets on X with $\wedge_{\alpha\in\Lambda} B_\alpha = 0$. Then $1 = 1 - \wedge_{\alpha\in\Lambda} B_\alpha \Rightarrow 1 = \vee_{\alpha\in\Lambda}(1 - B_\alpha)$. By given condition there exist a finite subset Λ_0 of Λ such that $1 = \vee_{\alpha\in\Lambda_0}(1 - B_\alpha) \Rightarrow 1 = (1 - \wedge_{\alpha\in\Lambda_0} B_\alpha)$. Hence, $\wedge_{\alpha\in\Lambda_0} B_\alpha \leq (\wedge_{\alpha\in\Lambda_0} B_\alpha) \wedge (1 - \wedge_{\alpha\in\Lambda_0} B_\alpha) = 0$. Consequently, $\wedge_{\alpha\in\Lambda_0} B_\alpha = 0$.

Pseudo near compactness of a fts can be characterized in terms of cluster points of fuzzy net, as seen in the following result.

Theorem 5.3.2 A fts X if fuzzy pseudo nearly compact iff every fuzzy net on X has a fuzzy ps-cluster point.

Proof. Let $\{U_\alpha : \alpha \in D\}$ be a fuzzy net in a fuzzy pseudo nearly compact fts X. For each $\alpha \in D$, let $F_\alpha = ps\text{-}cl[\bigvee\{U_\beta : \beta \in D, \alpha \leq \beta\}]$. Then $\mathcal{F} = \{F_\alpha : \alpha \in D\}$ is a family of *ps-ro* closed fuzzy sets with the property that for every finite subset D_0 of D, $\wedge\{F_\alpha : \alpha \in D_0\} \neq 0$. By Theorem (5.3.1), $\bigwedge\{F_\alpha : \alpha \in D\} \neq 0$. Let $x_\lambda \in \bigwedge\{F_\alpha : \alpha \in D\}$. Then for any *ps-ro* open *q-nbd.* A of x_λ and any $\alpha \in D, Aq\bigvee\{U_\beta : \alpha \leq \beta\}$.Thus there exist a $\beta \in D$ with $\alpha \leq \beta$ such that AqU_β. This shows that x_λ is a fuzzy ps-cluster point of the fuzzy net $\{U_\alpha : \alpha \in D\}$.

Conversely, Let $\mathcal{F}$ be a collection of *ps-ro* closed fuzzy sets on X satisfying the hypothesis. Let $\mathcal{F}^*$ denote the family of all finite intersection of members of $\mathcal{F}$ directed by the relation " $\prec$ "(say) such that for $F_1, F_2 \in \mathcal{F}^*, F_2 \prec F_1$ iff $F_1 \leq F_2$. Let us consider the fuzzy net $\mathcal{U} = \{F : F \in (\mathcal{F}^*, \prec)\}$ of fuzzy sets on X. By hypothesis, there exist a fuzzy point x_λ which is a fuzzy ps-cluster point of $\mathcal{U}$. we shall show that $x_\lambda \in \bigwedge \mathcal{F}$. In fact, let $F \in \mathcal{F}$ be arbitrary and A be any *ps-ro* open *q-nbd.* of x_λ. Since $F \in \mathcal{F}^*$ and x_λ is a fuzzy ps-cluster point of $\mathcal{U}$ there exist G (say) in $\mathcal{F}^*$ such that $G \prec F$ (i.e., $G \leq F$)

and GqA. Hence, FqA. Thus $x_\lambda \in ps\text{-}clF = F$. Hence, $\bigwedge \mathcal{F} \neq 0$. By Theorem (5.3.1) X is fuzzy pseudo nearly compact fts.

Theorem 5.3.3 For a fuzzy set A on a fts, the following are equivalent:

(a) Every fuzzy net in A has ps-cluster point in A.

(b) Every fuzzy net in A has a ps-convergent fuzzy subnet.

(c) Every fuzzy filterbase in A ps-adheres at some fuzzy point in A.

Proof. $(a) \Rightarrow (b)$: Let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in A having ps-cluster point at $x_\alpha \leq A$. Let $Q_{x_\alpha} = \{A : A$ is *ps-ro* open *q-nbd.* of $x_\alpha\}$. For any $B \in Q_{x_\alpha}$, there can be chosen some $n \in D$ such that $S_n qB$. Let E denote the set of all ordered pairs (n, B) with the property that $n \in D,\ B \in Q_{x_\alpha}$ and $S_n qB$. Then $(E, \succ)$ is a directed set where $(m, C) \succ (n, B)$ iff $m \geq n$ in D and $C \leq B$. Then $T : (E \succ) \to (X, \tau)$ given by $T(n, B) = S_n$, is a fuzzy subnet of $\{S_n : n \in (D, \geq)\}$. Let V be any *ps-ro* open *q-nbd.* of x_α. Then there exists $n \in D$ such that $(n, V) \in E$ and hence $S_n qV$. Now, for any $(m, U) \succ (n, V)$, $T(m, U) = S_m qU \leq V \Rightarrow T(m, U)qV$. Hence, $T \xrightarrow{ps} x_\alpha$.

$(b) \Rightarrow (a)$ If a fuzzy net $\{S_n : n \in (D, \geq)\}$ in A does not have any ps-cluster point, then there is a *ps-ro* open *q-nbd.* U of X_α and $n \in D$ such that $S_n \not q U, \forall\ m \geq n$. Then clearly no fuzzy subnet of the fuzzy net can ps-converge to x_α.

$(c) \Rightarrow (a)$ Let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in A. Consider the fuzzy filter base $\mathcal{F} = \{T_n : n \in D\}$ in A, generated by the fuzzy net, where $T_n = \{S_m : m \in (D, \geq)$ and $m \geq n\}$. By (c), there exist a fuzzy point $a_\alpha \leq A \wedge (ps\text{-}ad\mathcal{F})$. Then for each *ps-ro* open *q-nbd.* U of a_α and each $F \in \mathcal{F}$, UqF, i.e., $UqT_n, \forall\, n \in D$. Hence, the given fuzzy net has ps-cluster point at a_α.

$(a) \Rightarrow (c)$ Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a fuzzy filterbase in A. For each $\alpha \in \Lambda$, choose a fuzzy point $x_{F_\alpha} \leq F_\alpha$, and construct the fuzzy net $S = \{x_{F_\alpha} : F_\alpha \in \mathcal{F}\}$ in A with $(\mathcal{F}, \succ\!\succ)$ as domain, where for two members F_α, $F_\beta \in \mathcal{F}$, $F_\alpha \succ\!\succ F_\beta$ iff $F_\alpha \leq F_\beta$. By (a), the fuzzy net has a ps-cluster point say $x_t \leq A$, where $0 < t \leq 1$. Then for any *ps-ro* open *q-nbd.* U of x_t and any $F_\alpha \in \mathcal{F}$, there exists $F_\beta \in \mathcal{F}$ such that $F_\beta \succ\!\succ F_\alpha$ and $x_{F_\beta} qU$. Then $F_\beta qU$ and hence $F_\alpha qU$. Thus $\mathcal{F}$ adheres at x_t.

Using Theorem (5.3.3) along with what we have proved in Theorem (5.3.1) and (5.3.2) we obtain the following characterizations of a fuzzy pseudo nearly compact fts.

Theorem 5.3.4 In a fts (X, τ), the following are equivalent:

(a) X is fuzzy pseudo nearly compact.

(b) Every fuzzy net on X has ps-cluster point at some fuzzy point in

X.

(c) Every fuzzy net on X has a *ps*-convergent fuzzy subnet.

(d) Every fuzzy filterbase on X *ps*-adheres at some fuzzy point in X.

(e) For every $\{B_\alpha : \alpha \in \Lambda\}$ of *ps-ro* closed fuzzy sets on X with $\bigwedge_{\alpha\in\Lambda} B_\alpha = 0$, there exist a finite subset Λ_0 of Λ such that $\bigwedge_{\alpha\in\Lambda_0} B_\alpha = 0$.

Theorem 5.3.5 If a fts is fuzzy pseudo nearly compact, then every fuzzy filterbase on X with atmost one *ps*-adherent point is *ps*-convergent.

Proof. Let $\mathcal{F}$ be a fuzzy filterbase with atmost one *ps*-adherent point in a fuzzy pseudo nearly compact fts X. Then by Theorem (5.3.4), $\mathcal{F}$ has at least one *ps*-adherent point. Let x_α be the unique *ps*-adherent point of $\mathcal{F}$. If $\mathcal{F}$ do not *ps*-converge to x_α, then there is some *ps-ro* open *q-nbd.* U of x_α such that for each $F \in \mathcal{F}$ with $F \leq U$, $F \wedge (1-U) \neq 0$. Then $\mathcal{G} = \{F \wedge (1-U) : F \in \mathcal{F}\}$ is a fuzzy filterbase on X and hence has a *ps*-adherent point y_t(say) in X. Now, $U \not{q} G$, for all $G \in \mathcal{G}$, so that $x_\alpha \neq y_t$. Again, for each *ps-ro* open *q-nbd.* V of y_t and each $F \in \mathcal{F}$, $Vq(F \wedge (1-U)) \Rightarrow VqF \Rightarrow y_t$ is a *ps*-adherent point of $\mathcal{F}$, where $x_\alpha \neq y_t$. This shows that y_t is another *ps*-adherent point of $\mathcal{F}$, which is not the case.

In what follows, we observe how fuzzy pseudo near compactness of a fuzzy set on a fts be characterized.

Theorem 5.3.6 For a fuzzy set A on a fts (X, τ), the following are equivalent:

(a) A is fuzzy pseudo nearly compact.

(b) For every family $\mathcal{F}$ of *ps-ro* closed fuzzy sets on X with $\bigwedge\{F : F \in \mathcal{F}\} \wedge A = 0$, there exist a finite subcollection $\mathcal{F}_0$ of $\mathcal{B}$ such that $\bigwedge \mathcal{F}_0 \not{q} A$.

(c) If $\mathcal{B}$ is a *ps-ro* closed fuzzy filterbase on X such that each finite intersection of members of $\mathcal{B}$ is q-coincident with A, then $(\bigwedge \mathcal{B}) \wedge A \neq 0$.

Proof. $(a) \Rightarrow (b)$: Let A be a fuzzy pseudo nearly compact set on a fts (X, τ) and $\mathcal{F}$ be a family of *ps-ro* closed fuzzy sets on X such that $\bigwedge\{F : F \in \mathcal{F}\} \wedge A = 0$. Then for all $x \in supp(A)$, $inf\{F : F \in \mathcal{F}\} = 0$, so that the collection $\{(1 - F) : F \in \mathcal{F}\}$ is a cover of A by *ps-ro* open fuzzy sets. Hence, there is a finite subcollection $\mathcal{F}_0$ of $\mathcal{F}$ such that $A \leq \vee\{(1 - F) : F \in \mathcal{F}\}$. Then $\wedge\{F : F \in \mathcal{F}_0\} \leq 1 - A$ and hence $\bigwedge \mathcal{F}_0 \not{q} A$.

$(b) \Rightarrow (c)$: Straightforward and hence omitted.

