Chapter 5

Pseudo nearly compact fuzzy sets and *ps*-*ro* continuous functions

5.1 Introduction

By accepting the necessary condition obtained in Theorem(3.4.10) (i) in Chapter 3, for a fuzzy set to be starplus nearly compact as a basic definition of a compact-like covering property, we introduce the notion of pseudo near compactness. The pseudo near compactness has been studied via fuzzy nets, fuzzy filterbase and properties of *ps*-*ro* closed fuzzy sets.

We have further introduced two operators, called by us fuzzy *ps*-closure and fuzzy *ps*-interior. In section (5.3), we have formed a new type of fuzzy continuous-like function, named by us, pseudo fuzzy *ro*-continuous function. This we have studied with the aid of the preceding two operators. Apart from these operators, we have also studied them through other concepts like fuzzy points, *ps-ro* closed fuzzy sets and so on. Besides, we are able to show that this type of functions preserve pseudo near compactness of a fts.

5.2 *ps*-closure and *ps*-interior operators

Definition 5.2.1 The union of all *ps-ro* open fuzzy sets, each contained in a fuzzy set A on a fts X is called fuzzy *ps*-interior of A and is denoted by *ps-int*(A). So, *ps-int*$(A) = \vee\{B : B \leq A,\ B$ is *ps-ro* open fuzzy set on $X\}$

Definition 5.2.2 The intersection of all *ps-ro* closed fuzzy sets on a fts X, each containing a fuzzy set A on X is called fuzzy *ps*-closure of A and is denoted by *ps-cl*(A). So, *ps-cl*$(A) = \wedge\{B : A \leq B,\ B$ is *ps-ro* closed fuzzy set on $X\}$

Some properties of *ps-cl* and *ps-int* operators are furnished below. Since the proofs are straightforward, we only state the properties without proof.

Theorem 5.2.1 For any fuzzy set A on a fts (X, τ), the following hold:

(i) *ps-cl*(A) is the smallest *ps-ro* closed fuzzy set containing A.

(ii) $ps\text{-}cl(A) \leq ps\text{-}cl(B)$ if $A \leq B$.

(iii) $ps\text{-}cl(A) = A$ iff A is *ps-ro* closed.

(iv) $ps\text{-}cl(ps\text{-}cl(A)) = ps\text{-}cl(A)$.

(v) $ps\text{-}cl(A \vee B) = ps\text{-}cl(A) \vee ps\text{-}cl(B)$

Theorem 5.2.2 For any fuzzy set A on a *fts* (X, τ), the following hold:

(i) $ps\text{-}int(A)$ is the largest *ps-ro* open fuzzy set contained in A.

(ii) $ps\text{-}int(A) \leq ps\text{-}int(B)$ if $A \leq B$.

(iii) $ps\text{-}int(A) = A$ iff A is *ps-ro* open.

(iv) $ps\text{-}int(ps\text{-}int(A)) = ps\text{-}int(A)$.

(v) $ps\text{-}int(A \wedge B) = ps\text{-}int(A) \wedge ps\text{-}int(B)$

Definition 5.2.3 In a *fts* (X, τ), a fuzzy set A is said to be a

(i) *ps-ro nbd.* of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha \in B \leq A$. In addition, if A is *ps-ro* open fuzzy set, the *ps-ro nbd.* is called *ps-ro* open *nbd.*

(ii) *ps-ro* quasi neighborhood or simply *ps-ro q-nbd.* of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha q B \leq A$. In addition, if A is *ps-ro* open, the *ps-ro q-nbd.* is called *ps-ro* open *q-nbd.*.

Definition 5.2.4 A fuzzy point x_α where $0 < \alpha \leq 1$ is called fuzzy *ps*-cluster point of a fuzzy set A if for every *ps-ro* open *q-nbd.* U of

x_α, UqA.

Theorem 5.2.3 In a fts (X,τ), for any fuzzy set A, *ps-cl*(A) is the union of all fuzzy ps-cluster points of A.

