Chapter 6

Fuzzy continuous functions on Left fuzzy topological rings

6.1 Introduction

Topological ring in its various aspects has been widely studied in general topology. Concept of left fuzzy topological ring has recently been introduced by Deb Ray in [18], with the motive whether it embraces the hitherto known properties of topological rings. The prime result obtained in the above work is the characterization of left fuzzy topological rings in terms of the fundamental system of fuzzy neighbourhoods of the fuzzy point 0_α $(0 < \alpha \leq 1)$.

In this chapter, some properties of left fuzzy topological rings are reviewed from the categorical stand point and some results are obtained. Further, the collection of all fuzzy continuous functions on a fuzzy topological space, having values in left fuzzy topological ring has been studied both under algebraic and topological view-points.

6.2 Left fuzzy topological rings

In this section, left fuzzy topological ring as introduced by Deb Ray in [18], has been discussed in general and certain properties of the same from categorical view-point are interpreted.

As prerequisites, we state a few known definitions and results from [18].

Definition 6.2.1 [18] Let (X, τ) and (Y, σ) be two fuzzy topological spaces. A function $f : X \times X \to Y$ is said to be fuzzy left continuous if f is fuzzy continuous with respect to the fuzzy topology on the product $X \times X$ generated by the collection $\{U \times V : U, V \in \tau\}$ where $(U \times V)(s,t) = \begin{cases} V(t), & \text{if } U(s) > 0 \\ 0, & \text{otherwise} \end{cases}$

Definition 6.2.2 [18] Let R be a ring and τ be a fuzzy topology on R such that, for all $x, y \in R$,

i) $(x, y) \to x + y$ is fuzzy left continuous.

ii) $(x, y) \to x.y$ is fuzzy left continuous.

iii) $x \to -x$ is fuzzy continuous.

The pair (R, τ) is called a left fuzzy topological ring, (In short left ftr).

For non zero fuzzy sets U, V on R, the fuzzy sets $U+V$, UV and $-U$ are defined in [18] as follows :

$(U+V)(x) = sup\{V(x-s) : U(s) > 0\}$

$$(UV)(x) = \begin{cases} sup\{V(t) : x = st \text{ and } U(s) > 0\}, & \text{if } \{(s,t) \in R \times R : st = x\} \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$(-U)(x) = U(-x), \forall\, x \in R.$

Remark 6.2.1 In what follows, as in [18] we use left associativity of addition of fuzzy sets. i.e., for any three nonzero fuzzy sets U, V, W on R, $U+V+W = (U+V)+W$.

Remark 6.2.2 [18] Although $U+V \neq V+U$ and $UV \neq VU$ in general, for fuzzy points x_α, y_α on R $(0 < \alpha < 1)$, the following hold:

(i) $x_\alpha + y_\alpha = (x+y)_\alpha$

(ii) $x_\alpha \cdot y_\alpha = (xy)_\alpha$

(iii) $-x_\alpha = (-x)_\alpha$

(iv) $(-x)_\alpha + x_\alpha = 0_\alpha = x_\alpha + (-x)_\alpha$

In fact, the collection of fuzzy points (for each α) forms a ring with respect to these operations.

Theorem 6.2.1 [18] In a left ftr, for any fuzzy sets S_1, S_2, T_1 and T_2 with $S_1 \leq S_2$, $T_1 \leq T_2$, the following hold:

(i) $S_1 + T_1 \leq S_2 + T_2$

(ii) $S_1.T_1 \leq S_2.T_2$

(iii) $x_\alpha S_1 \leq x_\alpha T_1$

(iv) $S_1 x_\alpha \leq T_1 x_\alpha$, $\forall\, x \in R$ and $\alpha \in (0, 1]$

Theorem 6.2.2 [18] In a left ftr (R, τ), for each fuzzy set V, each $x \in R$ and $0 < \alpha \leq 1$,

$$(x_\alpha V)(z) = \begin{cases} sup\{V(t) : z = xt\}, & \text{if there is } t, \text{ such that } z = xt \\ 0, & \text{otherwise} \end{cases}$$

$$(V x_\alpha)(z) = \begin{cases} \alpha, & \text{if there is } s, \text{ such that } sx = z \\ 0, & \text{otherwise} \end{cases}$$

Example 6.2.1 [18] Let Z_3 be the ring of integers modulo 3. Define a fuzzy set A on Z_3 as $A(x) = 0.25$ for all $x \in Z_3$. Then clearly $\tau = \{0_X, 1_X, A\}$ is a fuzzy topology on Z_3. As $A + A = A$ and $A.A = A$, it is easy to see that (Z_3, τ) is a left ftr.

Theorem 6.2.3 [18] let R be a left ftr. If $\phi : R \to R$ is given by $\phi(x) = -x$ then ϕ is a fuzzy homeomorphism.

Corollary 6.2.1 [18] V is fuzzy open if and only if $-V$ is fuzzy open.

Corollary 6.2.2 [18] V is a fuzzy *nbd.* of 0_α if and only if $-V$ is a fuzzy *nbd.* of 0_α.

Theorem 6.2.4 [18] Suppose (R, τ) is a left fuzzy toopological ring. Then for each $a \in R$, $\phi_a : R \to R$ given by $\phi_a(x) = a + x$ is a fuzzy homeomorphism.

Theorem 6.2.5 [18] In a left ftr (R, τ), for each α with $0 < \alpha \leq 1$ and $x \in R$, V is fuzzy open (fuzzy closed) iff $x_\alpha + V$ is fuzzy open (respectively, fuzzy closed).

We show that *q-nbd.* of any fuzzy point x_α can also be characterized through *q-nbds.* of 0_α, for all $\alpha \in I_1$.

Theorem 6.2.6 In a left ftr R, V is a fuzzy *q-nbd.* of 0_α iff $-V$ is a fuzzy *q-nbd.* of 0_α.

Proof. Let V be a fuzzy *q-nbd.* of 0_α. There exists fuzzy open set A such that $0_\alpha q A \leq V$. i.e., $\alpha + A(0) > 1$ and $A \leq V$. For all $x \in R$, $A(-x) \leq V(-x) \Rightarrow -A \leq -V$. Now, $0_\alpha(0) + (-A)(0) = \alpha + A(-0) > 1$. Hence, $0_\alpha q(-A)$ and $-A \leq -V$. Using Corollary (6.2.1) $-V$ is a fuzzy *q-nbd.* of 0_α.

Conversely, let $-V$ is a fuzzy *q-nbd.* of 0_α. There exist fuzzy open set A such that $0_\alpha q A \leq -V$. As above, $-A \leq V$ and $0_\alpha q(-A)$. i.e., V is a fuzzy *q-nbd.* of 0_α.