$(c) \Rightarrow (a)$: If A is not fuzzy pseudo nearly compact in X, there exist a *ps-ro* open fuzzy cover $\mathcal{U}$ of A having no finite subcover of A. So, for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, there exist $x \in supp(A)$ such that $A(x) > sup\{U(x) : U \in \mathcal{U}\}$. i.e., $inf\{(1 - U(x)) : U \in \mathcal{U}\} >$

$1-A(x) \geq 0$. Thus $\{(1-U) : U \in \mathcal{U}\} = \mathcal{B}$(say) is a fuzzy *ps-ro* closed fuzzy filterbase on X having no finite subcollection $\mathcal{B}_0$ such that $\bigwedge\{B : B \in \mathcal{B}_0\} \not q A$. In fact otherwise $A \leq 1-\bigwedge\{(1-U) : U \in \mathcal{U}_0\} = \bigvee\{U : U \in \mathcal{U}_0\}$ for some finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, contradicting our hypothesis. Using (c), we then have $\bigwedge\{(1-U) : U \in \mathcal{U}\} \wedge A \neq 0$ and hence there is $x \in supp(A)$ such that $inf\{(1-U(x)) : U \in \mathcal{U}\} > 0$ i.e., $sup\{U(x) : U \in \mathcal{U}\} < 1$, which contradicts the fact that $\mathcal{U}$ is a cover of A.

Theorem 5.3.7 A fuzzy set A in a fuzzy pseudo nearly compact space X is fuzzy pseudo nearly compact if every fuzzy filterbase in A has a fuzzy *ps*-cluster point in A.

Proof. Let every fuzzy filterbase in a fuzzy set A in a fuzzy pseudo nearly compact space X has a fuzzy *ps*-cluster point in A. If A is not fuzzy pseudo nearly compact set on X, then there exists a *ps-ro* fuzzy open cover $\mathcal{U}$ of A such that for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, $A \geq \bigvee\{U : U \in \mathcal{U}_0\}$. Corresponding to each $U \in \mathcal{U}_0$, we define a fuzzy set B_U as follows :

$$B_U(x) = \begin{cases} min\{1 - U(x), A(x), |A(x) - U(x)|\}, & \text{for } x \in supp(A) \\ 0, & otherwise. \end{cases}$$

Again, for every finite subfamily $\{B_{U_1}, B_{U_2}, ..., B_{U_n}\}$ of $\mathcal{B} = \{B_U : U \in \mathcal{U}\}$, we have $sup\{U_i(x) : 1 \leq i \leq n\} < A(x) < 1$, for some $x \in$

$supp(A)$ so that $min\{A(x)-U_1(x), A(x)-U_2(x), ..., A(x)-U_n(x)\} > 0$. Hence, $\bigwedge_{i=1}^{n} B_{U_i} \neq 0$ and consequently $\mathcal{B} = \{B_U : U \in \mathcal{U}\}$ is a fuzzy filterbase in A. Now, for each fuzzy point x_α in A, there exists $U \in \mathcal{U}$ such that $x_\alpha q U$. Since $B_U \not{q} U$, $\mathcal{B}$ has no fuzzy ps-cluster point in A, which is a contradiction.

The converse of this theorem may not hold in general. Imposing some conditions on the filterbase, we get the converse as follows.

Theorem 5.3.8 Let A be a fuzzy pseudo nearly compact set on X and $\mathcal{B}$ a family of *ps-ro* open fuzzy sets contained in A such that every finite intersection of members of $\mathcal{B}$ is q-coincident with at least one member of $\mathcal{B}$. Then $\mathcal{B}$ has a fuzzy ps-cluster point in A.

Proof. If $\mathcal{B}$ has no fuzzy ps-cluster point in A, then proceeding as in the proof of Theorem(5.3.7) we construct a *ps-ro* open fuzzy cover $\mathcal{U}$ of A such that each V_x^n of $\mathcal{U}$ corresponds to a $B_x^n \in \mathcal{B}$ with $V_x^n \not{q} B_x^n$. Thus there exist finite subfamily $\{V_{x_1}^{n_1}, V_{x_2}^{n_2}, ..., V_{x_k}^{n_k}\}$ of $\mathcal{U}$ such that $A \leq \bigvee_{i=1}^{k} V_{x_i}^{n_i}$. Then $(\bigwedge_{i=1}^{k} V_{x_i}^{n_i}) \not{q} A$ with $B \leq A$, for all $B \in \mathcal{B}$. This contradicts the definition of $\mathcal{B}$.

One more characterization of fuzzy pseudo nearly compact set follows next.

Theorem 5.3.9 A fuzzy set on a fts (X, τ) is fuzzy pseudo nearly compact iff whenever $\mathcal{F}$ is a fuzzy filterbase with the property that for any finite subcollection $\mathcal{F}_0 = \{F_1, F_2, ..., F_n\}$ of $\mathcal{F}$ and for any *ps-ro* open fuzzy set U with $A \leq U$, $(\bigwedge \mathcal{F}_0)qU$ holds, then $\mathcal{F}$ has a fuzzy *ps*-cluster point in A.

Proof. Let A be a fuzzy pseudo nearly compact set on a fts (X, τ) and $\mathcal{F}$ a fuzzy filterbase on X having no fuzzy $ps-$cluster point in A. For each $x \in supp(A)$, there exists a positive integer m_x such that $\frac{1}{m_x} < A(x)$. For any positive integer $n \geq m_x$, as $x_{\frac{1}{n}} \leq A(x)$, $x_{\frac{1}{n}}$ is not a fuzzy *ps*-cluster point of $\mathcal{F}$. Hence, there is a *ps-ro* open *q-nbd.* V_x^n of $x_{\frac{1}{n}}$ and $B_x^n \in \mathcal{F}$ such that $V_x^n \not{q} B_x^n$. As $V_x^n(x) + \frac{1}{n} > 1$, we have $sup\{V_x^n(x) : n \geq m_x\} = 1$. Hence, the collection $\mathcal{U} = \{V_x^n : x \in supp(A), n \geq m_x > \frac{1}{A(x)}\}$, forms a *ps-ro* open fuzzy cover of A such that for each $V_x^n \in \mathcal{U}$, there exist $B_x^n \in \mathcal{F}$ with $V_x^n \not{q} B_x^n$. Since A is pseudo nearly compact fuzzy set on X, there exist a finite subcollection $V_{x_1}^{n_1}, V_{x_2}^{n_2}, ..., V_{x_k}^{n_k}$ of $\mathcal{U}$, such that $A \leq \bigvee_{i=1}^{k} V_{x_i}^{n_i} = V(\text{say})$. Then V is a *ps-ro* open fuzzy set such that $A \leq V$ and $V \not{q}(\bigwedge_{i=1}^{k} B_{x_i}^{n_i})$.

Conversely, let $\mathcal{F}$ be *ps-ro* closed fuzzy filterbase on X such that $\bigwedge\{F : F \in \mathcal{F}\} \wedge A = 0$. As for *ps-ro* closed fuzzy set $F \in \mathcal{F}$ we have $F = ps\text{-}cl(F)$, it follows that $\mathcal{F}$ has no fuzzy *ps*-cluster point in A. By hypothesis, there is a fuzzy *ps-ro* open set U with $A \leq U$ and there exists $\{F_1, F_2, ..., F_n\}$ of $\mathcal{F}$ such that $(\bigwedge_{i=1}^{n} F_i) \not{q} U$ and hence

$(\bigwedge_{i=1}^{n} F_i) \not{q} A$. Hence, by Theorem (5.3.6), A is fuzzy pseudo nearly compact set on X.

Theorem 5.3.10 In a fuzzy pseudo nearly compact space X every *ps-ro* closed fuzzy set is fuzzy pseudo nearly compact.

Proof. For any filterbase $\mathcal{B}$ in A, there exists a fuzzy point x_α on X, such that x_α is a fuzzy *ps*-cluster point of $\mathcal{B}$. Then for any $F \in \mathcal{B}$ we have $x_\alpha \leq ps\text{-}cl(F) \leq ps\text{-}cl(A) = A$. Hence, $\mathcal{B}$ has a fuzzy *ps*-cluster point in A and consequently, A is fuzzy pseudo nearly compact set on X.

We conclude this section with a few results on fuzzy pseudo near compactness which are analogous to its topological counter part, near compactness.

Theorem 5.3.11 In a fuzzy pseudo nearly compact space X, the complement of every pseudo regular open fuzzy set is fuzzy pseudo nearly compact.

Proof. Let A be a pseudo regular open fuzzy set on a fuzzy pseudo nearly compact space X. Hence, A is *ps-ro* open fuzzy set and $1-A$ is *ps-ro* closed fuzzy set. By Theorem (5.3.10), $1-A$ is fuzzy pseudo nearly compact in X.

Theorem 5.3.12 In a *fts* X, the finite union of fuzzy pseudo nearly compact sets is also so.

Proof. Straightforward and hence omitted.

5.4 Pseudo fuzzy ro-continuous functions

Definition 5.4.1 A function f from a fts X to a fts Y is pseudo fuzzy ro-continuous (ps-ro continuous, for short) if $f^{-1}(U)$ is ps-ro open fuzzy set on X, for each pseudo regular open fuzzy set U on Y.

Definition 5.4.2 [87] A function f from a topological space X to another topological space Y is called strong irresolute if $f^{-1}(U)$ is regular open in X for each regular open set U in Y.

The following Example shows that pseudo fuzzy δ-continuity does not imply ps-ro continuity.

Example 5.4.1 Let $X = \{x, y, z\}$ and the topology generated by $\{0, 1, \mu, \nu, \eta\}$, where $\mu(x) = 0.4$, $\mu(y) = 0.4$, $\mu(z) = 0.5$, $\nu(x) = 0.4$, $\nu(y) = 0.6$, $\nu(z) = 0.4$ and $\eta(x) = 0.5$, $\eta(y) = 0.5$, $\eta(z) = 0.6$. It is easy to see that (X, τ) is a fts. Let $f : (X, \tau) \to (X, \tau)$ be a function defined as $f(x) = x$, $f(y) = f(z) = y$. It can be checked that f is indeed a δ-continuous function from $(X, i_\alpha(\tau))$ to itself for each $\alpha \in I_1$. Hence, f is pseudo fuzzy δ-continuous. As worked out in Example (3.2.4), ν is a nontrivial pseudo regular open fuzzy set.

Also, the fuzzy set K given by $K(x) = 0.4, K(y) = 0.6, K(z) = 0.6$ is pseudo δ-open, without being *ps-ro* open. But, as $f^{-1}(\nu) = K$, f can not be *ps-ro* continuous. It is easy to see that f is pseudo fuzzy δ-continuous.

Theorem 5.4.1 A function $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is strong irresolute for each $\alpha \in I_1$, where $(X,\tau), (Y,\sigma)$ are fts, then $f : (X,\tau) \to (Y,\sigma)$ is *ps-ro* continuous.

Proof. Let μ be any pseudo regular open fuzzy set on (Y,σ). μ^α is regular open in $(Y, i_\alpha(\sigma))$. By the *strong irresolute*-ness of $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma)), f^{-1}(\mu^\alpha) = (f^{-1}(\mu))^\alpha$ is regular open in $(X, i_\alpha(\tau))$. Hence, $f^{-1}(\mu)$ is pseudo regular open and hence, *ps-ro* open fuzzy set on (X,τ), proving f to be *ps-ro* continuous.

We cite here, some characterizations of *ps-ro* continuity in terms of complements of pseudo regular open fuzzy sets, *ps-cl* and *ps-int* operators and of fuzzy points.