Proof. Let $B =$ *ps-cl*(A). Let x_α be any fuzzy point such that $x_\alpha \leq B$ and if possible let there be *ps-ro* open *q-nbd.* U of x_α such that $U \not{q} A$. Then there exist a *ps-ro* open fuzzy set V on X such that $x_\alpha q V \leq U$. Consequently, $V \not{q} A$. So that $A \leq 1 - V$. As $1 - V$ is *ps-ro* closed fuzzy set, $B \leq 1 - V$. As $x_\alpha \not\leq 1 - V$, we have $x_\alpha > B$, a contradiction. Conversely, suppose $x_\alpha \not\leq B$. Then there exists a *ps-ro* closed fuzzy set F containing A such that $x_\alpha > F$. So, $x_\alpha q(1-F)$ and $A \not{q}(1-F)$. Further, $(1-F)$ is *ps-ro* open fuzzy set, so that x_α is not ps-cluster point of A.

Theorem 5.2.4 For a fuzzy set A in a fts X, *ps-int*$(1-A) = 1-$*ps-cl*(A).

Proof. *ps-int*$(1-A) = \vee\{B : B \leq (1-A),\ B$ is *ps-ro* open on $X\}$

So, $1 -$ *ps-int*$(1-A)$

$= 1 - \vee\{B : A \leq (1-B),\ (1-B)$ is *ps-ro* closed on $X\}$

$= \wedge\{(1-B) : A \leq (1-B),\ (1-B)$ is *ps-ro* closed on $X\}$

$=$ *ps-cl*(A). Hence, *ps-int*$(1-A) = 1 -$ *ps-cl*(A).

Remark 5.2.1 It may be observed that a fuzzy set A on a fts (X,τ) is *ps-ro* open iff A is *ps-ro nbd.* of each of the fuzzy points contained in A.

We conclude this section with some definitions that we require in the next section.

Definition 5.2.5 Let $\{S_n : n \in D\}$ be a fuzzy net on a fts X. i.e., for each member n of a directed set $(D, \leq)$, S_n be a fuzzy set on X. A fuzzy point x_α on X is said to be a fuzzy *ps*-cluster point of the fuzzy net if for every $n \in D$ and every *ps-ro* open *q-nbd.* V of x_α, there exists $m \in D$, with $n \leq m$ such that $S_m q V$.

Definition 5.2.6 Let x_α be a fuzzy point on a fts X. A fuzzy net $\{S_n : n \in (D, \geq)\}$ on X is said to *ps*-converge to x_α, written as $S_n \xrightarrow{ps} x_\alpha$ if for each *ps-ro* open *q-nbd.* W of x_α, there exists $m \in D$ such that $S_n q W$ for all $n \geq m$,$(n \in D)$.

Definition 5.2.7 A collection $\mathcal{B}$ of fuzzy sets on a fts (X, τ) is said to form a fuzzy filter base in X if for every finite subcollection $\{B_1, B_2, ..., B_n\}$ of $\mathcal{B}$, $\bigwedge_{i=1}^{n} B_i \neq 0$. If in addition, the members of $\mathcal{B}$ are *ps-ro* open (closed) fuzzy sets then $\mathcal{B}$ is called a *ps-ro* open fuzzy (respectively, *ps-ro* closed fuzzy) filter base in X. If every member of a fuzzy filterbase $\mathcal{B}$ on X is contained in some fuzzy set A in X, then $\mathcal{B}$ is called a fuzzy filterbase in A.

Definition 5.2.8 A fuzzy filterbase $\mathcal{B}$ on a fts (X, τ) is said to have a fuzzy ps-cluster point in a fuzzy set A if there exist a fuzzy point x_α in A such that $x_\alpha \leq \bigwedge\{ps\text{-}cl(U) : U \in \mathcal{B}\}$.

Definition 5.2.9 Let x_α be a fuzzy point on a fts X. A fuzzy filterbase $\mathcal{B}$ is said

(i) to ps-adhere at x_α written as $x_\alpha \leq ps\text{-}ad.\mathcal{B}$ if for each *ps-ro* open *q-nbd.* U of x_α and each $B \in \mathcal{B}$, BqU.

(ii) to ps-converge to x_α, written as $\mathcal{B} \xrightarrow{ps} x_\alpha$ if for each *ps-ro* open *q-nbd.* U of x_α, there coresponds some $B \in \mathcal{B}$ such that $B \leq U$.

5.3 Pseudo near compactness

Definition 5.3.1 Let A be a fuzzy set on a fts X. A collection $\mathcal{U}$ of fuzzy sets on X is called a cover of A if $sup\{U : U \in \mathcal{U}\} \geq A$. If in addition, the members of $\mathcal{U}$ are *ps-ro* open fuzzy sets on X, then $\mathcal{U}$ is called a *ps-ro* open cover of A. In particular, if $A = 1_X$, we get the definition of *ps-ro* open cover of the fts X. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in a fts is said to have a finite *ps-ro* open subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\vee\{U : U \in \mathcal{U}_0\} \geq A$.