Theorem 6.2.7 In a left ftr (R, τ), for each α with $0 < \alpha \leq 1$ and $x \in R$, if V is a fuzzy q-$nbd.$ (fuzzy open q-$nbd.$ or fuzzy closed q-$nbd.$) of 0_α, then $x_\alpha + V$ is a fuzzy q-$nbd.$ (fuzzy open q-$nbd.$ or fuzzy closed q-$nbd.$) of x_α. Moreover, any fuzzy q-$nbd.$ of x_α is precisely of the form $x_\alpha + V$, where V is a fuzzy q-$nbd.$ of 0_α.

Proof. If V is a fuzzy q-$nbd.$ of 0_α, there is a fuzzy open set A such that $0_\alpha q A \leq V$. i.e., $\alpha + A(0) > 1$ and $A \leq V$. By Theorem (6.2.5) $x_\alpha + A$ is a fuzzy open set. By Theorem (6.2.1) $x_\alpha + A \leq x_\alpha + V$. We verify that $x_\alpha q(x_\alpha + A)$. Now,

$\alpha + (x_\alpha + A)(x)$

$= \alpha + sup\{A(x - s) : x_\alpha(s) > 0\}$

$= \alpha + A(0) > 1$

This shows $(x_\alpha + A)$ is fuzzy open, such that $x_\alpha q(x_\alpha + A) \leq x_\alpha + V$. Hence, $x_\alpha + V$ is a fuzzy q-$nbd.$ of x_α. Suppose, V^* is any fuzzy q-$nbd.$ of x_α. Then there is a fuzzy open set U^* such that $x_\alpha q U^* \leq V^*$. i.e., $\alpha + U^*(x) > 1$ and $U^*(y) \leq V^*(y), \forall\, y$. Consider $U = (-x)_\alpha + U^*$ and $V = (-x)_\alpha + V^*$. Then U is a fuzzy open set. To show $0_\alpha q U \leq V$. Now,

$0_\alpha(0) + U(0)$

$= \alpha + [(-x)_\alpha + U^*](x)$

$= \alpha + U^*(0) > 1.$

So, $0_\alpha qU$. As $(-x)_\alpha \leq (-x)_\alpha$ and $U^* \leq V^*$, $U \leq V$. Hence, $0_\alpha qU$ and $U \leq V$. Again, $x_\alpha + U = x_\alpha + (-x)_\alpha + V^* = 0_\alpha + V^* = V^*$. This completes the proof.

A topological ring is "homogeneous", a function defined on it is continuous throughout its domain of definition whenever it is continuous at 0. The following theorem reflects a similar behaviour of left ftr.

Theorem 6.2.8 Let (R, τ) and (S, σ) be left fuzzy topoogical rings and $f : R \to S$ be a ring homomorphism. Then $f : (R, \tau) \to (S, \sigma)$ is fuzzy continuous iff f is fuzzy continuous at 0_α, where $0 < \alpha \leq 1$.

Proof. Let $f : (R, \tau) \to (S, \sigma)$ be fuzzy continuous. In particular f is fuzzy continuous at 0_α. Conversely, let $f : (R, \tau) \to (S, \sigma)$ be fuzzy continuous at $0_\alpha, \forall\ \alpha \in (0, 1]$. For any fuzzy open set U containing $(f(0))_\alpha = f(0_\alpha)$ in S, there exist fuzzy open set V containing 0_α on R such that $f(V) \leq U$. Let x_α be fuzzy point on R and B be any fuzzy open set on S containing the fuzzy point $(f(x))_\alpha$ on S. Now, $x_\alpha + V$ is fuzzy open set containing x_α. As B is a fuzzy open set on S containing $(f(x))_\alpha$, we have $B = (f(x))_\alpha + U$. To show $f((x)_\alpha + V) \leq (f(x))_\alpha + U$. $[f((x)_\alpha + V)](z)$

$= \sup[(x_\alpha + V)(t) : f(t) = z]$

$= \sup[V(t - x) : f(t) = z]$

$= \sup[V(p) : f(x+p) = z]$

$= \sup[V(p) : f(x) + f(p) = z]$

$= \sup[V(p) : f(p) = z - f(x)]$

$= f(V)(z - f(x))$

$\leq U(z - f(x))$

$= [f(x))_\alpha + U](z).$

Hence, $f((x)_\alpha + V) \leq (f(x))_\alpha + U$.

Using the language of categories, we obtain the following :

Theorem 6.2.9 The collection of all left ftr and fuzzy continuous homomorphisms form a category.

Proof. Consider the collection of all left ftr as objects and for each pair of objects X,Y, the set of all arrows as the collection of fuzzy continuous homomorphisms from X to Y. Then it is easy to observe that taking composition of arrows as the usual composition of functions, one gets:

(i) composition of arrows is associative and

(ii) for each object X, $id : X \to X$ given by $id(x) = x$ is the identity arrow. Consequently, it forms a category.

Remark 6.2.3 The category mentioned in Theorem (6.2.9), will henceforth be referred to as FTR.

Remark 6.2.4 It is well known that corresponding to any topological space (X, τ), one can obtain the characteristic fuzzy topological space (X, τ_f).

Theorem 6.2.10 If (X, τ) is a topological ring then (X, τ_f) is a left ftr.

Proof. For a topological space (X, τ), it is known that (X, τ_f) is a fuzzy topological space. We have to show the following :

(i) $\forall\ x, y \in Z, (x, y) \to x + y$ is left fuzzy continuous.

(ii) $\forall\ x, y \in Z, (x, y) \to x.y$ is left fuzzy continuous.

(iii) $\forall\ x \in Z, x \to -x$ is fuzzy continuous.

We show that '+' is left fuzzy continuous. Let μ be a fuzzy open set on (X, τ_f) with $(x + y)_\alpha \leq \mu$. Then $\mu = \chi_A$ for some $A \in \tau$. Hence, $(x + y)_\alpha \leq \chi_A \Rightarrow \alpha \leq \chi_A(x + y) \Rightarrow x + y \in A$. Since (X, τ) is a topological ring, there exist open sets B, $C \in \tau$ such that $x \in B$, $y \in C$ and $B + C \subseteq A$. Then $x_\alpha \leq \chi_B$ and $y_\alpha \leq \chi_C$, where χ_B, $\chi_C \in \tau_f$. Now to complete the proof, we show $\chi_B + \chi_C \leq \chi_A = \mu$. Now, $\forall\ z \in X$,

$(\chi_B + \chi_C)(z)$

$= sup\{\chi_C(z - t) : \chi_B(t) > 0\}$

$= sup\{\chi_C(z - t) : t \in B\}$

$$= \begin{cases} 1, & \text{for } t \in B \text{ and } z - t \in C \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \text{for } z \in B + C \\ 0, & \text{otherwise} \end{cases}$$

$$= \chi_{B+C}(z)$$

$$\leq \chi_A(z).$$

Hence, '+' is left fuzzy continuous. Proceeding in a similar manner, (ii) and (iii) can be obtained. Hence, (X, τ_f) is a left fts.