Theorem 5.4.2 A function $f : (X,\tau) \to (Y,\sigma)$ is *ps-ro* continuous iff $f^{-1}(\mu)$ is *ps-ro* closed fuzzy set on a fts (X,τ), where $1-\mu$ is pseudo regular open fuzzy set on a fts (Y,σ).

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be *ps-ro* continuous and μ be such that $1-\mu$ is pseudo regular open fuzzy set on (Y,σ). As $f : (X,\tau) \to (Y,\sigma)$ is pseudo fuzzy *ro*-continuous, $f^{-1}(1-\mu)$ is *ps-ro* open fuzzy set on X. Now,

$(1 - f^{-1}(1 - \mu))(x)$

$= 1 - f^{-1}(1 - \mu)(x)$

$= 1 - (1 - \mu)(f(x))$

$= \mu f(x)$

$= f^{-1}(\mu)(x)$. Hence, $f^{-1}(\mu)$ is *ps-ro* closed fuzzy set on (X, τ).

Conversely, Let μ be any pseudo regular open fuzzy set and so $f^{-1}(1-\mu)$ is *ps-ro* closed. Then $1 - f^{-1}(1 - \mu)$ is *ps-ro* open fuzzy set on $(Y, i_\alpha(\sigma)), \forall\ \alpha \in I_1$. Hence, f is *ps-ro* continuous.

Theorem 5.4.3 Let (X, τ) and (Y, σ) be two fts. For a function $f : X \to Y$, the following are equivalent:

(a)f is *ps-ro* continuous.

(b) Inverse image of each *ps-ro* open fuzzy sets on Y under f is *ps-ro* open on X.

(c) For each fuzzy point x_α on X and each *ps-ro* open *nbd.* V of $f(x_\alpha)$, there exists a *ps-ro* open fuzzy set U on X, such that $x_\alpha \leq U$ and $f(U) \leq V$.

(d)For each *ps-ro* closed fuzzy set F on Y, $f^{-1}(F)$ is *ps-ro* closed on X.

(e) For each fuzzy point x_α on X, the inverse image under f of every *ps-ro nbd.* of $f(x_\alpha)$ on Y is a *ps-ro nbd.* of x_α on X.

(f)For all fuzzy set A on X, $f(ps\text{-}cl(A)) \leq ps\text{-}cl(f(A))$.

(g)For all fuzzy set B on Y, $ps\text{-}cl(f^{-1}(B)) \leq f^{-1}(ps\text{-}cl(B))$.

(h)For all fuzzy set B on Y, $f^{-1}(ps\text{-}int(B)) \leq ps\text{-}int(f^{-1}(B))$.

Proof. $(a) \Rightarrow (b)$ Let f be *ps-ro* continuous, and μ be any *ps-ro* open fuzzy set on Y. Then $\mu = \vee \mu_i$, where μ_i is pseudo regular open fuzzy set on Y, for each i. Now, $f^{-1}(\mu) = f^{-1}(\vee_i \mu_i) = \vee_i f^{-1}(\mu_i)$. Since f is *ps-ro* continuous, $f^{-1}(\mu_i)$ is *ps-ro* open fuzzy set and consequently, $f^{-1}(\mu)$ is *ps-ro* open on X.

$(b) \Rightarrow (c)$ Let V be any *ps-ro* open *nbd.* of $f(x_\alpha)$ on Y. Then there is a *ps-ro* open fuzzy set V_1 on Y such that $f(x_\alpha) \leq V_1 \leq V$. By (b) $f^{-1}(V_1)$ is *ps-ro* open fuzzy set on X. Again, $x_\alpha \leq f^{-1}(V_1) \leq f^{-1}(V)$. So, $f^{-1}(V)$ is a *ps-ro nbd.* of x_α, such that $f(f^{-1}(V)) \leq V$, as desired.

$(c) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y and $x_\alpha \leq f^{-1}(V)$. Then $f(x_\alpha) \leq V$ and so by (c), there exists *ps-ro* open fuzzy set U on X such that $x_\alpha \leq U$ and $f(U) \leq V$. Hence, $x_\alpha \leq U \leq f^{-1}(V)$. i.e., $f^{-1}(V)$ is a *ps-ro nbd.* of each of the fuzzy points contained in it. Thus $f^{-1}(V)$ is *ps-ro* open fuzzy set on X.

$(b) \Leftrightarrow (d)$ Obvious.

$(b) \Rightarrow (e)$ Suppose, W is a *ps-ro* open *nbd.* of $f(x_\alpha)$. Then there exists a *ps-ro* open fuzzy set U on Y such that $f(x_\alpha) \leq U \leq W$. Then $x_\alpha \leq f^{-1}(U) \leq f^{-1}(W)$. By (b), $f^{-1}(U)$ is *ps-ro* open fuzzy set on X and hence the result is obtained.

$(e) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y. If $x_\alpha \leq f^{-1}(V)$ then $f(x_\alpha) \leq V$ and so $f^{-1}(V)$ is a *ps-ro nbd.* of x_α.

$(d) \Rightarrow (f)$ $ps\text{-}cl(f(A))$ being a *ps-ro* closed fuzzy set on Y, $f^{-1}(ps\text{-}cl(f(A)))$ is *ps-ro* closed fuzzy set on X. Again,

$f(A) \leq ps\text{-}cl(f(A))$

$\Rightarrow A \leq f^{-1}(ps\text{-}cl(f(A)))$.

As $ps\text{-}cl(A)$ is the smallest *ps-ro* closed fuzzy set on X containing A, $ps\text{-}cl(A) \leq f^{-1}(ps\text{-}cl(f(A)))$. Hence,

$f(ps\text{-}cl(A)) \leq ff^{-1}(ps\text{-}cl(f(A))) \leq ps\text{-}cl(f(A))$.

$(f) \Rightarrow (d)$ For any *ps-ro* closed fuzzy set B on Y,

$f(ps\text{-}cl(f^{-1}(B)))$

$\leq ps\text{-}cl(f(f^{-1}(B)))$

$\leq ps\text{-}cl(B) = B$.

Hence, $ps\text{-}cl(f^{-1}(B)) \leq f^{-1}(B) \leq ps\text{-}cl(f^{-1}(B))$.

Thus, $f^{-1}(B)$ is *ps-ro* closed fuzzy set on X.

$(f) \Rightarrow (g)$ For any fuzzy set B on Y,

$f(ps\text{-}cl(f^{-1}(B)))$

$\leq ps\text{-}cl(f(f^{-1}(B)))$

$\leq ps\text{-}cl(B)$.

Hence, $ps\text{-}cl(f^{-1}(B)) \leq f^{-1}(ps\text{-}cl(B))$.

$(g) \Rightarrow (f)$ Let $B = f(A)$ for some fuzzy set A on X. Then

$ps\text{-}cl(f^{-1}(B)) \leq f^{-1}(ps\text{-}cl(B))$

$\Rightarrow ps\text{-}cl(A) \leq ps\text{-}cl(f^{-1}(B)) \leq f^{-1}(ps\text{-}cl(f(A)))$.

So, $f(ps\text{-}cl(A)) \leq ps\text{-}cl(f(A))$.

$(b) \Rightarrow (h)$ For any fuzzy set B on Y, $f^{-1}(ps\text{-}int(B))$ is *ps-ro* open fuzzy set on X. Also, $f^{-1}(ps\text{-}int(B)) \leq f^{-1}(B)$.

So, $f^{-1}(ps\text{-}int(B)) \leq ps\text{-}int(f^{-1}(B))$.

$(h) \Rightarrow (b)$ Let B be any *ps-ro* open fuzzy set on Y. So, $ps\text{-}int(B) = B$. Now, $f^{-1}(ps\text{-}int(B)) \leq ps\text{-}int(f^{-1}(B))$

$\Rightarrow f^{-1}(B) \leq ps\text{-}int(f^{-1}(B)) \leq f^{-1}(B)$.

Hence, $f^{-1}(B)$ is *ps-ro* open fuzzy set on X.

We observe next that in terms of *ps-ro* open *q-nbds.* of fuzzy points also *ps-ro* continuity can be characterized.

Theorem 5.4.4 Let (X, τ) and (Y, σ) be two *fts*. A function $f : X \rightarrow Y$ is f is *ps-ro* continuous iff for every fuzzy point x_α on X and every *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$ there exists a *ps-ro* open fuzzy set U on X with $x_\alpha qU$ and $f(U) \leq V$.

Proof. Let f be *ps-ro* continuous and x_α a fuzzy point on X, V a *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$. So,

$V(f(x)) + \alpha > 1$

$\Rightarrow f^{-1}(V)(x) + \alpha > 1$

$\Rightarrow (f^{-1}(V))(x) + \alpha > 1$

$\Rightarrow x_\alpha q(f^{-1}(V))$.

Now, $ff^{-1}(V) \leq V$ is always true. Choosing $U = f^{-1}(V)$ we have, $f(U) \leq V$ with $x_\alpha qU$.

Conversely, let the condition hold. Let V be any *ps-ro* open fuzzy set on Y. To prove $f^{-1}(V)$ is *ps-ro* open fuzzy set on X, we shall prove $1 - f^{-1}(V)$ is *ps-ro* closed fuzzy set on X. Let x_α be any fuzzy point on X such that $x_\alpha > 1_X - f^{-1}(V)$. So,

$(1 - f^{-1}(V))(x) < \alpha$

$\Rightarrow f^{-1}(V)(x) + \alpha > 1$

$\Rightarrow V(f(x)) + \alpha > 1$

$\Rightarrow f(x_\alpha)qV$.

By given condition, there exists a *ps-ro* open fuzzy set on U such that $x_\alpha qU$ and $f(U) \leq V$. Now,

$U(t) + (1 - f^{-1}(V))(t)$

$= U(t) + 1 - V(f(t))$

$\leq V(f(t)) + 1 - V(f(t)) = 1, \forall\, t.$

Hence, $U \not{q}(1 - f^{-1}(V))$. Consequently, x_α is not a *ps*-cluster point of $1 - f^{-1}(V)$. This proves $1 - f^{-1}(V)$ is a *ps-ro* closed fuzzy set on X

Lemma 5.4.1 [5] Let Z, X, Y be fts and $f_1 : Z \to X$ and $f_2 :$

$Z \to Y$ be two functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1 U_2 are fuzzy sets on Z, X, Y respectively such that $f(B) \leq U_1 \times U_2$, then $f_1(B) \leq U_1$ and $f_2(B) \leq U_2$.

Theorem 5.4.5 Let X, Y, Z be fts. For any functions $f_1 : Z \to X$ and $f_2 : Z \to Y$, a function $f : Z \to X \times Y$ is defined as $f(x) = (f_1(x), f_2(x))$ for $x \in Z$, where $X \times Y$ is endowded with the product fuzzy topology. If f is *ps-ro* continuous then f_1 and f_2 are both *ps-ro* continuous.

Proof. Let U_1 be a *ps-ro q-nbd.* of $f_1(x_\alpha)$ on X, for any fuzzy point x_α on Z. Then $U_1 \times 1_Y$ is a *ps-ro q-nbd.* of $f(x_\alpha) = (f_1(x_\alpha), f_2(x_\alpha))$ on $X \times Y$. By *ps-ro* continuity of f, there exists *ps-ro q-nbd.* V of x_α on Z such that $f(V) \leq U_1 \times 1_Y$. Then $f(V)(t) \leq (U_1 \times 1_Y)(t) = U_1(t) \wedge 1_Y(t) = U_1(t), \forall\, t \in Z$. So, $f_1(V) \leq U_1$. Hence, f_1 is *ps-ro* continuous. Similarly, it can be shown that f_2 is also *ps-ro* continuous.