Definition 5.3.2 A fuzzy set A on a fts (X, τ) is called fuzzy pseudo nearly compact set if every covering of A by *ps-ro* open fuzzy sets has a finite subcover. Clearly, for $A = X$, the fts (X, τ) becomes fuzzy pseudo nearly compact fts.

Remark 5.3.1 It is easy to observe, as pseudo regular open fuzzy sets form a base for *ps-ro* open fuzzy topology, replacing *ps-ro* open cover by pseudo regular open cover, we may obtain pseudo near compactness.

Theorem 5.3.1 A fts (X, τ) is fuzzy pseudo nearly compact iff every $\{B_\alpha : \alpha \in \Lambda\}$ of *ps-ro* closed fuzzy sets on X with $\wedge_{\alpha \in \Lambda} B_\alpha = 0$, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha \in \Lambda_0} B_\alpha = 0$.

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a *ps-ro* open cover of X. Now, $\wedge_{\alpha \in \Lambda}(1 - U_\alpha) = (1 - \vee_{\alpha \in \Lambda} U_\alpha) = 0$. As $\{1 - U_\alpha : \alpha \in \Lambda\}$ is a collection of *ps-ro* closed fuzzy sets on X, by given condition, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha \in \Lambda_0}(1 - U_\alpha) = 0 \Rightarrow 1 - \vee_{\alpha \in \Lambda_0} U_\alpha = 0$. i.e., $1 = \vee_{\alpha \in \Lambda_0} U_\alpha$. So, X is fuzzy pseudo nearly compact.

Conversely, Let $\{B_\alpha : \alpha \in \Lambda\}$ be a family of *ps-ro* closed fuzzy sets on X with $\wedge_{\alpha \in \Lambda} B_\alpha = 0$. Then $1 = 1 - \wedge_{\alpha \in \Lambda} B_\alpha \Rightarrow 1 = \vee_{\alpha \in \Lambda}(1 - B_\alpha)$. By given condition there exist a finite subset Λ_0 of Λ such that $1 = \vee_{\alpha \in \Lambda_0}(1 - B_\alpha) \Rightarrow 1 = (1 - \wedge_{\alpha \in \Lambda_0} B_\alpha)$. Hence, $\wedge_{\alpha \in \Lambda_0} B_\alpha \leq (\wedge_{\alpha \in \Lambda_0} B_\alpha) \wedge (1 - \wedge_{\alpha \in \Lambda_0} B_\alpha) = 0$. Consequently, $\wedge_{\alpha \in \Lambda_0} B_\alpha = 0$.

Pseudo near compactness of a fts can be characterized in terms of cluster points of fuzzy net, as seen in the following result.

Theorem 5.3.2 A fts X if fuzzy pseudo nearly compact iff every fuzzy net on X has a fuzzy ps-cluster point.

Proof. Let $\{U_\alpha : \alpha \in D\}$ be a fuzzy net in a fuzzy pseudo nearly compact fts X. For each $\alpha \in D$, let $F_\alpha = ps\text{-}cl[\bigvee\{U_\beta : \beta \in D, \alpha \leq \beta\}]$. Then $\mathcal{F} = \{F_\alpha : \alpha \in D\}$ is a family of ps-ro closed fuzzy sets with the property that for every finite subset D_0 of D, $\wedge\{F_\alpha : \alpha \in D_0\} \neq 0$. By Theorem (5.3.1), $\bigwedge\{F_\alpha : \alpha \in D\} \neq 0$. Let $x_\lambda \in \bigwedge\{F_\alpha : \alpha \in D\}$. Then for any ps-ro open q-$nbd.$ A of x_λ and any $\alpha \in D, Aq\bigvee\{U_\beta : \alpha \leq \beta\}$.Thus there exist a $\beta \in D$ with $\alpha \leq \beta$ such that AqU_β. This shows that x_λ is a fuzzy ps-cluster point of the fuzzy net $\{U_\alpha : \alpha \in D\}$.