Though the following result is known in advance, we sketch a proof of the same.

Theorem 6.2.11 If f is a continuous homomorphism from a topological ring (X, τ) to a topological ring (Y, σ) then $f : (X, \tau_f) \to (Y, \sigma_f)$ is a fuzzy continuous homomorphism between the corresponding left ftr.

Proof. Since $f : (X, \tau) \to (Y, \sigma)$ is a homomorphism, $f : (X, \tau_f) \to (Y, \sigma_f)$ remains a homomorphism. Let $\mu \in \sigma_f$. Then there is some $U \in \sigma$ such that $\mu = \chi_U$. Now, $f^{-1}(\mu) = f^{-1}(\chi_U) = \chi_{f^{-1}(U)}$. Since $U \in \sigma$ and f is continuous, $f^{-1}(U) \in \tau$. Hence, $\chi_{f^{-1}(U)} \in \tau_f$. Consequently, $f : (X, \tau_f) \to (Y, \sigma_f)$ is a fuzzy continuous homomorphism.

We express the above in terms of categories as follows:

Theorem 6.2.12 If *TopRng* is the category of topological rings and continuous homomorphisms, then *TopRng* is a full subcategory of *FTR*.

Proof. In the light of the Theorems (6.2.10) and (6.2.11), any object of *TopRng* can be viewed as an object of *FTR* and any morphism between two objects of *TopRng* is a morphism between the corresponding objects of *FTR*. Hence, *TopRng* is a subcategory of *FTR*. Now, consider the inclusion functor $i : TopRng \to FTR$ that sends (X, τ) to its characteristic *fts* (X, τ_f) and $f : (X, \tau) \to (Y, \sigma)$ to $f : (X, \tau_f) \to (Y, \sigma_f)$. To show that the functor i is full. Let (X, τ) and (Y, σ) be two objects in *TopRng* and $f^* : (X, \tau_f) \to (Y, \sigma_f)$ a morphism in *FTR*. If $U \in \sigma$ then $\chi_U \in \sigma_f$ and so, $f^{*-1}(\chi_U) = \chi_{f^{*-1}(U)} \in \tau_f$ which in turn gives $f^{*-1}(U) \in \tau$. Hence, there exist $f^* : (X, \tau) \to (Y, \sigma)$ a morphism in *TopRng* such that $i(f^*) = f^*$. i.e., i is full. Consequently, *TopRng* is a full subcategory of *FTR*.

Theorem 6.2.13 Let (Z, σ) be a left *ftr* then $\forall\ \alpha \in I_1, (Z, i_\alpha(\sigma))$ is a topological ring.

Proof. We need to show the following :

(i) $\forall\ x, y \in Z, (x, y) \to x + y$ is continuous.

(ii) $\forall\ x, y \in Z, (x, y) \to x.y$ is continuous.

(iii) $\forall\ x \in Z, x \to -x$ is continuous.

Let $x, y \in Z$ and A be any open set in $(Z, i_\alpha(\sigma))$ containing $x+y$. There exist fuzzy open set μ in (Z, σ) such that $\mu^\alpha = A$. So, $(x+y)_\alpha < \mu$. As, (Z, σ) is a left ftr, there exist fuzzy open sets U and V such that $x_\alpha < U$, $y_\alpha < V$ and $U + V \leq \mu$. Then $x \in U^\alpha$ and $y \in V^\alpha$. We shall show that $U^\alpha + V^\alpha \subseteq A$. Let $z \in U^\alpha + V^\alpha$. Then $z = s + t$ where $s \in U^\alpha$ and $t \in V^\alpha$. i.e., $U(s) > \alpha$ and $V(t) > \alpha$. Now,

$(U+V)(z)$

$= sup\{V(z-p) : U(p) > 0\}$

$\geq V(t),\ where\ U(s) > 0\ and\ z = s + t$

$> \alpha$. So, $\mu(z) > \alpha$, $z \in \mu^\alpha = A$. Hence, $U^\alpha + V^\alpha \subseteq A$. This proves '+' is continuous. The proof for '.' is continuous is similar and hence omitted. Now, we shall prove that $x \to -x$ is continuous. Let $x \in Z$ and A be an open set on $(Z, i_\alpha(\sigma))$ containing $-x$. There is a fuzzy open set μ on (Z, σ) such that $\mu^\alpha = A$. So, $(-x) \in \mu^\alpha \Rightarrow (-x)_\alpha < \mu$. As (Z, σ) is left ftr, there exist fuzzy open set U containing x_α such that $x_\alpha < U$ and $-U \leq \mu$. We shall show $(-U)^\alpha \subseteq A$. Let $z \in (-U)^\alpha \Rightarrow \mu(z) \geq -U(z) > \alpha$. So, $z \in \mu^\alpha$. Hence, $(-U)^\alpha \subseteq A$.

Theorem 6.2.14 [80] A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy continuous iff $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is continuous for each $\alpha \in I_1$, where (X, τ), (Y, σ) are fts.

Theorem 6.2.15 A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy continuous homomorphism iff $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is continuous homomorphism for each $\alpha \in I_1$, where $(X, \tau), (Y, \sigma)$ are left ftr.

Proof. Immediate from Theorem (6.2.14)

In view of Theorem (6.2.13) and Theorem (6.2.15), we get:

Theorem 6.2.16 For each $\alpha \in I_1$, $i_\alpha : FTR \to TopRng$ is a covariant functor.

6.3 Left ftr-valued fuzzy continuous functions

In this section, our objective is to study the collection of all left ftr-valued fuzzy continuous functions on a fuzzy topological space. We find that the ring operations on the co-domain space induce a ring structure on this collection of functions.

In what follows, unless it is explicitly mentioned, the rings are non commutative and without unity.

Theorem 6.3.1 Let (Y, τ_Y) be a fts and (Z, τ_Z) be a left ftr. If $FC(Y, Z)$ stands for all fuzzy continuous functions from Y to Z, then $\forall\ f, g \in FC(Y, Z) \Rightarrow f + g, fg, -f \in FC(Y, Z)$

Proof. Let y_α be a fuzzy point on Y and U be any fuzzy open set on Z such that $(f + g)(y_\alpha) \leq U$. Now, for any $z \in Z$,

$[(f+g)(y_\alpha)](z)$

$= sup\{y_\alpha(t) : (f+g)(t) = z\}$

$$= \begin{cases} \alpha, & \text{for } z = (f+g)(y) \\ 0, & \text{otherwise} \end{cases}$$

So, $[(f+g)(y_\alpha)]$

$= ((f+g)(y))_\alpha$

$= (f(y) + g(y))_\alpha$

$= ((f(y))_\alpha + (g(y))_\alpha$.