Lemma 5.4.2 [3] Let X, Y be fts and $g : X \to X \times Y$ be the graph of the function $f : X \to Y$. Then if A, B are fuzzy sets on X and Y respectively, $g^{-1}(A \times B) = A \wedge f^{-1}(B)$.

Theorem 5.4.6 Let $f : X \to Y$ be a function from a fts X to another fts Y and $g : X \to X \times Y$ be the graph of the function f. Then f is *ps-ro* continuous if g is so.

Proof. Let g be *ps-ro* continuous and B be *ps-ro* open fuzzy set on Y. By Lemma (5.4.2), $f^{-1}(B) = 1_X \wedge f^{-1}(B) = g^{-1}(1_X \times B)$. Now, as $1_X \times B$ is *ps-ro* open fuzzy set on $X \times Y$,$f^{-1}(B)$ becomes *ps-ro* open fuzzy set on X. Hence, f is *ps-ro* continuous.

Finally, we show that pseudo near compactness is preserved by *ps-ro* continuous function.

Theorem 5.4.7 In a fts X, the *ps-ro* continuous image of fuzzy pseudo nearly compact set is also so.

Proof. Suppose, A is fuzzy pseudo nearly compact set on X and $f : X \to Y$ is a *ps-ro* continuous function from a fts X into another fts Y. Let $B = f(A)$ and $\{U_\alpha : \alpha \in \Lambda\}$ be a *ps-ro* open cover of B in Y. By *ps-ro* continuity of f, each $f^{-1}(U_\alpha)$ is *ps-ro* open fuzzy set on X with $\bigvee f^{-1}(U_\alpha) = f^{-1}(\bigvee U_\alpha) \geq f^{-1}(B) \geq A$. As A is fuzzy pseudo nearly compact, there is a finite subset Λ_0 of Λ such that $\bigvee(f^{-1}(U_\alpha) : \alpha \in \Lambda_0) \geq A$. Consequently, $B = f(A) \leq f(\bigvee f^{-1}(U_\alpha) : \alpha \in \Lambda_0) \leq \bigvee(ff^{-1}(U_\alpha) : \alpha \in \Lambda_0) \leq \bigvee(U_\alpha : \alpha \in \Lambda_0)$, as desired.

Chapter 6

Fuzzy continuous functions on Left fuzzy topological rings

6.1 Introduction

Topological ring in its various aspects has been widely studied in general topology. Concept of left fuzzy topological ring has recently been introduced by Deb Ray in [18], with the motive whether it embraces the hitherto known properties of topological rings. The prime result obtained in the above work is the characterization of left fuzzy topological rings in terms of the fundamental system of fuzzy neighbourhoods of the fuzzy point 0_α $(0 < \alpha \leq 1)$.

In this chapter, some properties of left fuzzy topological rings are reviewed from the categorical stand point and some results are obtained. Further, the collection of all fuzzy continuous functions on a fuzzy topological space, having values in left fuzzy topological ring has been studied both under algebraic and topological view-points.

6.2 Left fuzzy topological rings

In this section, left fuzzy topological ring as introduced by Deb Ray in [18], has been discussed in general and certain properties of the same from categorical view-point are interpreted.

As prerequisites, we state a few known definitions and results from [18].

Definition 6.2.1 [18] Let (X, τ) and (Y, σ) be two fuzzy topological spaces. A function $f : X \times X \to Y$ is said to be fuzzy left continuous if f is fuzzy continuous with respect to the fuzzy topology on the product $X \times X$ generated by the collection $\{U \times V : U, V \in \tau\}$ where $(U \times V)(s,t) = \begin{cases} V(t), & \text{if } U(s) > 0 \\ 0, & \text{otherwise} \end{cases}$

Definition 6.2.2 [18] Let R be a ring and τ be a fuzzy topology on R such that, for all $x, y \in R$,

i) $(x, y) \to x + y$ is fuzzy left continuous.

ii) $(x, y) \to x.y$ is fuzzy left continuous.

iii) $x \to -x$ is fuzzy continuous.

The pair (R, τ) is called a left fuzzy topological ring, (In short left ftr).

For non zero fuzzy sets U, V on R, the fuzzy sets $U+V$, UV and $-U$ are defined in [18] as follows :

$(U+V)(x) = sup\{V(x-s) : U(s) > 0\}$

$$(UV)(x) = \begin{cases} sup\{V(t) : x = st \text{ and } U(s) > 0\}, & \text{if } \{(s,t) \in R \times R : st = x\} \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$(-U)(x) = U(-x), \forall\, x \in R.$

Remark 6.2.1 In what follows, as in [18] we use left associativity of addition of fuzzy sets. i.e., for any three nonzero fuzzy sets U, V, W on R, $U+V+W = (U+V)+W$.

Remark 6.2.2 [18] Although $U+V \neq V+U$ and $UV \neq VU$ in general, for fuzzy points x_α, y_α on R $(0 < \alpha < 1)$, the following hold:

(i) $x_\alpha + y_\alpha = (x+y)_\alpha$

(ii) $x_\alpha \cdot y_\alpha = (xy)_\alpha$

(iii) $-x_\alpha = (-x)_\alpha$

(iv) $(-x)_\alpha + x_\alpha = 0_\alpha = x_\alpha + (-x)_\alpha$

In fact, the collection of fuzzy points (for each α) forms a ring with respect to these operations.

Theorem 6.2.1 [18] In a left ftr, for any fuzzy sets S_1, S_2, T_1 and T_2 with $S_1 \leq S_2$, $T_1 \leq T_2$, the following hold:

(i) $S_1 + T_1 \leq S_2 + T_2$

(ii) $S_1.T_1 \leq S_2.T_2$

(iii) $x_\alpha S_1 \leq x_\alpha T_1$

(iv) $S_1 x_\alpha \leq T_1 x_\alpha$, $\forall\, x \in R$ and $\alpha \in (0, 1]$

Theorem 6.2.2 [18] In a left ftr (R, τ), for each fuzzy set V, each $x \in R$ and $0 < \alpha \leq 1$,

$$(x_\alpha V)(z) = \begin{cases} sup\{V(t) : z = xt\}, & \text{if there is } t, \text{ such that } z = xt \\ 0, & \text{otherwise} \end{cases}$$

$$(V x_\alpha)(z) = \begin{cases} \alpha, & \text{if there is } s, \text{ such that } sx = z \\ 0, & \text{otherwise} \end{cases}$$

Example 6.2.1 [18] Let Z_3 be the ring of integers modulo 3. Define a fuzzy set A on Z_3 as $A(x) = 0.25$ for all $x \in Z_3$. Then clearly $\tau = \{0_X, 1_X, A\}$ is a fuzzy topology on Z_3. As $A + A = A$ and $A.A = A$, it is easy to see that (Z_3, τ) is a left ftr.

Theorem 6.2.3 [18] let R be a left ftr. If $\phi : R \to R$ is given by $\phi(x) = -x$ then ϕ is a fuzzy homeomorphism.

Corollary 6.2.1 [18] V is fuzzy open if and only if $-V$ is fuzzy open.

Corollary 6.2.2 [18] V is a fuzzy *nbd.* of 0_α if and only if $-V$ is a fuzzy *nbd.* of 0_α.

Theorem 6.2.4 [18] Suppose (R, τ) is a left fuzzy toopological ring. Then for each $a \in R$, $\phi_a : R \to R$ given by $\phi_a(x) = a + x$ is a fuzzy homeomorphism.

Theorem 6.2.5 [18] In a left ftr (R, τ), for each α with $0 < \alpha \leq 1$ and $x \in R$, V is fuzzy open (fuzzy closed) iff $x_\alpha + V$ is fuzzy open (respectively, fuzzy closed).

We show that *q-nbd.* of any fuzzy point x_α can also be characterized through *q-nbds.* of 0_α, for all $\alpha \in I_1$.

Theorem 6.2.6 In a left ftr R, V is a fuzzy *q-nbd.* of 0_α iff $-V$ is a fuzzy *q-nbd.* of 0_α.

Proof. Let V be a fuzzy *q-nbd.* of 0_α. There exists fuzzy open set A such that $0_\alpha q A \leq V$. i.e., $\alpha + A(0) > 1$ and $A \leq V$. For all $x \in R$, $A(-x) \leq V(-x) \Rightarrow -A \leq -V$. Now, $0_\alpha(0) + (-A)(0) = \alpha + A(-0) > 1$. Hence, $0_\alpha q(-A)$ and $-A \leq -V$. Using Corollary (6.2.1) $-V$ is a fuzzy *q-nbd.* of 0_α.
Conversely, let $-V$ is a fuzzy *q-nbd.* of 0_α. There exist fuzzy open set A such that $0_\alpha q A \leq -V$. As above, $-A \leq V$ and $0_\alpha q(-A)$. i.e., V is a fuzzy *q-nbd.* of 0_α.

Theorem 6.2.7 In a left ftr (R, τ), for each α with $0 < \alpha \leq 1$ and $x \in R$, if V is a fuzzy *q-nbd.* (fuzzy open *q-nbd.* or fuzzy closed *q-nbd.*) of 0_α, then $x_\alpha + V$ is a fuzzy *q-nbd.* (fuzzy open *q-nbd.* or fuzzy closed *q-nbd.*) of x_α. Moreover, any fuzzy *q-nbd.* of x_α is precisely of the form $x_\alpha + V$, where V is a fuzzy *q-nbd.* of 0_α.

Proof. If V is a fuzzy *q-nbd.* of 0_α, there is a fuzzy open set A such that $0_\alpha q A \leq V$. i.e., $\alpha + A(0) > 1$ and $A \leq V$. By Theorem (6.2.5) $x_\alpha + A$ is a fuzzy open set. By Theorem (6.2.1) $x_\alpha + A \leq x_\alpha + V$. We verify that $x_\alpha q(x_\alpha + A)$. Now,

$\alpha + (x_\alpha + A)(x)$

$= \alpha + sup\{A(x - s) : x_\alpha(s) > 0\}$

$= \alpha + A(0) > 1$

This shows $(x_\alpha + A)$ is fuzzy open, such that $x_\alpha q(x_\alpha + A) \leq x_\alpha + V$. Hence, $x_\alpha + V$ is a fuzzy *q-nbd.* of x_α. Suppose, V^* is any fuzzy *q-nbd.* of x_α. Then there is a fuzzy open set U^* such that $x_\alpha q U^* \leq V^*$. i.e., $\alpha + U^*(x) > 1$ and $U^*(y) \leq V^*(y), \forall\, y$. Consider $U = (-x)_\alpha + U^*$ and $V = (-x)_\alpha + V^*$. Then U is a fuzzy open set. To show $0_\alpha q U \leq V$. Now,

$0_\alpha(0) + U(0)$

$= \alpha + [(-x)_\alpha + U^*](x)$

$= \alpha + U^*(0) > 1.$

So, $0_\alpha qU$. As $(-x)_\alpha \leq (-x)_\alpha$ and $U^* \leq V^*$, $U \leq V$. Hence, $0_\alpha qU$ and $U \leq V$. Again, $x_\alpha + U = x_\alpha + (-x)_\alpha + V^* = 0_\alpha + V^* = V^*$. This completes the proof.