Conversely, Let $\mathcal{F}$ be a collection of ps-ro closed fuzzy sets on X satisfying the hypothesis. Let $\mathcal{F}^*$ denote the family of all finite intersection of members of $\mathcal{F}$ directed by the relation " $\prec$ "(say) such that for $F_1, F_2 \in \mathcal{F}^*, F_2 \prec F_1$ iff $F_1 \leq F_2$. Let us consider the fuzzy net $\mathcal{U} = \{F : F \in (\mathcal{F}^*, \prec)\}$ of fuzzy sets on X. By hypothesis, there exist a fuzzy point x_λ which is a fuzzy ps-cluster point of $\mathcal{U}$. we shall show that $x_\lambda \in \bigwedge \mathcal{F}$. In fact, let $F \in \mathcal{F}$ be arbitrary and A be any ps-ro open q-$nbd.$ of x_λ. Since $F \in \mathcal{F}^*$ and x_λ is a fuzzy ps-cluster point of $\mathcal{U}$ there exist G (say) in $\mathcal{F}^*$ such that $G \prec F$ (i.e., $G \leq F$)

and GqA. Hence, FqA. Thus $x_\lambda \in ps\text{-}clF = F$. Hence, $\bigwedge \mathcal{F} \neq 0$. By Theorem (5.3.1) X is fuzzy pseudo nearly compact fts.

Theorem 5.3.3 For a fuzzy set A on a fts, the following are equivalent:

(a) Every fuzzy net in A has ps-cluster point in A.

(b) Every fuzzy net in A has a ps-convergent fuzzy subnet.

(c) Every fuzzy filterbase in A ps-adheres at some fuzzy point in A.

Proof. $(a) \Rightarrow (b)$: Let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in A having ps-cluster point at $x_\alpha \leq A$. Let $Q_{x_\alpha} = \{A : A$ is *ps-ro* open *q-nbd.* of $x_\alpha\}$. For any $B \in Q_{x_\alpha}$, there can be chosen some $n \in D$ such that $S_n qB$. Let E denote the set of all ordered pairs (n, B) with the property that $n \in D$, $B \in Q_{x_\alpha}$ and $S_n qB$. Then $(E, \succ)$ is a directed set where $(m, C) \succ (n, B)$ iff $m \geq n$ in D and $C \leq B$. Then $T : (E \succ) \rightarrow (X, \tau)$ given by $T(n, B) = S_n$, is a fuzzy subnet of $\{S_n : n \in (D, \geq)\}$. Let V be any *ps-ro* open *q-nbd.* of x_α. Then there exists $n \in D$ such that $(n, V) \in E$ and hence $S_n qV$. Now, for any $(m, U) \succ (n, V)$, $T(m, U) = S_m qU \leq V \Rightarrow T(m, U)qV$. Hence, $T \xrightarrow{ps} x_\alpha$.

$(b) \Rightarrow (a)$ If a fuzzy net $\{S_n : n \in (D, \geq)\}$ in A does not have any ps-cluster point, then there is a *ps-ro* open *q-nbd.* U of X_α and $n \in D$ such that $S_n \not{q} U, \forall\, m \geq n$. Then clearly no fuzzy subnet of the fuzzy net can ps-converge to x_α.

$(c) \Rightarrow (a)$ Let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in A. Consider the fuzzy filter base $\mathcal{F} = \{T_n : n \in D\}$ in A, generated by the fuzzy net, where $T_n = \{S_m : m \in (D, \geq)$ and $m \geq n\}$. By (c), there exist a fuzzy point $a_\alpha \leq A \wedge (ps\text{-}ad\mathcal{F})$. Then for each *ps-ro* open *q-nbd.* U of a_α and each $F \in \mathcal{F}$, UqF, i.e., $UqT_n, \forall\, n \in D$. Hence, the given fuzzy net has ps-cluster point at a_α.

$(a) \Rightarrow (c)$ Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a fuzzy filterbase in A. For each $\alpha \in \Lambda$, choose a fuzzy point $x_{F_\alpha} \leq F_\alpha$, and construct the fuzzy net $S = \{x_{F_\alpha} : F_\alpha \in \mathcal{F}\}$ in A with $(\mathcal{F}, \succ\!\succ)$ as domain, where for two members F_α, $F_\beta \in \mathcal{F}$, $F_\alpha \succ\!\succ F_\beta$ iff $F_\alpha \leq F_\beta$. By (a), the fuzzy net has a ps-cluster point say $x_t \leq A$, where $0 < t \leq 1$. Then for any *ps-ro* open *q-nbd.* U of x_t and any $F_\alpha \in \mathcal{F}$, there exists $F_\beta \in \mathcal{F}$ such that $F_\beta \succ\!\succ F_\alpha$ and $x_{F_\beta} qU$. Then $F_\beta qU$ and hence $F_\alpha qU$. Thus $\mathcal{F}$ adheres at x_t.