Hence, $(f(y))_\alpha + (g(y))_\alpha \le U$ in Z. As Z is a left ftr, there exist fuzzy open sets V_1 and W_1 on Z such that $(f(y))_\alpha \le V_1$, $(g(y))_\alpha \le W_1$ and $V_1 + W_1 \le U$.

Again, as $f(y_\alpha) = (f(y))_\alpha$ and $g(y_\alpha) = (g(y))_\alpha$, $f(y_\alpha) \le V_1$, $g(y_\alpha) \le W_1$. By fuzzy continuity of f and g, there exist fuzzy open sets V_2 and W_2 on Y with $y_\alpha \le V_2$ and $y_\alpha \le W_2$ such that $f(V_2) \le V_1$ and $g(W_2) \le W_1$. Choose, $S = V_2 \wedge W_2$. Clearly, S is a fuzzy open set containing y_α. Then for any $z \in Z$,

$f(S)(z)$

$= sup\{(V_2 \wedge W_2)(t) : f(t) = z\}$

$\le sup\{V_2(t) : f(t) = z\}$

$= f(V_2)(z)$ and similarly, $g(S)(z) \le g(W_2)(z)$. Consequently, $f(S) \le V_1$ and $g(S) \le W_1$, so that $f(S) + g(S) \le V_1 + W_1 \le U$. Now, to complete the proof, it is to show that $(f+g)(S) \leq f(S)+g(S)$. Now,

$(f(S)+g(S))(z)$

$= sup\{g(S)(z-x) : f(S)(x) > 0\}$

$= sup[sup\{S(q) : g(q) = z-x\} : f(S)(x) > 0]$

$= sup\{S(q) : g(q) = z-x \text{ and } \exists\, t \text{ with } S(t) > 0,\ f(t) = x\}$

$= sup\{S(q) : \exists\, t \text{ with } S(t) > 0 \text{ and } g(q) = z - f(t)\}$

$\geq sup\{S(q) : g(q) = z - f(q)\}$

$= sup\{S(q) : f(q) + g(q) = z\}$

$= sup\{S(q) : (f+g)(q) = z\}$

$= (f+g)(S)(z)$.

Hence, $f+g \in FC(Y, Z)$. Let y_α be a fuzzy point on Y and U be any fuzzy open set on Z such that $(fg)(y_\alpha) \leq U$. Now, for any $z \in Z$,

$[(fg)(y_\alpha)](z)$

$= sup\{y_\alpha(t) : (fg)(t) = z\}$

$$= \begin{cases} \alpha, & for \text{ z = (fg)(y)} \\ 0, & otherwise \end{cases}$$

. and so, $[(fg)(y_\alpha)]$

$= ((fg)(y))_\alpha$

$= (f(y).g(y))_\alpha$

$= ((f(y))_\alpha.(g(y))_\alpha$.

Hence, $(f(y))_\alpha.(g(y))_\alpha \leq U$ in Z. As Z is a left ftr, there exist fuzzy open sets V_1 and W_1 on Z such that $(f(y))_\alpha \leq V_1$, $(g(y))_\alpha \leq W_1$ and $V_1.W_1 \leq U$. Again, as $f(y_\alpha) = (f(y))_\alpha$ and $g(y_\alpha) = (g(y))_\alpha$, $f(y_\alpha) \leq V_1$, $g(y_\alpha) \leq W_1$. By fuzzy continuity of f and g, there exist fuzzy open sets V_2 and W_2 in Y with $y_\alpha \leq V_2$ and $y_\alpha \leq W_2$ such that $f(V_2) \leq V_1$ and $g(W_2) \leq W_1$. Choose, $S = V_2 \wedge W_2$. Clearly, S is a fuzzy open set containing y_α. Then for any $z \in Z$, $f(S)(z) \leq f(V_2)(z)$ and similarly, $g(S)(z) \leq g(W_2)(z)$. Consequently, $f(S) \leq V_1$ and $g(S) \leq W_1$, so that $f(S).g(S) \leq V_1.W_1 \leq U$. Now, to complete the proof, it is to show that $(f.g)(S) \leq f(S).g(S)$. Now,

$(f(S).g(S))(z)$

$= sup\{g(S)(p) : f(S)(q) > 0,\ qp = z\}$

$= sup[sup\{S(t) : g(t) = p\} : f(S)(q) > 0,\ qp = z]$

$= sup\{S(t) : t \in B\}$, where

$B = \{t \in Y : \exists\ m \in Y$ such that $f(m)g(t) = z$ and $S(m) > 0\}$ Also,

$(f.g)(S)(z)$

$= sup\{S(p) : (f.g)(p) = z\}$

$= sup\{S(p) : f(p).g(p) = z\}$

$= sup\{S(p) : p \in A\}$,where $A = \{t \in Y : f(t)g(t) = z\}$

For all $p \in A$, if $S(p) = 0$, then $(f.g)(S)(z) = 0 \leq (f(S).g(S))(z)$ holds. For otherwise, there exists some $p \in A$ such that $S(p) > 0$ and consequently, $p \in B$. Hence, in such case, from above we conclude that $(f.g)(S)(z) = 0 \leq (f(S).g(S))(z)$, as desired.

Let U be any fuzzy open set on Z. Then $\forall\ z \in Z$, $(-f)^{-1}(U)(z) = U((-f)(z)) = U(-f(z)) = (-U)(f(z)) = f^{-1}(-U)(z)$. As U is fuzzy open iff $-U$ is fuzzy open and $f \in FC(Y,Z)$, we get $-f \in FC(Y,Z)$.

Definition 6.3.1 [51] Let (X,τ) be a fts and $r \in [0,1]$. By r^* we mean a fuzzy set on X such that $r^*(x) = r$, for every $x \in X$. The fts (X,τ) is called fully stratified if $r^* \in \tau$, for every $r \in [0,1]$.

Theorem 6.3.2 [56] Let Y and Z be two fts such that Y is fully stratified. Then every $r \in [0,1]$, $r^* : Y \to Z$, given by $r^*(y) = r, \forall\ y \in Y$ is fuzzy continuous.

Corollary 6.3.1 Let Y be a fts and Z be a left ftr with additive identity 0. The function $0^* : Y \to Z$ given by $0^*(y) = 0, \forall\ y \in Y$ is fuzzy continuous. Further, if Z contains identity 1, then the function given by $1^*(y) = 1, \forall\ y \in Y$ is also fuzzy continuous.

Theorem 6.3.3 If (Y,τ_Y) is a fully stratified fts and (Z,τ_Z) is a left ftr, then $FC(Y,Z)$ forms a ring with respect to the usual addition and multiplication of functions.