A topological ring is "homogeneous", a function defined on it is continuous throughout its domain of definition whenever it is continuous at 0. The following theorem reflects a similar behaviour of left ftr.

Theorem 6.2.8 Let (R, τ) and (S, σ) be left fuzzy topoogical rings and $f : R \longrightarrow S$ be a ring homomorphism. Then $f : (R, \tau) \longrightarrow (S, \sigma)$ is fuzzy continuous iff f is fuzzy continuous at 0_α, where $0 < \alpha \leq 1$.

Proof. Let $f : (R, \tau) \rightarrow (S, \sigma)$ be fuzzy continuous. In particular f is fuzzy continuous at 0_α. Conversely, let $f : (R, \tau) \rightarrow (S, \sigma)$ be fuzzy continuous at $0_\alpha, \forall\ \alpha \in (0, 1]$. For any fuzzy open set U containing $(f(0))_\alpha = f(0_\alpha)$ in S, there exist fuzzy open set V containing 0_α on R such that $f(V) \leq U$. Let x_α be fuzzy point on R and B be any fuzzy open set on S containing the fuzzy point $(f(x))_\alpha$ on S. Now, $x_\alpha + V$ is fuzzy open set containing x_α. As B is a fuzzy open set on S containing $(f(x))_\alpha$, we have $B = (f(x))_\alpha + U$. To show $f((x)_\alpha + V) \leq (f(x))_\alpha + U$. $[f((x)_\alpha + V)](z)$

$= \sup[(x_\alpha + V)(t) : f(t) = z]$

$= \sup[V(t - x) : f(t) = z]$

$= \sup[V(p) : f(x+p) = z]$

$= \sup[V(p) : f(x) + f(p) = z]$

$= \sup[V(p) : f(p) = z - f(x)]$

$= f(V)(z - f(x))$

$\leq U(z - f(x))$

$= [f(x))_\alpha + U](z).$

Hence, $f((x)_\alpha + V) \leq (f(x))_\alpha + U.$

Using the language of categories, we obtain the following :

Theorem 6.2.9 The collection of all left ftr and fuzzy continuous homomorphisms form a category.

Proof. Consider the collection of all left ftr as objects and for each pair of objects X,Y, the set of all arrows as the collection of fuzzy continuous homomorphisms from X to Y. Then it is easy to observe that taking composition of arrows as the usual composition of functions, one gets:

(i) composition of arrows is associative and

(ii) for each object X, $id : X \to X$ given by $id(x) = x$ is the identity arrow. Consequently, it forms a category.

Remark 6.2.3 The category mentioned in Theorem (6.2.9), will henceforth be referred to as FTR.

Remark 6.2.4 It is well known that corresponding to any topological space (X, τ), one can obtain the characteristic fuzzy topological space (X, τ_f).

Theorem 6.2.10 If (X, τ) is a topological ring then (X, τ_f) is a left ftr.

Proof. For a topological space (X, τ), it is known that (X, τ_f) is a fuzzy topological space. We have to show the following :

(i) $\forall\ x, y \in Z, (x, y) \rightarrow x + y$ is left fuzzy continuous.

(ii) $\forall\ x, y \in Z, (x, y) \rightarrow x.y$ is left fuzzy continuous.

(iii) $\forall\ x \in Z, x \rightarrow -x$ is fuzzy continuous.

We show that '+' is left fuzzy continuous. Let μ be a fuzzy open set on (X, τ_f) with $(x + y)_\alpha \leq \mu$. Then $\mu = \chi_A$ for some $A \in \tau$. Hence, $(x + y)_\alpha \leq \chi_A \Rightarrow \alpha \leq \chi_A(x + y) \Rightarrow x + y \in A$. Since (X, τ) is a topological ring, there exist open sets B, $C \in \tau$ such that $x \in B$, $y \in C$ and $B + C \subseteq A$. Then $x_\alpha \leq \chi_B$ and $y_\alpha \leq \chi_C$, where χ_B, $\chi_C \in \tau_f$. Now to complete the proof, we show $\chi_B + \chi_C \leq \chi_A = \mu$.

Now, $\forall\ z \in X$,

$(\chi_B + \chi_C)(z)$

$= sup\{\chi_C(z - t) : \chi_B(t) > 0\}$

$= sup\{\chi_C(z - t) : t \in B\}$

$$= \begin{cases} 1, & \text{for } t \in B \text{ and } z - t \in C \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \text{for } z \in B + C \\ 0, & \text{otherwise} \end{cases}$$

$$= \chi_{B+C}(z)$$

$$\leq \chi_A(z).$$

Hence, '+' is left fuzzy continuous. Proceeding in a similar manner, (ii) and (iii) can be obtained. Hence, (X, τ_f) is a left fts.

Though the following result is known in advance, we sketch a proof of the same.

Theorem 6.2.11 If f is a continuous homomorphism from a topological ring (X, τ) to a topological ring (Y, σ) then $f : (X, \tau_f) \to (Y, \sigma_f)$ is a fuzzy continuous homomorphism between the corresponding left ftr.

Proof. Since $f : (X, \tau) \to (Y, \sigma)$ is a homomorphism, $f : (X, \tau_f) \to (Y, \sigma_f)$ remains a homomorphism. Let $\mu \in \sigma_f$. Then there is some $U \in \sigma$ such that $\mu = \chi_U$. Now, $f^{-1}(\mu) = f^{-1}(\chi_U) = \chi_{f^{-1}(U)}$. Since $U \in \sigma$ and f is continuous, $f^{-1}(U) \in \tau$. Hence, $\chi_{f^{-1}(U)} \in \tau_f$. Consequently, $f : (X, \tau_f) \to (Y, \sigma_f)$ is a fuzzy continuous homomorphism.

We express the above in terms of categories as follows:

Theorem 6.2.12 If $TopRng$ is the category of topological rings and continuous homomorphisms, then $TopRng$ is a full subcategory of FTR.

Proof. In the light of the Theorems (6.2.10) and (6.2.11), any object of $TopRng$ can be viewed as an object of FTR and any morphism between two objects of $TopRng$ is a morphism between the corresponding objects of FTR. Hence, $TopRng$ is a subcategory of FTR. Now, consider the inclusion functor $i : TopRng \to FTR$ that sends (X, τ) to its characteristic fts (X, τ_f) and $f : (X, \tau) \to (Y, \sigma)$ to $f : (X, \tau_f) \to (Y, \sigma_f)$. To show that the functor i is full. Let (X, τ) and (Y, σ) be two objects in $TopRng$ and $f^* : (X, \tau_f) \to (Y, \sigma_f)$ a morphism in FTR. If $U \in \sigma$ then $\chi_U \in \sigma_f$ and so, $f^{*-1}(\chi_U) = \chi_{f^{*-1}(U)} \in \tau_f$ which in turn gives $f^{*-1}(U) \in \tau$. Hence, there exist $f^* : (X, \tau) \to (Y, \sigma)$ a morphism in $TopRng$ such that $i(f^*) = f^*$. i.e., i is full. Consequently, $TopRng$ is a full subcategory of FTR.

Theorem 6.2.13 Let (Z, σ) be a left ftr then $\forall\ \alpha \in I_1, (Z, i_\alpha(\sigma))$ is a topological ring.

Proof. We need to show the following :

(i) $\forall\ x, y \in Z, (x, y) \to x + y$ is continuous.

(ii) $\forall\ x, y \in Z, (x, y) \to x.y$ is continuous.

(iii) $\forall\ x \in Z, x \to -x$ is continuous.

Let $x, y \in Z$ and A be any open set in $(Z, i_\alpha(\sigma))$ containing $x+y$. There exist fuzzy open set μ in (Z, σ) such that $\mu^\alpha = A$. So, $(x+y)_\alpha < \mu$. As, (Z, σ) is a left ftr, there exist fuzzy open sets U and V such that $x_\alpha < U$, $y_\alpha < V$ and $U + V \leq \mu$. Then $x \in U^\alpha$ and $y \in V^\alpha$. We shall show that $U^\alpha + V^\alpha \subseteq A$. Let $z \in U^\alpha + V^\alpha$. Then $z = s + t$ where $s \in U^\alpha$ and $t \in V^\alpha$. i.e., $U(s) > \alpha$ and $V(t) > \alpha$. Now,

$(U+V)(z)$

$= sup\{V(z-p) : U(p) > 0\}$

$\geq V(t),\ where\ U(s) > 0\ and\ z = s + t$

$> \alpha$. So, $\mu(z) > \alpha$, $z \in \mu^\alpha = A$. Hence, $U^\alpha + V^\alpha \subseteq A$. This proves '+' is continuous. The proof for '.' is continuous is similar and hence omitted. Now, we shall prove that $x \to -x$ is continuous. Let $x \in Z$ and A be an open set on $(Z, i_\alpha(\sigma))$ containing $-x$. There is a fuzzy open set μ on (Z, σ) such that $\mu^\alpha = A$. So, $(-x) \in \mu^\alpha \Rightarrow (-x)_\alpha < \mu$. As (Z, σ) is left ftr, there exist fuzzy open set U containing x_α such that $x_\alpha < U$ and $-U \leq \mu$. We shall show $(-U)^\alpha \subseteq A$. Let $z \in (-U)^\alpha \Rightarrow \mu(z) \geq -U(z) > \alpha$. So, $z \in \mu^\alpha$. Hence, $(-U)^\alpha \subseteq A$.

Theorem 6.2.14 [80] A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy continuous iff $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is continuous for each $\alpha \in I_1$, where (X, τ), (Y, σ) are fts.

Theorem 6.2.15 A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy continuous homomorphism iff $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is continuous homomorphism for each $\alpha \in I_1$, where $(X, \tau), (Y, \sigma)$ are left ftr.

Proof. Immediate from Theorem (6.2.14)

In view of Theorem (6.2.13) and Theorem (6.2.15), we get:

Theorem 6.2.16 For each $\alpha \in I_1$, $i_\alpha : FTR \to TopRng$ is a covariant functor.

6.3 Left ftr-valued fuzzy continuous functions

In this section, our objective is to study the collection of all left ftr-valued fuzzy continuous functions on a fuzzy topological space. We find that the ring operations on the co-domain space induce a ring structure on this collection of functions.

In what follows, unless it is explicitly mentioned, the rings are non commutative and without unity.

Theorem 6.3.1 Let (Y, τ_Y) be a fts and (Z, τ_Z) be a left ftr. If $FC(Y, Z)$ stands for all fuzzy continuous functions from Y to Z, then $\forall\ f, g \in FC(Y, Z) \Rightarrow f + g, fg, -f \in FC(Y, Z)$

Proof. Let y_α be a fuzzy point on Y and U be any fuzzy open set on Z such that $(f + g)(y_\alpha) \leq U$. Now, for any $z \in Z$,

$[(f+g)(y_\alpha)](z)$

$= sup\{y_\alpha(t) : (f+g)(t) = z\}$

$= \begin{cases} \alpha, & \text{for } z = (f+g)(y) \\ 0, & \text{otherwise} \end{cases}$

So, $[(f+g)(y_\alpha)]$

$= ((f+g)(y))_\alpha$

$= (f(y)+g(y))_\alpha$

$= ((f(y))_\alpha + (g(y))_\alpha.$

Hence, $(f(y))_\alpha + (g(y))_\alpha \leq U$ in Z. As Z is a left ftr, there exist fuzzy open sets V_1 and W_1 on Z such that $(f(y))_\alpha \leq V_1$, $(g(y))_\alpha \leq W_1$ and $V_1 + W_1 \leq U$.