Using Theorem (5.3.3) along with what we have proved in Theorem (5.3.1) and (5.3.2) we obtain the following characterizations of a fuzzy pseudo nearly compact fts.

Theorem 5.3.4 In a fts (X, τ), the following are equivalent:

(a) X is fuzzy pseudo nearly compact.

(b) Every fuzzy net on X has ps-cluster point at some fuzzy point in

X.

(c) Every fuzzy net on X has a *ps*-convergent fuzzy subnet.

(d) Every fuzzy filterbase on X *ps*-adheres at some fuzzy point in X.

(e) For every $\{B_\alpha : \alpha \in \Lambda\}$ of *ps-ro* closed fuzzy sets on X with $\bigwedge_{\alpha\in\Lambda} B_\alpha = 0$, there exist a finite subset Λ_0 of Λ such that $\bigwedge_{\alpha\in\Lambda_0} B_\alpha = 0$.

Theorem 5.3.5 If a fts is fuzzy pseudo nearly compact, then every fuzzy filterbase on X with atmost one *ps*-adherent point is *ps*-convergent.

Proof. Let $\mathcal{F}$ be a fuzzy filterbase with atmost one *ps*-adherent point in a fuzzy pseudo nearly compact fts X. Then by Theorem (5.3.4), $\mathcal{F}$ has at least one *ps*-adherent point. Let x_α be the unique *ps*-adherent point of $\mathcal{F}$. If $\mathcal{F}$ do not *ps*-converge to x_α, then there is some *ps-ro* open *q-nbd.* U of x_α such that for each $F \in \mathcal{F}$ with $F \leq U$, $F \wedge (1-U) \neq 0$. Then $\mathcal{G} = \{F \wedge (1-U) : F \in \mathcal{F}\}$ is a fuzzy filterbase on X and hence has a *ps*-adherent point y_t(say) in X. Now, $U \not{q} G$, for all $G \in \mathcal{G}$, so that $x_\alpha \neq y_t$. Again, for each *ps-ro* open *q-nbd.* V of y_t and each $F \in \mathcal{F}$, $Vq(F \wedge (1-U)) \Rightarrow VqF \Rightarrow y_t$ is a *ps*-adherent point of $\mathcal{F}$, where $x_\alpha \neq y_t$. This shows that y_t is another *ps*-adherent point of $\mathcal{F}$, which is not the case.

In what follows, we observe how fuzzy pseudo near compactness of a fuzzy set on a fts be characterized.

Theorem 5.3.6 For a fuzzy set A on a fts (X, τ), the following are equivalent:

(a) A is fuzzy pseudo nearly compact.

(b) For every family $\mathcal{F}$ of *ps-ro* closed fuzzy sets on X with $\bigwedge\{F : F \in \mathcal{F}\} \wedge A = 0$, there exist a finite subcollection $\mathcal{F}_0$ of $\mathcal{B}$ such that $\bigwedge \mathcal{F}_0 \not{q} A$.

(c) If $\mathcal{B}$ is a *ps-ro* closed fuzzy filterbase on X such that each finite intersection of members of $\mathcal{B}$ is q-coincident with A, then $(\bigwedge \mathcal{B}) \wedge A \neq 0$.

Proof. $(a) \Rightarrow (b)$: Let A be a fuzzy pseudo nearly compact set on a fts (X, τ) and $\mathcal{F}$ be a family of *ps-ro* closed fuzzy sets on X such that $\bigwedge\{F : F \in \mathcal{F}\} \wedge A = 0$. Then for all $x \in supp(A)$, $inf\{F : F \in \mathcal{F}\} = 0$, so that the collection $\{(1 - F) : F \in \mathcal{F}\}$ is a cover of A by *ps-ro* open fuzzy sets. Hence, there is a finite subcollection $\mathcal{F}_0$ of $\mathcal{F}$ such that $A \leq \vee\{(1 - F) : F \in \mathcal{F}\}$. Then $\wedge\{F : F \in \mathcal{F}_0\} \leq 1 - A$ and hence $\bigwedge \mathcal{F}_0 \not{q} A$.

$(b) \Rightarrow (c)$: Straightforward and hence omitted.