Proof. By Theorem(6.3.1), $\forall\ f,g \in FC(Y,Z), f+g, fg, -f \in FC(Y,Z)$. Since Z is a ring, it is clear that '+' is associative and commutative while '.' is associative. It is easy to verify that in

$FC(Y, Z)$, '.' is distributive over '+' . Now, $\forall\ f \in FC(Y, Z)$ and $z \in Z$, $(f + 0^*)(z) = f(z) + 0^*(z) = f(z) + 0 = f(z)$ and $(f + (-f))(z) = f(z) + (-f)(z) = 0 = 0^*(z)$ prove that $FC(Y, Z)$ is a ring.

Theorem 6.3.4 Let $FC(Y, Z)$ be the ring of fuzzy continuous functions from a fully stratified fts (Y, τ_Y) to a left ftr (Z, τ_Z).

1. $FC(Y, Z)$ is commutative, if Z is commutative.

2. $FC(Y, Z)$ contains identity, if Z contains identity.

Proof. $\forall\ f, g \in FC(Y, Z)$ and $\forall\ y \in Y$, $(fg)(y) = f(y).g(y) = g(y).f(y) = (gf)(y)$. Again, $\forall\ f \in FC(Y, Z)$ and $\forall\ y \in Y$, $(f.1^*)(y) = f(y).1^*(y) = f(y).1 = f(y) = 1.f(y) = 1^*(y).f(y) = (1^*.f)(y)$, showing that 1^* is the identity in $FC(Y, Z)$

Definition 6.3.2 A fuzzy set μ on a ring R is called a left (right) fuzzy ideal of R iff $\forall\ x, y \in R$

1. $\mu(x - y) = min\{\mu(x), \mu(y)\}$

2. $\mu(xy) \geq \mu(y)$ (respectively, $\mu(xy) \geq \mu(x)$)

A left as well as right fuzzy ideal is called a fuzzy ideal.

Theorem 6.3.5 Let $FC(Y, Z)$ be the ring of fuzzy continuous functions from a fully stratified fts (Y, τ_Y) to a left ftr (Z, τ_Z). If Z_1

is a fuzzy ideal on Z, then the fuzzy set I on $FC(Y, Z)$, given by $I(f) = inf\{Z_1(f(y)) : y \in Y\}, \forall\ f \in FC(Y, Z)$, is a fuzzy ideal in $FC(Y, Z)$.

Proof. For all $f, h \in FC(Y, Z)$,

$I(f - h)$

$= inf\{Z_1((f - h)(y)) : y \in Y\}$

$= inf\{Z_1(f(y) - h(y)) : y \in Y\}$

$= inf[min\{Z_1(f(y), Z_1(h(y))\} : y \in Y]$

$= min[inf\{Z_1(f(y)) : y \in Y\}, inf\{Z_1(h(y)) : y \in Y\}]$

$= min\{I(f), I(h)\}$ and

$I(fh)$

$= inf\{Z_1((fh)(y)) : y \in Y\}$

$= inf\{Z_1(f(y)h(y)) : y \in Y\}$

$\geq inf\{Z_1(h(y)) : y \in Y\}$

$= I(h)$.

Hence, I is left fuzzy ideal on $FC(Y, Z)$. Similarly, it can be shown that I is a right fuzzy ideal on $FC(Y, Z)$. Hence, I is a fuzzy ideal on $FC(Y, Z)$

We show next how the algebraic nature of $FC(Y, Z)$ helps in determining the fuzzy connectedness of Y. Some relevent definitions and results are furnished here, before we establish the main result.

Definition 6.3.3 [32] A fts (X, τ) is fuzzy disconnected if there exist fuzzy sets U and V such that $U \vee V = 1, U \not{q} \overline{V}$ and $V \not{q} \overline{U}$.

Lemma 6.3.1 If (X, τ) is fuzzy disconnected fts then there exist fuzzy closed sets C and D such that $C \vee D = 1$ and $C \not{q} D$.

Proof. Let (X, τ) be fuzzy disconnected. There exist fuzzy sets A and B such that $A \vee B = 1, A \not{q} \overline{B}$ and $B \not{q} \overline{A}$. i.e., $\forall\ y \in Y$, $A(y) \vee B(y) = 1,\ A(y) + \overline{B}(y) \leq 1$ and $\overline{A}(y) + B(y) \leq 1$. Hence, for each $y \in Y$ we have either $[A(y) = 1$ and $B(y) = 0]$ or $[A(y) = 0$ and $B(y) = 1]$. We shall prove that the fuzzy closed sets $1 - int(clA)$ and $1 - int(clB)$ are the required sets. Now, $\forall\ y \in Y$ if $A(y) = 1$ then $A(y) \leq 1 - \overline{B}(y) \leq 1 - int(clB)(y)$, i.e., $1 - int(clB) = 1$ and if $B(y) = 1$ then similarly, we have $1 - int(clA) = 1$, showing $(1 - int(clA)) \vee (1 - int(clB)) = 1$ and $(1 - int(clA)) \not{q} (1 - int(clB))$.

Theorem 6.3.6 If a fts (X, τ) is fuzzy disconnected then $\forall\ \alpha \in I_1, (X, i_\alpha(\tau))$ is disconnected.

Proof. Let (X, τ) be fuzzy disconnected. By Lemma (6.3.1), there exist fuzzy open sets A and B on X such that $A \vee B = 1$ and $A \not{q} B$. Hence, for each $x \in X$ we have either $[A(x) = 1$ and $B(x) = 0]$ or $[A(x) = 0$ and $B(x) = 1]$. Now, $\forall\ \alpha \in I_1$, A^α and B^α are open in $(X, i_\alpha(\tau))$ with $A^\alpha \cup B^\alpha = (A \vee B)^\alpha = X$. If possible let $z \in A^\alpha \cap B^\alpha$. Then, $A(z) > \alpha$ and $B(z) > \alpha$, which is not possible.

Hence, $A^\alpha \cap B^\alpha = \Phi$ and so, $(X, i_\alpha(\tau))$ is disconnected.

Lemma 6.3.2 Let $C(Y,Z)$ denote the ring of continuous functions from a topological space $(Y, i_\alpha(\tau))$ to a topological ring $(Z, i_\alpha(\sigma))$, for each $\alpha \in I_1$. If Y is disconnected then there exist $f \in C(Y,Z)$ such that $f \neq 0, 1$ and $f^2 = f$.

Proof. If Y is disconnected, there exist nonempty disjoint closed sets A, B such that $Y = A \cup B$. Defining $f : Y \to Z$ by $f(y) = \begin{cases} 1, & if\ y \in A \\ 0 & if\ y \in B \end{cases}$, we get the desired non trivial idempotent.

Theorem 6.3.7 Let Z be a left ftr with 1 and without zero divisor such that 0_α is fuzzy closed for each $\alpha \in (0,1]$. If Y is any fully stratified fts such that the ring $FC(Y,Z)$ has some nontrivial idempotent element then Y is fuzzy disconnected.