Again, as $f(y_\alpha) = (f(y))_\alpha$ and $g(y_\alpha) = (g(y))_\alpha$, $f(y_\alpha) \leq V_1$, $g(y_\alpha) \leq W_1$. By fuzzy continuity of f and g, there exist fuzzy open sets V_2 and W_2 on Y with $y_\alpha \leq V_2$ and $y_\alpha \leq W_2$ such that $f(V_2) \leq V_1$ and $g(W_2) \leq W_1$. Choose, $S = V_2 \wedge W_2$. Clearly, S is a fuzzy open set containing y_α. Then for any $z \in Z$,

$f(S)(z)$

$= sup\{(V_2 \wedge W_2)(t) : f(t) = z\}$

$\leq sup\{V_2(t) : f(t) = z\}$

$= f(V_2)(z)$ and similarly, $g(S)(z) \leq g(W_2)(z)$. Consequently, $f(S) \leq V_1$ and $g(S) \leq W_1$, so that $f(S) + g(S) \leq V_1 + W_1 \leq U$. Now, to complete the proof, it is to show that $(f+g)(S) \leq f(S)+g(S)$. Now,

$(f(S)+g(S))(z)$

$= sup\{g(S)(z-x) : f(S)(x) > 0\}$

$= sup[sup\{S(q) : g(q) = z-x\} : f(S)(x) > 0]$

$= sup\{S(q) : g(q) = z-x \text{ and } \exists\, t \text{ with } S(t) > 0,\ f(t) = x\}$

$= sup\{S(q) : \exists\, t \text{ with } S(t) > 0 \text{ and } g(q) = z - f(t)\}$

$\geq sup\{S(q) : g(q) = z - f(q)\}$

$= sup\{S(q) : f(q) + g(q) = z\}$

$= sup\{S(q) : (f+g)(q) = z\}$

$= (f+g)(S)(z)$.

Hence, $f+g \in FC(Y,Z)$. Let y_α be a fuzzy point on Y and U be any fuzzy open set on Z such that $(fg)(y_\alpha) \leq U$. Now, for any $z \in Z$,

$[(fg)(y_\alpha)](z)$

$= sup\{y_\alpha(t) : (fg)(t) = z\}$

$$= \begin{cases} \alpha, & for\ \text{z} = \text{(fg)(y)} \\ 0, & otherwise \end{cases}$$

and so, $[(fg)(y_\alpha)]$

$= ((fg)(y))_\alpha$

$= (f(y).g(y))_\alpha$

$= ((f(y))_\alpha.(g(y))_\alpha.$

Hence, $(f(y))_\alpha.(g(y))_\alpha \leq U$ in Z. As Z is a left ftr, there exist fuzzy open sets V_1 and W_1 on Z such that $(f(y))_\alpha \leq V_1$, $(g(y))_\alpha \leq W_1$ and $V_1.W_1 \leq U$. Again, as $f(y_\alpha) = (f(y))_\alpha$ and $g(y_\alpha) = (g(y))_\alpha$, $f(y_\alpha) \leq V_1$, $g(y_\alpha) \leq W_1$. By fuzzy continuity of f and g, there exist fuzzy open sets V_2 and W_2 in Y with $y_\alpha \leq V_2$ and $y_\alpha \leq W_2$ such that $f(V_2) \leq V_1$ and $g(W_2) \leq W_1$. Choose, $S = V_2 \wedge W_2$. Clearly, S is a fuzzy open set containing y_α. Then for any $z \in Z$, $f(S)(z) \leq f(V_2)(z)$ and similarly, $g(S)(z) \leq g(W_2)(z)$. Consequently, $f(S) \leq V_1$ and $g(S) \leq W_1$, so that $f(S).g(S) \leq V_1.W_1 \leq U$. Now, to complete the proof, it is to show that $(f.g)(S) \leq f(S).g(S)$. Now,

$(f(S).g(S))(z)$

$= sup\{g(S)(p) : f(S)(q) > 0,\ qp = z\}$

$= sup[sup\{S(t) : g(t) = p\} : f(S)(q) > 0,\ qp = z]$

$= sup\{S(t) : t \in B\}$, where

$B = \{t \in Y : \exists\ m \in Y$ such that $f(m)g(t) = z$ and $S(m) > 0\}$ Also,

$(f.g)(S)(z)$

$= sup\{S(p) : (f.g)(p) = z\}$

$= sup\{S(p) : f(p).g(p) = z\}$

$= sup\{S(p) : p \in A\}$,where $A = \{t \in Y : f(t)g(t) = z\}$

For all $p \in A$, if $S(p) = 0$, then $(f.g)(S)(z) = 0 \leq (f(S).g(S))(z)$ holds. For otherwise, there exists some $p \in A$ such that $S(p) > 0$ and consequently, $p \in B$. Hence, in such case, from above we conclude that $(f.g)(S)(z) = 0 \leq (f(S).g(S))(z)$, as desired.

Let U be any fuzzy open set on Z. Then $\forall\ z \in Z$, $(-f)^{-1}(U)(z) = U((-f)(z)) = U(-f(z)) = (-U)(f(z)) = f^{-1}(-U)(z)$. As U is fuzzy open iff $-U$ is fuzzy open and $f \in FC(Y, Z)$, we get $-f \in FC(Y, Z)$.

Definition 6.3.1 [51] Let (X, τ) be a fts and $r \in [0, 1]$. By r^* we mean a fuzzy set on X such that $r^*(x) = r$, for every $x \in X$. The fts (X, τ) is called fully stratified if $r^* \in \tau$, for every $r \in [0, 1]$.

Theorem 6.3.2 [56] Let Y and Z be two fts such that Y is fully stratified. Then every $r \in [0, 1]$, $r^* : Y \to Z$, given by $r^*(y) = r, \forall\ y \in Y$ is fuzzy continuous.

Corollary 6.3.1 Let Y be a fts and Z be a left ftr with additive identity 0. The function $0^* : Y \to Z$ given by $0^*(y) = 0, \forall\ y \in Y$ is fuzzy continuous. Further, if Z contains identity 1, then the function given by $1^*(y) = 1, \forall\ y \in Y$ is also fuzzy continuous.

Theorem 6.3.3 If (Y, τ_Y) is a fully stratified fts and (Z, τ_Z) is a left ftr, then $FC(Y, Z)$ forms a ring with respect to the usual addition and multiplication of functions.

Proof. By Theorem(6.3.1), $\forall\ f, g \in FC(Y, Z), f + g, fg, -f \in FC(Y, Z)$. Since Z is a ring, it is clear that '+' is associative and commutative while '.' is associative. It is easy to verify that in

$FC(Y, Z)$, '.' is distributive over '+' . Now, $\forall\ f \in FC(Y, Z)$ and $z \in Z$, $(f + 0^*)(z) = f(z) + 0^*(z) = f(z) + 0 = f(z)$ and $(f + (-f))(z) = f(z) + (-f)(z) = 0 = 0^*(z)$ prove that $FC(Y, Z)$ is a ring.

Theorem 6.3.4 Let $FC(Y, Z)$ be the ring of fuzzy continuous functions from a fully stratified fts (Y, τ_Y) to a left ftr (Z, τ_Z).

1. $FC(Y, Z)$ is commutative, if Z is commutative.

2. $FC(Y, Z)$ contains identity, if Z contains identity.

Proof. $\forall\ f, g \in FC(Y, Z)$ and $\forall\ y \in Y$, $(fg)(y) = f(y).g(y) = g(y).f(y) = (gf)(y)$. Again, $\forall\, f \in FC(Y, Z)$ and $\forall\, y \in Y$, $(f.1^*)(y) = f(y).1^*(y) = f(y).1 = f(y) = 1.f(y) = 1^*(y).f(y) = (1^*.f)(y)$, showing that 1^* is the identity in $FC(Y, Z)$

Definition 6.3.2 A fuzzy set μ on a ring R is called a left (right) fuzzy ideal of R iff $\forall\ x, y \in R$

1. $\mu(x - y) = min\{\mu(x), \mu(y)\}$

2. $\mu(xy) \geq \mu(y)$ (respectively, $\mu(xy) \geq \mu(x)$)

A left as well as right fuzzy ideal is called a fuzzy ideal.

Theorem 6.3.5 Let $FC(Y, Z)$ be the ring of fuzzy continuous functions from a fully stratified fts (Y, τ_Y) to a left ftr (Z, τ_Z). If Z_1

is a fuzzy ideal on Z, then the fuzzy set I on $FC(Y,Z)$, given by $I(f) = inf\{Z_1(f(y)) : y \in Y\}, \forall\ f \in FC(Y,Z)$, is a fuzzy ideal in $FC(Y,Z)$.

Proof. For all $f, h \in FC(Y,Z)$,

$I(f-h)$

$= inf\{Z_1((f-h)(y)) : y \in Y\}$

$= inf\{Z_1(f(y) - h(y)) : y \in Y\}$

$= inf[min\{Z_1(f(y), Z_1(h(y))\} : y \in Y]$

$= min[inf\{Z_1(f(y)) : y \in Y\}, inf\{Z_1(h(y)) : y \in Y\}]$

$= min\{I(f), I(h)\}$ and

$I(fh)$

$= inf\{Z_1((fh)(y)) : y \in Y\}$

$= inf\{Z_1(f(y)h(y)) : y \in Y\}$

$\geq inf\{Z_1(h(y)) : y \in Y\}$

$= I(h)$.

Hence, I is left fuzzy ideal on $FC(Y,Z)$. Similarly, it can be shown that I is a right fuzzy ideal on $FC(Y,Z)$. Hence, I is a fuzzy ideal on $FC(Y,Z)$

We show next how the algebraic nature of $FC(Y,Z)$ helps in determining the fuzzy connectedness of Y. Some relevent definitions and results are furnished here, before we establish the main result.

Definition 6.3.3 [32] A fts (X, τ) is fuzzy disconnected if there exist fuzzy sets U and V such that $U \vee V = 1, U \not{q} \overline{V}$ and $V \not{q} \overline{U}$.

Lemma 6.3.1 If (X, τ) is fuzzy disconnected fts then there exist fuzzy closed sets C and D such that $C \vee D = 1$ and $C \not{q} D$.

Proof. Let (X, τ) be fuzzy disconnected. There exist fuzzy sets A and B such that $A \vee B = 1, A \not{q} \overline{B}$ and $B \not{q} \overline{A}$. i.e., $\forall\, y \in Y$, $A(y) \vee B(y) = 1,\ A(y) + \overline{B}(y) \leq 1$ and $\overline{A}(y) + B(y) \leq 1$. Hence, for each $y \in Y$ we have either $[A(y) = 1$ and $B(y) = 0]$ or $[A(y) = 0$ and $B(y) = 1]$. We shall prove that the fuzzy closed sets $1 - int(clA)$ and $1 - int(clB)$ are the required sets. Now, $\forall\, y \in Y$ if $A(y) = 1$ then $A(y) \leq 1 - \overline{B}(y) \leq 1 - int(clB)(y)$, i.e., $1 - int(clB) = 1$ and if $B(y) = 1$ then similarly, we have $1 - int(clA) = 1$, showing $(1 - int(clA)) \vee (1 - int(clB)) = 1$ and $(1 - int(clA)) \not{q} (1 - int(clB))$.