$(c) \Rightarrow (a)$: If A is not fuzzy pseudo nearly compact in X, there exist a *ps-ro* open fuzzy cover $\mathcal{U}$ of A having no finite subcover of A. So, for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, there exist $x \in supp(A)$ such that $A(x) > sup\{U(x) : U \in \mathcal{U}\}$. i.e., $inf\{(1 - U(x)) : U \in \mathcal{U}\} >$

$1-A(x) \geq 0$. Thus $\{(1-U) : U \in \mathcal{U}\} = \mathcal{B}$(say) is a fuzzy *ps-ro* closed fuzzy filterbase on X having no finite subcollection $\mathcal{B}_0$ such that $\bigwedge\{B : B \in \mathcal{B}_0\} \not q A$. In fact otherwise $A \leq 1-\bigwedge\{(1-U) : U \in \mathcal{U}_0\} = \bigvee\{U : U \in \mathcal{U}_0\}$ for some finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, contradicting our hypothesis. Using (*c*), we then have $\bigwedge\{(1-U) : U \in \mathcal{U}\} \wedge A \neq 0$ and hence there is $x \in supp(A)$ such that $inf\{(1-U(x)) : U \in \mathcal{U}\} > 0$ i.e., $sup\{U(x) : U \in \mathcal{U}\} < 1$, which contradicts the fact that $\mathcal{U}$ is a cover of A.

Theorem 5.3.7 A fuzzy set A in a fuzzy pseudo nearly compact space X is fuzzy pseudo nearly compact if every fuzzy filterbase in A has a fuzzy *ps*-cluster point in A.

Proof. Let every fuzzy filterbase in a fuzzy set A in a fuzzy pseudo nearly compact space X has a fuzzy *ps*-cluster point in A. If A is not fuzzy pseudo nearly compact set on X, then there exists a *ps-ro* fuzzy open cover $\mathcal{U}$ of A such that for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, $A \geq \bigvee\{U : U \in \mathcal{U}_0\}$. Corresponding to each $U \in \mathcal{U}_0$, we define a fuzzy set B_U as follows :

$$B_U(x) = \begin{cases} min\{1 - U(x), A(x), |A(x) - U(x)|\}, & \text{for } x \in supp(A) \\ 0, & otherwise. \end{cases}$$

Again, for every finite subfamily $\{B_{U_1}, B_{U_2}, ..., B_{U_n}\}$ of $\mathcal{B} = \{B_U : U \in \mathcal{U}\}$, we have $sup\{U_i(x) : 1 \leq i \leq n\} < A(x) < 1$, for some $x \in$

$supp(A)$ so that $min\{A(x)-U_1(x), A(x)-U_2(x), ..., A(x)-U_n(x)\} > 0$. Hence, $\bigwedge_{i=1}^{n} B_{U_i} \neq 0$ and consequently $\mathcal{B} = \{B_U : U \in \mathcal{U}\}$ is a fuzzy filterbase in A. Now, for each fuzzy point x_α in A, there exists $U \in \mathcal{U}$ such that $x_\alpha q U$. Since $B_U \not{q} U$, $\mathcal{B}$ has no fuzzy *ps*-cluster point in A, which is a contradiction.

The converse of this theorem may not hold in general. Imposing some conditions on the filterbase, we get the converse as follows.

Theorem 5.3.8 Let A be a fuzzy pseudo nearly compact set on X and $\mathcal{B}$ a family of *ps-ro* open fuzzy sets contained in A such that every finite intersection of members of $\mathcal{B}$ is q-coincident with at least one member of $\mathcal{B}$. Then $\mathcal{B}$ has a fuzzy *ps*-cluster point in A.

Proof. If $\mathcal{B}$ has no fuzzy *ps*-cluster point in A, then proceeding as in the proof of Theorem(5.3.7) we construct a *ps-ro* open fuzzy cover $\mathcal{U}$ of A such that each V_x^n of $\mathcal{U}$ corresponds to a $B_x^n \in \mathcal{B}$ with $V_x^n \not{q} B_x^n$. Thus there exist finite subfamily $\{V_{x_1}^{n_1}, V_{x_2}^{n_2}, ..., V_{x_k}^{n_k}\}$ of $\mathcal{U}$ such that $A \leq \bigvee_{i=1}^{k} V_{x_i}^{n_i}$. Then $(\bigwedge_{i=1}^{k} V_{x_i}^{n_i}) \not{q} A$ with $B \leq A$, for all $B \in \mathcal{B}$. This contradicts the definition of $\mathcal{B}$.