Proof. Let $f \in FC(Y,Z)$ is such that $f^2 = f$ and $f \neq 0, 1$. To show Y is fuzzy disconnected. $\forall\, y \in Y, f^2(y) = f(y) \Rightarrow f(y)(1 - f(y)) = 0$. As Z has no zero divisor, for each $y \in Y$ we have, $f(y) = 0$ or $f(y) = 1$. 0_α for all $\alpha \in I_1$ is fuzzy closed in Z. Consider, $\alpha = 1$. As $1_\alpha = 0_\alpha + 1_\alpha$ and 0_α is fuzzy closed, using Theorem (6.2.5), 1_α is fuzzy closed in Z. f being fuzzy continuous, $f^{-1}(0_\alpha)$ and $f^{-1}(1_\alpha)$ are fuzzy closed in Y. Now, $[f^{-1}(0_\alpha) \vee f^{-1}(1_\alpha)](y)$

$= sup[f^{-1}(0_\alpha)(y), f^{-1}(1_\alpha)(y)]$

$= sup[(0_\alpha)(f(y)), (1_\alpha)f((y))]$

$= \alpha = 1.$

Hence, $f^{-1}(0_\alpha) \vee f^{-1}(1_\alpha) = 1_Y$. Clearly, for all $x \in Y$, $f^{-1}(0_\alpha)(x) + f^{-1}(1_\alpha)(x) = \alpha = 1$. i.e., $f^{-1}(0_\alpha) \not{q} f^{-1}(1_\alpha)$. This shows that Y is disconnected.

Theorem 6.3.8 Let Z be any left ftr and Y be any fuzzy disconnected fts then the ring $FC(Y, Z)$ has some nontrivial idempotent element.

Proof. Let Y be fuzzy disconnected. By Theorem (6.3.6), $\forall\ \alpha \in I_1, (X, i_\alpha(\tau))$ is disconnected. By Theorem (6.2.13), $\forall\ \alpha \in I_1, (Z, i_\alpha(\sigma))$ is a topological ring. Now, by Lemma (6.3.2), there exist a continuous function $f : (Y, i_\alpha(\tau)) \rightarrow (Z, i_\alpha(\sigma))$ such that $f \neq 0, 1$ and $f^2 = f$. By Theorem (6.2.14), $f : (Y, \tau) \rightarrow (Z, \sigma)$ is fuzzy continuous such that $f \neq 0, 1$ and $f^2 = f$.

Combining Theorem (6.3.7) and Theorem (6.3.8) we have:

Theorem 6.3.9 Let Z be a left ftr with 1 and without zero divisor such that 0_α is fuzzy closed for each $\alpha \in (0, 1]$. If Y is any fully stratified fts, then the ring $FC(Y, Z)$ has exactly two idempotents iff Y is fuzzy connected.

Theorem 6.3.10 Let X and Y be two fully stratified fts and $f : X \to Y$ be a fuzzy continuous function. For any left ftr Z, $f^* : FC(Y, Z) \to FC(X, Z)$ given by $f^*(g) = g \circ f$ is a ring homomorphism.

Proof. Straightforward.

Theorem 6.3.11 If X is any fully stratified fts and Z_1, Z_2 are left ftr, then every fuzzy continuous ring homomorphism $\phi : Z_1 \to Z_2$ induces a ring homomorphism $\hat{\phi} : FC(X, Z_1) \to FC(X, Z_2)$ given by $\hat{\phi}(f) = \phi \circ f$.

Proof. Immediate.

Reframing the results discussed above in the language of categories, we obtain the following functors:

Theorem 6.3.12 If FTS is the category of fully stratified fuzzy topological spaces and fuzzy continuous functions; *Rng* is the category of all rings and ring homomorphisms, then

(i)$FC(-, Z) : FTS \to Rng$ given by $Y \to FC(Y, Z)$ is a contravariant functor, for each left ftr Z.

(ii)$FC(X, -) : Rng \to Rng$ given by $Z \to FC(X, Z)$ is a covariant functor, for each fully stratified fuzzy topological space X.

6.4 $FC(Y, Z)$ as topological and Left fuzzy topological rings

In the previous chapters we have seen how a collection of functions can be fuzzy topologized. As $FC(Y, Z)$ is a class of functions, it is possible to equip it with various topologies and fuzzy topologies.

Here, we observe the interplay between its ring structure and its topological and fuzzy topological behaviour.

Definition 6.4.1 [35] Let U be a fuzzy open set on a fts Z and y_α ($\alpha \in (0,1]$) be a fuzzy point on a fts Y. By $[y_\alpha, U]$ we denote the subset of $FC(Y, Z)$ where $[y_\alpha, U] = \{f \in FC(Y, Z) : f(y_\alpha) \leq U\}$. The collection of all such $[y_\alpha, U]$ forms a subbase for some topology on $FC(Y, Z)$, called fuzzy-point fuzzy-open topology ($fp - fo$), denoted by, τ_{fp-fo}.[1]

Theorem 6.4.1 If (Y, τ_Y) is a fully stratified fts and (Z, τ_Z) is a left ftr, then $(FC(Y, Z), \tau_{(fp-fo)})$ is a topological ring.

Proof. It is clear that $FC(Y, Z)$ is a ring and $FC(Y, Z)$ is a topological space with respect to $\tau_{(fp-fo)}$. We need to show that

(i) $(f, g) \to f + g$ is continuous.

(ii) $(f, g) \to f.g$ is continuous.

[1]In [35], the name of this topology was fuzzy-point open topology, denoted by τ_{F-p-o}

(iii) $f \to -f$ is continuous. Let $[y_\alpha, U]$ be a subbasic open set containing $f+g$. Then $(f+g)(y_\alpha) \leq U$. Now, as $(f+g)(y_\alpha) = (f(y))_\alpha + (g(y))_\alpha$, $(f(y))_\alpha + (g(y))_\alpha \leq U$ in Z. Z being a left ftr, there exist fuzzy open sets V, W on Z such that, $(f(y))_\alpha \leq V$, $(g(y))_\alpha \leq W$ and $V + W \leq U$. Again, $f(y_\alpha) = (f(y))_\alpha \leq V \Rightarrow f \in [y_\alpha, V]$ and similarly, $g \in [y_\alpha, W]$. For continuity of $f+g$, we need to show that $[y_\alpha, V] + [y_\alpha, W] \subseteq [y_\alpha, U]$.

Let $\xi \in [y_\alpha, V] + [y_\alpha, W]$. Then there exist $\eta \in [y_\alpha, V]$ and $\psi \in [y_\alpha, W]$, such that $\xi = \eta + \psi$.

Now, $\eta \in [y_\alpha, V], \psi \in [y_\alpha, W]$

$\Rightarrow \eta(y_\alpha) \leq V,\ \psi(y_\alpha) \leq W$

$\Rightarrow (\eta + \psi)(y_\alpha) \leq (V + W) \leq U$

$\Rightarrow \xi(y_\alpha) \leq U$

$\Rightarrow \xi \in [y_\alpha, U]$.