Theorem 6.3.6 If a fts (X, τ) is fuzzy disconnected then $\forall\, \alpha \in I_1, (X, i_\alpha(\tau))$ is disconnected.

Proof. Let (X, τ) be fuzzy disconnected. By Lemma (6.3.1), there exist fuzzy open sets A and B on X such that $A \vee B = 1$ and $A \not{q} B$. Hence, for each $x \in X$ we have either $[A(x) = 1$ and $B(x) = 0]$ or $[A(x) = 0$ and $B(x) = 1]$. Now, $\forall\, \alpha \in I_1$, A^α and B^α are open in $(X, i_\alpha(\tau))$ with $A^\alpha \cup B^\alpha = (A \vee B)^\alpha = X$. If possible let $z \in A^\alpha \cap B^\alpha$. Then, $A(z) > \alpha$ and $B(z) > \alpha$, which is not possible.

Hence, $A^\alpha \cap B^\alpha = \Phi$ and so, $(X, i_\alpha(\tau))$ is disconnected.

Lemma 6.3.2 Let $C(Y, Z)$ denote the ring of continuous functions from a topological space $(Y, i_\alpha(\tau))$ to a topological ring $(Z, i_\alpha(\sigma))$, for each $\alpha \in I_1$. If Y is disconnected then there exist $f \in C(Y, Z)$ such that $f \neq 0, 1$ and $f^2 = f$.

Proof. If Y is disconnected, there exist nonempty disjoint closed sets A, B such that $Y = A \cup B$. Defining $f : Y \to Z$ by $f(y) =$
$$\begin{cases} 1, & if\ y \in A \\ 0 & if\ y \in B \end{cases},$$
we get the desired non trivial idempotent.

Theorem 6.3.7 Let Z be a left ftr with 1 and without zero divisor such that 0_α is fuzzy closed for each $\alpha \in (0, 1]$. If Y is any fully stratified fts such that the ring $FC(Y, Z)$ has some nontrivial idempotent element then Y is fuzzy disconnected.

Proof. Let $f \in FC(Y, Z)$ is such that $f^2 = f$ and $f \neq 0, 1$. To show Y is fuzzy disconnected. $\forall\, y \in Y, f^2(y) = f(y) \Rightarrow f(y)(1 - f(y)) = 0$. As Z has no zero divisor, for each $y \in Y$ we have, $f(y) = 0$ or $f(y) = 1$. 0_α for all $\alpha \in I_1$ is fuzzy closed in Z. Consider, $\alpha = 1$. As $1_\alpha = 0_\alpha + 1_\alpha$ and 0_α is fuzzy closed, using Theorem (6.2.5), 1_α is fuzzy closed in Z. f being fuzzy continuous, $f^{-1}(0_\alpha)$ and $f^{-1}(1_\alpha)$ are fuzzy closed in Y. Now, $[f^{-1}(0_\alpha) \vee f^{-1}(1_\alpha)](y)$

$= sup[f^{-1}(0_\alpha)(y), f^{-1}(1_\alpha)(y)]$

$= sup[(0_\alpha)(f(y)), (1_\alpha)f((y))]$

$= \alpha = 1.$

Hence, $f^{-1}(0_\alpha) \vee f^{-1}(1_\alpha) = 1_Y$. Clearly, for all $x \in Y$, $f^{-1}(0_\alpha)(x) + f^{-1}(1_\alpha)(x) = \alpha = 1$. i.e., $f^{-1}(0_\alpha) \not{q} f^{-1}(1_\alpha)$. This shows that Y is disconnected.

Theorem 6.3.8 Let Z be any left ftr and Y be any fuzzy disconnected fts then the ring $FC(Y, Z)$ has some nontrivial idempotent element.

Proof. Let Y be fuzzy disconnected. By Theorem (6.3.6), $\forall\ \alpha \in I_1, (X, i_\alpha(\tau))$ is disconnected. By Theorem (6.2.13), $\forall\ \alpha \in I_1, (Z, i_\alpha(\sigma))$ is a topological ring. Now, by Lemma (6.3.2), there exist a continuous function $f : (Y, i_\alpha(\tau)) \to (Z, i_\alpha(\sigma))$ such that $f \neq 0, 1$ and $f^2 = f$. By Theorem (6.2.14), $f : (Y, \tau) \to (Z, \sigma)$ is fuzzy continuous such that $f \neq 0, 1$ and $f^2 = f$.

Combining Theorem (6.3.7) and Theorem (6.3.8) we have:

Theorem 6.3.9 Let Z be a left ftr with 1 and without zero divisor such that 0_α is fuzzy closed for each $\alpha \in (0, 1]$. If Y is any fully stratified fts, then the ring $FC(Y, Z)$ has exactly two idempotents iff Y is fuzzy connected.

Theorem 6.3.10 Let X and Y be two fully stratified fts and $f : X \to Y$ be a fuzzy continuous function. For any left ftr Z, $f^* : FC(Y, Z) \to FC(X, Z)$ given by $f^*(g) = g \circ f$ is a ring homomorphism.

Proof. Straightforward.

Theorem 6.3.11 If X is any fully stratified fts and Z_1, Z_2 are left ftr, then every fuzzy continuous ring homomorphism $\phi : Z_1 \to Z_2$ induces a ring homomorphism $\hat{\phi} : FC(X, Z_1) \to FC(X, Z_2)$ given by $\hat{\phi}(f) = \phi \circ f$.

Proof. Immediate.

Reframing the results discussed above in the language of categories, we obtain the following functors:

Theorem 6.3.12 If FTS is the category of fully stratified fuzzy topological spaces and fuzzy continuous functions; *Rng* is the category of all rings and ring homomorphisms, then

(i)$FC(-, Z) : FTS \to$ *Rng* given by $Y \to FC(Y, Z)$ is a contravariant functor, for each left ftr Z.

(ii)$FC(X, -)$: *Rng* $\to$ *Rng* given by $Z \to FC(X, Z)$ is a covariant functor, for each fully stratified fuzzy topological space X.

6.4 $FC(Y, Z)$ as topological and Left fuzzy topological rings

In the previous chapters we have seen how a collection of functions can be fuzzy topologized. As $FC(Y, Z)$ is a class of functions, it is possible to equip it with various topologies and fuzzy topologies.

Here, we observe the interplay between its ring structure and its topological and fuzzy topological behaviour.

Definition 6.4.1 [35] Let U be a fuzzy open set on a fts Z and y_α ($\alpha \in (0, 1]$) be a fuzzy point on a fts Y. By $[y_\alpha, U]$ we denote the subset of $FC(Y, Z)$ where $[y_\alpha, U] = \{f \in FC(Y, Z) : f(y_\alpha) \leq U\}$. The collection of all such $[y_\alpha, U]$ forms a subbase for some topology on $FC(Y, Z)$, called fuzzy-point fuzzy-open topology ($fp - fo$), denoted by, τ_{fp-fo}.[1]

Theorem 6.4.1 If (Y, τ_Y) is a fully stratified fts and (Z, τ_Z) is a left ftr, then $(FC(Y, Z), \tau_{(fp-fo)})$ is a topological ring.

Proof. It is clear that $FC(Y, Z)$ is a ring and $FC(Y, Z)$ is a topological space with respect to $\tau_{(fp-fo)}$. We need to show that

(i) $(f, g) \to f + g$ is continuous.

(ii) $(f, g) \to f.g$ is continuous.

[1]In [35], the name of this topology was fuzzy-point open topology, denoted by τ_{F-p-o}

(iii) $f \to -f$ is continuous. Let $[y_\alpha, U]$ be a subbasic open set containing $f+g$. Then $(f+g)(y_\alpha) \leq U$. Now, as $(f+g)(y_\alpha) = (f(y))_\alpha + (g(y))_\alpha$, $(f(y))_\alpha + (g(y))_\alpha \leq U$ in Z. Z being a left ftr, there exist fuzzy open sets V, W on Z such that, $(f(y))_\alpha \leq V$, $(g(y))_\alpha \leq W$ and $V + W \leq U$. Again, $f(y_\alpha) = (f(y))_\alpha \leq V \Rightarrow f \in [y_\alpha, V]$ and similarly, $g \in [y_\alpha, W]$. For continuity of $f+g$, we need to show that $[y_\alpha, V] + [y_\alpha, W] \subseteq [y_\alpha, U]$.

Let $\xi \in [y_\alpha, V] + [y_\alpha, W]$. Then there exist $\eta \in [y_\alpha, V]$ and $\psi \in [y_\alpha, W]$, such that $\xi = \eta + \psi$.

Now, $\eta \in [y_\alpha, V], \psi \in [y_\alpha, W]$

$\Rightarrow \eta(y_\alpha) \leq V,\ \psi(y_\alpha) \leq W$

$\Rightarrow (\eta + \psi)(y_\alpha) \leq (V + W) \leq U$

$\Rightarrow \xi(y_\alpha) \leq U$

$\Rightarrow \xi \in [y_\alpha, U]$.

The proof for the product fg to be continuous is similar and hence omitted.

Now, for any $f \in FC(Y, Z)$ and any subbasic open set $[y_\alpha, U]$ containing $-f$, we get, $(-f)(y_\alpha) \leq U \Rightarrow (-f(y))_\alpha \leq U$. It is easy to see that $(-f(y))_\alpha \leq U \Rightarrow (f(y))_\alpha \leq -U$. Since U is fuzzy open iff $-U$ is fuzzy open and $(f(y_\alpha) = (f(y))_\alpha$, $f \in [y_\alpha, -U]$. We now show that $-[y_\alpha, U] \leq [y_\alpha, -U]$.

Let $\psi \in -[y_\alpha, U]$. Then there is some $\eta \in [y_\alpha, U]$ such that $\psi = -\eta$. $\eta \in [y_\alpha, U] \Rightarrow \eta(y_\alpha) \leq U \Rightarrow (-(-\eta))(y_\alpha) \leq U \Rightarrow (-\psi)(y_\alpha) \leq U \Rightarrow \psi(y_\alpha) \leq -U \Rightarrow \psi \in [y_\alpha, -U]$, as desired.

Definition 6.4.2 Let $FC(Y,Z)$ denote the collection of all fuzzy continuous functions from a $fts(Y,\tau_Y)$ to another fts (Z,τ_Z). By y_U we mean a fuzzy set on $FC(Y,Z)$, given by $y_U(f) = U(f(y))$, for every $f \in FC(Y,Z)$. The fuzzy point open topology (FPO) on $FC(Y,Z)$ is generated by fuzzy sets of the form y_U where $y \in Y$ and U is a fuzzy open set on Z [35].

Theorem 6.4.2 Let (Y,τ_Y) be fully stratified fts and (Z,τ_Z) be a left ftr. Then $FC(Y,Z)$ endowed with fuzzy point-open topology (FPO) is a left ftr .

Proof. It is clear that $FC(Y,Z)$ is a ring. We need to show that (i) $(f,g) \rightarrow f+g$ is left fuzzy continuous.

(ii) $(f,g) \rightarrow f.g$ is left fuzzy continuous.

(iii) $f \rightarrow -f$ is fuzzy continuous.