One more characterization of fuzzy pseudo nearly compact set follows next.

Theorem 5.3.9 A fuzzy set on a fts (X, τ) is fuzzy pseudo nearly compact iff whenever $\mathcal{F}$ is a fuzzy filterbase with the property that for any finite subcollection $\mathcal{F}_0 = \{F_1, F_2, ..., F_n\}$ of $\mathcal{F}$ and for any *ps-ro* open fuzzy set U with $A \leq U$, $(\bigwedge \mathcal{F}_0)qU$ holds, then $\mathcal{F}$ has a fuzzy *ps*-cluster point in A.

Proof. Let A be a fuzzy pseudo nearly compact set on a fts (X, τ) and $\mathcal{F}$ a fuzzy filterbase on X having no fuzzy $ps-$cluster point in A. For each $x \in supp(A)$, there exists a positive integer m_x such that $\frac{1}{m_x} < A(x)$. For any positive integer $n \geq m_x$, as $x_{\frac{1}{n}} \leq A(x)$, $x_{\frac{1}{n}}$ is not a fuzzy *ps*-cluster point of $\mathcal{F}$. Hence, there is a *ps-ro* open *q-nbd.* V_x^n of $x_{\frac{1}{n}}$ and $B_x^n \in \mathcal{F}$ such that $V_x^n \not{q} B_x^n$. As $V_x^n(x) + \frac{1}{n} > 1$, we have $sup\{V_x^n(x) : n \geq m_x\} = 1$. Hence, the collection $\mathcal{U} = \{V_x^n : x \in supp(A), n \geq m_x > \frac{1}{A(x)}\}$, forms a *ps-ro* open fuzzy cover of A such that for each $V_x^n \in \mathcal{U}$, there exist $B_x^n \in \mathcal{F}$ with $V_x^n \not{q} B_x^n$. Since A is pseudo nearly compact fuzzy set on X, there exist a finite subcollection $V_{x_1}^{n_1}, V_{x_2}^{n_2}, ..., V_{x_k}^{n_k}$ of $\mathcal{U}$, such that $A \leq \bigvee_{i=1}^k V_{x_i}^{n_i} = V(\text{say})$. Then V is a *ps-ro* open fuzzy set such that $A \leq V$ and $V \not{q} (\bigwedge_{i=1}^k B_{x_i}^{n_i})$.

Conversely, let $\mathcal{F}$ be *ps-ro* closed fuzzy filterbase on X such that $\bigwedge\{F : F \in \mathcal{F}\} \wedge A = 0$. As for *ps-ro* closed fuzzy set $F \in \mathcal{F}$ we have $F = ps\text{-}cl(F)$, it follows that $\mathcal{F}$ has no fuzzy *ps*-cluster point in A. By hypothesis, there is a fuzzy *ps-ro* open set U with $A \leq U$ and there exists $\{F_1, F_2, ..., F_n\}$ of $\mathcal{F}$ such that $(\bigwedge_{i=1}^n F_i) \not{q} U$ and hence

$(\bigwedge_{i=1}^{n} F_i)\ \not{q} A$. Hence, by Theorem (5.3.6), A is fuzzy pseudo nearly compact set on X.

Theorem 5.3.10 In a fuzzy pseudo nearly compact space X every *ps-ro* closed fuzzy set is fuzzy pseudo nearly compact.

Proof. For any filterbase $\mathcal{B}$ in A, there exists a fuzzy point x_α on X, such that x_α is a fuzzy *ps*-cluster point of $\mathcal{B}$. Then for any $F \in \mathcal{B}$ we have $x_\alpha \leq ps\text{-}cl(F) \leq ps\text{-}cl(A) = A$. Hence, $\mathcal{B}$ has a fuzzy *ps*-cluster point in A and consequently, A is fuzzy pseudo nearly compact set on X.

We conclude this section with a few results on fuzzy pseudo near compactness which are analogous to its topological counter part, near compactness.

Theorem 5.3.11 In a fuzzy pseudo nearly compact space X, the complement of every pseudo regular open fuzzy set is fuzzy pseudo nearly compact.

Proof. Let A be a pseudo regular open fuzzy set on a fuzzy pseudo nearly compact space X. Hence, A is *ps-ro* open fuzzy set and $1 - A$ is *ps-ro* closed fuzzy set. By Theorem (5.3.10), $1 - A$ is fuzzy pseudo nearly compact in X.