The proof for the product fg to be continuous is similar and hence omitted.

Now, for any $f \in FC(Y, Z)$ and any subbasic open set $[y_\alpha, U]$ containing $-f$, we get, $(-f)(y_\alpha) \leq U \Rightarrow (-f(y))_\alpha \leq U$. It is easy to see that $(-f(y))_\alpha \leq U \Rightarrow (f(y))_\alpha \leq -U$. Since U is fuzzy open iff $-U$ is fuzzy open and $(f(y_\alpha) = (f(y))_\alpha$, $f \in [y_\alpha, -U]$. We now show that $-[y_\alpha, U] \leq [y_\alpha, -U]$.

Let $\psi \in -[y_\alpha, U]$. Then there is some $\eta \in [y_\alpha, U]$ such that $\psi = -\eta$. $\eta \in [y_\alpha, U] \Rightarrow \eta(y_\alpha) \leq U \Rightarrow (-(-\eta))(y_\alpha) \leq U \Rightarrow (-\psi)(y_\alpha) \leq U \Rightarrow \psi(y_\alpha) \leq -U \Rightarrow \psi \in [y_\alpha, -U]$, as desired.

Definition 6.4.2 Let $FC(Y, Z)$ denote the collection of all fuzzy continuous functions from a $fts(Y, \tau_Y)$ to another fts (Z, τ_Z). By y_U we mean a fuzzy set on $FC(Y, Z)$, given by $y_U(f) = U(f(y))$, for every $f \in FC(Y, Z)$. The fuzzy point open topology (FPO) on $FC(Y, Z)$ is generated by fuzzy sets of the form y_U where $y \in Y$ and U is a fuzzy open set on Z [35].

Theorem 6.4.2 Let (Y, τ_Y) be fully stratified fts and (Z, τ_Z) be a left ftr. Then $FC(Y, Z)$ endowed with fuzzy point-open topology (FPO) is a left ftr .

Proof. It is clear that $FC(Y, Z)$ is a ring. We need to show that (i) $(f, g) \to f + g$ is left fuzzy continuous.

(ii) $(f, g) \to f.g$ is left fuzzy continuous.

(iii) $f \to -f$ is fuzzy continuous.

Let y_U be any subbasic open set containing $(f + g)_\alpha$. We have to find fuzzy open sets y_V, y_W in $FC(Y, Z)$ such that $y_V + y_W \leq y_U$, $f_\alpha \leq y_V$ and $g_\alpha \leq y_W$. Now,$(f+g)_\alpha \leq y_U \Rightarrow y_U(f+g) \geq \alpha \Rightarrow U[(f+g)(y)] \geq \alpha \Rightarrow U[(f(y)+g(y)] \geq \alpha$. It is easy to see that $(f(y)+g(y))_\alpha \leq U$ in Z. As $(f(y))_\alpha + (f(y))_\alpha = (f(y)+g(y))_\alpha$ and Z is a left ftr, there exist fuzzy open sets V and W in Z such that $(f(y))_\alpha \leq V, (g(y))_\alpha \leq W$ and $V + W \leq U$. Now, we verify that $f_\alpha \leq y_V$ and $g_\alpha \leq y_W$.

$f_\alpha(f) = \alpha \leq V(f(y)) = y_V(f)$ and for $h \neq f, f_\alpha(h) = 0 \leq y_V(h)$. Hence, $f_\alpha \leq y_V$. Similarly, we can prove $g_\alpha \leq y_W$. In order to complete the proof it is to show that $y_V + y_W \leq y_U$. Now,

$y_U(\phi)$

$= (V + W)(\phi(y))$

$= sup\{W(\phi(y) - t) : V(t) > 0\}$

$= sup\{W(\phi(y) - t) : t \in A\}$, Where $A = \{t \in Z : V(t) > 0\}$.

$(y_V + y_W)(\phi)$

$= sup\{y_W(\phi - \psi) : y_V(\psi) > 0\}$

$= sup[W\{(\phi - \psi)(y)\} : V(\psi(y)) > 0]$

$= sup[W\{\phi(y) - \psi(y)\} : V(\psi(y)) > 0]$

$= sup[W\{\phi(y) - \psi(y)\} : \psi(y) \in B]$, Where $B = \{\psi(y) \in Z : V(\psi(y) > 0\} \subseteq A$.

Hence, $y_U(\phi) \geq (y_V + y_V)(\phi)$, for all $\phi \in FC(Y, Z)$. i.e., $y_U \geq y_V + y_W$. Hence, $(f, g) \to f + g$ is fuzzy continuous. The proof for the $(f, g) \to f.g$ is fuzzy continuous is similar, so omitted.

Now, to prove $f \to -f$ is fuzzy continuous, let us consider a fuzzy open set y_U containing $(-f)_\alpha$. Hence, $(-f)_\alpha \leq y_U$

$\Rightarrow \alpha \leq y_U(-f)$

$\Rightarrow \alpha \leq U(-f)(y)$

$\Rightarrow \alpha \leq (-U)(f(y))$

$\Rightarrow \alpha \leq y_{-U}(f)$

Also, $f_\alpha(h) = 0 \leq y_{-U}(h), \forall\ h \neq f$. Hence, $f_\alpha \leq y_{-U}$. If U is fuzzy open then $-U$ is also so and consequently, y_{-U} is a subbasic open set on $FC(Y, Z)$ that contains f_α. We have to show that $-y_{-U} \leq y_U$. In fact, $-y_{-U}(\psi) = -(-U)(\psi(y)) = U(\psi(y)) = y_U(\psi)$, showing $-y_{-U} = y_U$. This completes the Theorem.

Theorem 6.4.3 Let (Y, τ_Y) be fully stratified fts and (Z, τ_Z) be a left ftr. Then $FC(Y, Z)$ endowed with fuzzy compact open topology is a left ftr

Proof. It is clear that $FC(Y, Z)$ is a ring. We need to show that (i) $(f, g) \to f + g$ is left fuzzy continuous.

(ii) $(f, g) \to f.g$ is left fuzzy continuous.

(iii) $f \to -f$ is fuzzy continuous.

Let K_U be a subbasic open set containing $(f+g)_\alpha$. Hence, $(f+g)_\alpha \leq K_U$

$\Rightarrow K_U(f + g) \geq \alpha$

$\Rightarrow inf\{U(f + g)(y) : y \in supp(K)\} \geq \alpha$. Hence, for all $y \in supp(K), U(f(y) + g(y)) \geq \alpha$, i.e., $(f(y) + g(y))_\alpha \leq U$. As Z is left ftr, there exist fuzzy open sets V and W on Z such that

$(f(y))_\alpha \leq V, (g(y))_\alpha \leq W$ and $V + W \leq U$. First we shall prove that, $f_\alpha \leq K_V$.