Let y_U be any subbasic open set containing $(f+g)_\alpha$. We have to find fuzzy open sets y_V, y_W in $FC(Y,Z)$ such that $y_V + y_W \leq y_U, f_\alpha \leq y_V$ and $g_\alpha \leq y_W$. Now,$(f+g)_\alpha \leq y_U \Rightarrow y_U(f+g) \geq \alpha \Rightarrow U[(f+g)(y)] \geq \alpha \Rightarrow U[(f(y)+g(y)] \geq \alpha$. It is easy to see that $(f(y)+g(y))_\alpha \leq U$ in Z. As $(f(y))_\alpha + (f(y))_\alpha = (f(y)+g(y))_\alpha$ and Z is a left ftr, there exist fuzzy open sets V and W in Z such that $(f(y))_\alpha \leq V, (g(y))_\alpha \leq W$ and $V + W \leq U$. Now, we verify that $f_\alpha \leq y_V$ and $g_\alpha \leq y_W$.

$f_\alpha(f) = \alpha \leq V(f(y)) = y_V(f)$ and for $h \neq f, f_\alpha(h) = 0 \leq y_V(h)$.

Hence, $f_\alpha \leq y_V$. Similarly, we can prove $g_\alpha \leq y_W$. In order to complete the proof it is to show that $y_V + y_W \leq y_U$. Now,

$y_U(\phi)$

$= (V + W)(\phi(y))$

$= sup\{W(\phi(y) - t) : V(t) > 0\}$

$= sup\{W(\phi(y) - t) : t \in A\}$, Where $A = \{t \in Z : V(t) > 0\}$.

$(y_V + y_W)(\phi)$

$= sup\{y_W(\phi - \psi) : y_V(\psi) > 0\}$

$= sup[W\{(\phi - \psi)(y)\} : V(\psi(y)) > 0]$

$= sup[W\{\phi(y) - \psi(y)\} : V(\psi(y)) > 0]$

$= sup[W\{\phi(y) - \psi(y)\} : \psi(y) \in B]$, Where $B = \{\psi(y) \in Z : V(\psi(y) > 0\} \subseteq A$.

Hence, $y_U(\phi) \geq (y_V + y_V)(\phi)$, for all $\phi \in FC(Y, Z)$. i.e., $y_U \geq y_V + y_W$.

Hence, $(f, g) \to f + g$ is fuzzy continuous. The proof for the $(f, g) \to f.g$ is fuzzy continuous is similar, so omitted.

Now, to prove $f \to -f$ is fuzzy continuous, let us consider a fuzzy open set y_U containing $(-f)_\alpha$. Hence, $(-f)_\alpha \leq y_U$

$\Rightarrow \alpha \leq y_U(-f)$

$\Rightarrow \alpha \leq U(-f)(y)$

$\Rightarrow \alpha \leq (-U)(f(y))$

$\Rightarrow \alpha \leq y_{-U}(f)$

Also, $f_\alpha(h) = 0 \leq y_{-U}(h), \forall\ h \neq f$. Hence, $f_\alpha \leq y_{-U}$. If U is fuzzy open then $-U$ is also so and consequently, y_{-U} is a subbasic open set on $FC(Y, Z)$ that contains f_α. We have to show that $-y_{-U} \leq y_U$. In fact, $-y_{-U}(\psi) = -(-U)(\psi(y)) = U(\psi(y)) = y_U(\psi)$, showing $-y_{-U} = y_U$. This completes the Theorem.

Theorem 6.4.3 Let (Y, τ_Y) be fully stratified fts and (Z, τ_Z) be a left ftr. Then $FC(Y, Z)$ endowed with fuzzy compact open topology is a left ftr

Proof. It is clear that $FC(Y, Z)$ is a ring. We need to show that (i) $(f, g) \rightarrow f + g$ is left fuzzy continuous.

(ii) $(f, g) \rightarrow f.g$ is left fuzzy continuous.

(iii) $f \rightarrow -f$ is fuzzy continuous.

Let K_U be a subbasic open set containing $(f+g)_\alpha$. Hence, $(f+g)_\alpha \leq K_U$

$\Rightarrow K_U(f + g) \geq \alpha$

$\Rightarrow inf\{U(f + g)(y) : y \in supp(K)\} \geq \alpha$. Hence, for all $y \in supp(K), U(f(y) + g(y)) \geq \alpha$, i.e., $(f(y) + g(y))_\alpha \leq U$. As Z is left ftr, there exist fuzzy open sets V and W on Z such that

$(f(y))_\alpha \leq V, (g(y))_\alpha \leq W$ and $V + W \leq U$. First we shall prove that, $f_\alpha \leq K_V$.

As, $f_\alpha(f) = \alpha$ and $\forall\ y \in supp(K), V(f(y)) \geq \alpha$ and $K_V(f) = inf\{V(f(y)) : y \in supp(K)\} \geq \alpha$. Hence, $K_V(f) \geq f_\alpha(f)$. If $f \neq h, f_\alpha(h) = 0 \leq K_V(h)$. So, $f_\alpha \leq K_V$. Similarly, it can be proved that $g_\alpha \leq K_W$. Consequently, to complete the proof we have to show $K_V + K_W \leq K_U$. Now,

$(K_V + K_W)(\phi) = sup\{K_W(\phi - \psi) : \psi \in A\}$, Where

$A = \{\psi \in FC(Y, Z) : K_V(\psi) > 0\}$

$= \{\psi \in FC(Y, Z) : inf\{V(\psi(y)) > 0 : y \in supp(K)\}\}$

$\subseteq \{\psi \in FC(Y, Z) : V(\psi(y)) > 0\}$

$= B_y, for\ each\ y \in supp(K)$. Hence,

$(K_V + K_W)(\phi)$

$\leq sup\{K_W(\phi - \psi) : \psi \in B_y\}, \forall\ y \in supp(K)$

$= sup[inf\{W(\phi - \psi)(z) : z \in supp(K)\} : \psi \in B_y], \forall\ y \in supp(K)$

$= sup\{W(\phi - \psi)(y) : \psi \in B_y\}, \forall\ y \in supp(K)$

$\leq inf[sup\{W(\phi - \psi)(y) : \psi \in B_y\},\ y \in supp(K)]$

$= inf\{(V + W)(\phi(y)) : y \in supp(K)\}$

$\leq inf\{U(\phi(y)) : y \in supp(K)\}$

$= K_U(\phi)$.

Hence, $(f, g) \rightarrow f + g$ is fuzzy continuous.

The proof for $(f, g) \to f.g$ is fuzzy continuous is similar and hence omitted. Now, to prove $f \to -f$ is fuzzy continuous, let us consider a subbasic open set K_U containing $(-f)_\alpha$. Hence, $(-f)_\alpha \leq K_U$

$\Rightarrow \alpha \leq K_U(-f)$

$\Rightarrow \alpha \leq inf\{U((-f)(x)) : x \in supp(K)\}$

$\Rightarrow \alpha \leq inf\{U(-f(x)) : x \in supp(K)\}$

$\Rightarrow \alpha \leq inf\{-U(f(x)) : x \in supp(K)\}$

$\Rightarrow f_\alpha(f) \leq K_{-U}(f)$

Also, $f_\alpha(h) = 0 \leq K_{-U}(h), \forall\ h \neq f$. Hence, $f_\alpha \leq K_{-U}$. If U is fuzzy open then $-U$ is also so and consequently, K_{-U} is a subbasic open set on $FC(Y, Z)$ that contains f_α. We have to show that $-K_{-U} \leq K_U$. In fact,

$-K_{-U}(g)$

$= K_{-U}(-g)$

$= inf\{(-U)(-g(x)) : x \in supp(K)\}$

$= inf\{U(g(x)) : x \in supp(K)\}$

$= K_U(g)$. This shows that $-K_{-U} = K_U$. Hence, $f \to -f$ is fuzzy continuous. This completes the Theorem.

Theorem 6.4.4 Let (Y, τ_Y) be a fully stratified fts and (Z, τ_Z) be a left ftr. Then $FC(Y, Z)$ endowed with fuzzy nearly compact regular open topology is a left ftr.

Proof. Follows as Theorem (6.4.3).

The induced homomorphism $f^* : FC(Y, Z) \to FC(X, Z)$ given by $f^*(g) = g \circ f$ as observed in Theorem (6.3.10), becomes fuzzy continuous homomorphism when $FC(Y, Z)$ and $FC(X, Z)$ are endowed with fuzzy compact open topology.

Theorem 6.4.5 Let X and Y be two fully stratified fts and Z be a left ftr. If $FC(Y, Z)$ and $FC(X, Z)$ are endowed with fuzzy compact open topology and $f^* : FC(Y, Z) \to FC(X, Z)$ given by $f^*(g) = g \circ f$ is a ring homomorphism induced from a fuzzy continuous function $f : X \to Y$, then f^* is fuzzy continuous.

Proof. Let K_μ be a subbasic fuzzy open set on $FC(X, Z)$. So, K is fuzzy compact on X and μ is fuzzy open on Z. As f is fuzzy continuous, $f(K)$is fuzzy compact on Y. We observe that $y \in supp(f(K))$ iff there exist $t \in supp(K)$ such that $f(t) = y$. Now, $(f(K))_\mu(g)$

$= inf\{\mu(g(y)) : y \in supp(f(K))\}$

$= inf\{\mu(g(f(t))) : t \in supp(f(K))\}$

$= K_\mu(g \circ f)$

$= f^{*-1}(K_\mu)(g), \forall\ g \in FC(Y, Z)$.

This completes the proof.

Theorem 6.4.6 If X is any fully stratified fts and Z_1, Z_2 are left ftr, then the ring homomorphism $\hat{\phi} : FC(X, Z_1) \to FC(X, Z_2)$ given by $\hat{\phi}(f) = \phi \circ f$ induced by a fuzzy continuous ring homomorphism $\phi : Z_1 \to Z_2$ is also fuzzy continuous, if both $FC(X, Z_1)$ and $FC(X, Z_2)$ have fuzzy compact open topology.

Proof. Let K_μ on $FC(X, Z_2)$. Now, $\hat{\phi}^{-1}(K_\mu)(g)$

$= K_\mu(\hat{\phi}(g))$

$= K_\mu(\phi \circ g)$

$= inf\{\mu(\phi(g(x))) : x \in supp(K)\}$

$= inf\{(\phi^{-1}(\mu))(g(x)) : x \in supp(K)\}$

$= K_{\phi^{-1}(\mu)}(g)$.

As, ϕ is fuzzy continuous and μ is fuzzy open on Z_2, $\phi^{-1}(\mu)$ is fuzzy open on Z_1. So, $K_{\phi^{-1}(\mu)}$ is subbasic fuzzy open on $FC(X, Z_1)$. Hence, $\hat{\phi}$ is fuzzy continuous.

Finally, we state the above results in the light of categories as follows:

Theorem 6.4.7 Let FTS be the category of fully stratified fuzzy topological spaces and fuzzy continuous functions and FTR the category of all left ftr and fuzzy continuous ring homomorphisms. Then
(i)$FC(-, Z) : FTS \to FTR$ given by $Y \to FC(Y, Z)$ is a contravariant functor, for each left ftr Z.
(ii)$FC(X, -) : FTR \to FTR$ given by $Z \to FC(X, Z)$ is a covariant functor, for each fully stratified fuzzy topological space X.

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