As, $f_\alpha(f) = \alpha$ and $\forall\ y \in supp(K), V(f(y)) \geq \alpha$ and

$K_V(f) = inf\{V(f(y)) : y \in supp(K)\} \geq \alpha$. Hence, $K_V(f) \geq f_\alpha(f)$.

If $f \neq h, f_\alpha(h) = 0 \leq K_V(h)$. So, $f_\alpha \leq K_V$. Similarly, it can be proved that $g_\alpha \leq K_W$. Consequently, to complete the proof we have to show $K_V + K_W \leq K_U$. Now,

$(K_V + K_W)(\phi) = sup\{K_W(\phi - \psi) : \psi \in A\}$, Where

$A = \{\psi \in FC(Y, Z) : K_V(\psi) > 0\}$

$= \{\psi \in FC(Y, Z) : inf\{V(\psi(y)) > 0 : y \in supp(K)\}\}$

$\subseteq \{\psi \in FC(Y, Z) : V(\psi(y)) > 0\}$

$= B_y, for\ each\ y \in supp(K)$. Hence,

$(K_V + K_W)(\phi)$

$\leq sup\{K_W(\phi - \psi) : \psi \in B_y\}, \forall\ y \in supp(K)$

$= sup[inf\{W(\phi - \psi)(z) : z \in supp(K)\} : \psi \in B_y], \forall\ y \in supp(K)$

$= sup\{W(\phi - \psi)(y) : \psi \in B_y\}, \forall\ y \in supp(K)$

$\leq inf[sup\{W(\phi - \psi)(y) : \psi \in B_y\},\ y \in supp(K)]$

$= inf\{(V + W)(\phi(y)) : y \in supp(K)\}$

$\leq inf\{U(\phi(y)) : y \in supp(K)\}$

$= K_U(\phi)$.

Hence, $(f, g) \rightarrow f + g$ is fuzzy continuous.

The proof for $(f, g) \to f.g$ is fuzzy continuous is similar and hence omitted. Now, to prove $f \to -f$ is fuzzy continuous, let us consider a subbasic open set K_U containing $(-f)_\alpha$. Hence, $(-f)_\alpha \leq K_U$

$\Rightarrow \alpha \leq K_U(-f)$

$\Rightarrow \alpha \leq inf\{U((-f)(x)) : x \in supp(K)\}$

$\Rightarrow \alpha \leq inf\{U(-f(x)) : x \in supp(K)\}$

$\Rightarrow \alpha \leq inf\{-U(f(x)) : x \in supp(K)\}$

$\Rightarrow f_\alpha(f) \leq K_{-U}(f)$

Also, $f_\alpha(h) = 0 \leq K_{-U}(h), \forall\ h \neq f$. Hence, $f_\alpha \leq K_{-U}$. If U is fuzzy open then $-U$ is also so and consequently, K_{-U} is a subbasic open set on $FC(Y, Z)$ that contains f_α. We have to show that $-K_{-U} \leq K_U$. In fact,

$-K_{-U}(g)$

$= K_{-U}(-g)$

$= inf\{(-U)(-g(x)) : x \in supp(K)\}$

$= inf\{U(g(x)) : x \in supp(K)\}$

$= K_U(g)$. This shows that $-K_{-U} = K_U$. Hence, $f \to -f$ is fuzzy continuous. This completes the Theorem.

Theorem 6.4.4 Let (Y, τ_Y) be a fully stratified fts and (Z, τ_Z) be a left ftr. Then $FC(Y, Z)$ endowed with fuzzy nearly compact regular open topology is a left ftr.

Proof. Follows as Theorem (6.4.3).

The induced homomorphism $f^* : FC(Y, Z) \to FC(X, Z)$ given by $f^*(g) = g \circ f$ as observed in Theorem (6.3.10), becomes fuzzy continuous homomorphism when $FC(Y, Z)$ and $FC(X, Z)$ are endowed with fuzzy compact open topology.

Theorem 6.4.5 Let X and Y be two fully stratified fts and Z be a left ftr. If $FC(Y, Z)$ and $FC(X, Z)$ are endowed with fuzzy compact open topology and $f^* : FC(Y, Z) \to FC(X, Z)$ given by $f^*(g) = g \circ f$ is a ring homomorphism induced from a fuzzy continuous function $f : X \to Y$, then f^* is fuzzy continuous.

Proof. Let K_μ be a subbasic fuzzy open set on $FC(X, Z)$. So, K is fuzzy compact on X and μ is fuzzy open on Z. As f is fuzzy continuous, $f(K)$is fuzzy compact on Y. We observe that $y \in supp(f(K))$ iff there exist $t \in supp(K)$ such that $f(t) = y$. Now, $(f(K))_\mu(g)$

$= inf\{\mu(g(y)) : y \in supp(f(K))\}$

$= inf\{\mu(g(f(t))) : t \in supp(f(K))\}$

$= K_\mu(g \circ f)$

$= f^{*-1}(K_\mu)(g), \forall\ g \in FC(Y, Z)$.

This completes the proof.

Theorem 6.4.6 If X is any fully stratified fts and Z_1, Z_2 are left ftr, then the ring homomorphism $\hat{\phi} : FC(X, Z_1) \to FC(X, Z_2)$ given by $\hat{\phi}(f) = \phi \circ f$ induced by a fuzzy continuous ring homomorphism $\phi : Z_1 \to Z_2$ is also fuzzy continuous, if both $FC(X, Z_1)$ and $FC(X, Z_2)$ have fuzzy compact open topology.

Proof. Let K_μ on $FC(X, Z_2)$. Now, $\hat{\phi}^{-1}(K_\mu)(g)$

$= K_\mu(\hat{\phi}(g))$

$= K_\mu(\phi \circ g)$

$= inf\{\mu(\phi(g(x))) : x \in supp(K)\}$

$= inf\{(\phi^{-1}(\mu))(g(x)) : x \in supp(K)\}$

$= K_{\phi^{-1}(\mu)}(g)$.

As, ϕ is fuzzy continuous and μ is fuzzy open on Z_2, $\phi^{-1}(\mu)$ is fuzzy open on Z_1. So, $K_{\phi^{-1}(\mu)}$ is subbasic fuzzy open on $FC(X, Z_1)$. Hence, $\hat{\phi}$ is fuzzy continuous.

Finally, we state the above results in the light of categories as follows:

Theorem 6.4.7 Let FTS be the category of fully stratified fuzzy topological spaces and fuzzy continuous functions and FTR the category of all left ftr and fuzzy continuous ring homomorphisms. Then (i)$FC(-, Z) : FTS \to FTR$ given by $Y \to FC(Y, Z)$ is a contravariant functor, for each left ftr Z.

(ii)$FC(X, -) : FTR \to FTR$ given by $Z \to FC(X, Z)$ is a covariant functor, for each fully stratified fuzzy topological space X.