Theorem 5.3.12 In a *fts* X, the finite union of fuzzy pseudo nearly compact sets is also so.

Proof. Straightforward and hence omitted.

5.4 Pseudo fuzzy ro-continuous functions

Definition 5.4.1 A function f from a fts X to a fts Y is pseudo fuzzy ro-continuous (ps-ro continuous, for short) if $f^{-1}(U)$ is ps-ro open fuzzy set on X, for each pseudo regular open fuzzy set U on Y.

Definition 5.4.2 [87] A function f from a topological space X to another topological space Y is called strong irresolute if $f^{-1}(U)$ is regular open in X for each regular open set U in Y.

The following Example shows that pseudo fuzzy δ-continuity does not imply ps-ro continuity.

Example 5.4.1 Let $X = \{x, y, z\}$ and the topology generated by $\{0, 1, \mu, \nu, \eta\}$, where $\mu(x) = 0.4$, $\mu(y) = 0.4$, $\mu(z) = 0.5$, $\nu(x) = 0.4$, $\nu(y) = 0.6$, $\nu(z) = 0.4$ and $\eta(x) = 0.5$, $\eta(y) = 0.5$, $\eta(z) = 0.6$. It is easy to see that (X, τ) is a fts. Let $f : (X, \tau) \to (X, \tau)$ be a function defined as $f(x) = x$, $f(y) = f(z) = y$. It can be checked that f is indeed a δ-continuous function from $(X, i_\alpha(\tau))$ to itself for each $\alpha \in I_1$. Hence, f is pseudo fuzzy δ-continuous. As worked out in Example (3.2.4), ν is a nontrivial pseudo regular open fuzzy set.

Also, the fuzzy set K given by $K(x) = 0.4, K(y) = 0.6, K(z) = 0.6$ is pseudo δ-open, without being *ps-ro* open. But, as $f^{-1}(\nu) = K$, f can not be *ps-ro* continuous. It is easy to see that f is pseudo fuzzy δ-continuous.

Theorem 5.4.1 A function $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is strong irresolute for each $\alpha \in I_1$, where $(X,\tau), (Y,\sigma)$ are fts, then $f : (X,\tau) \to (Y,\sigma)$ is *ps-ro* continuous.

Proof. Let μ be any pseudo regular open fuzzy set on (Y,σ). μ^α is regular open in $(Y, i_\alpha(\sigma))$. By the *strong irresolute*-ness of $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma)), f^{-1}(\mu^\alpha) = (f^{-1}(\mu))^\alpha$ is regular open in $(X, i_\alpha(\tau))$. Hence, $f^{-1}(\mu)$ is pseudo regular open and hence, *ps-ro* open fuzzy set on (X,τ), proving f to be *ps-ro* continuous.

We cite here, some characterizations of *ps-ro* continuity in terms of complements of pseudo regular open fuzzy sets, *ps-cl* and *ps-int* operators and of fuzzy points.

Theorem 5.4.2 A function $f : (X,\tau) \to (Y,\sigma)$ is *ps-ro* continuous iff $f^{-1}(\mu)$ is *ps-ro* closed fuzzy set on a fts (X,τ), where $1-\mu$ is pseudo regular open fuzzy set on a fts (Y,σ).

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be *ps-ro* continuous and μ be such that $1-\mu$ is pseudo regular open fuzzy set on (Y,σ). As $f : (X,\tau) \to (Y,\sigma)$ is pseudo fuzzy *ro*-continuous, $f^{-1}(1-\mu)$ is *ps-ro* open fuzzy set on X. Now,

$(1 - f^{-1}(1 - \mu))(x)$

$= 1 - f^{-1}(1 - \mu)(x)$

$= 1 - (1 - \mu)(f(x))$

$= \mu f(x)$

$= f^{-1}(\mu)(x)$. Hence, $f^{-1}(\mu)$ is *ps-ro* closed fuzzy set on (X, τ).

Conversely, Let μ be any pseudo regular open fuzzy set and so $f^{-1}(1-\mu)$ is *ps-ro* closed. Then $1 - f^{-1}(1 - \mu)$ is *ps-ro* open fuzzy set on $(Y, i_\alpha(\sigma)), \forall\ \alpha \in I_1$. Hence, f is *ps-ro* continuous.

Theorem 5.4.3 Let (X, τ) and (Y, σ) be two fts. For a function $f : X \to Y$, the following are equivalent:

(a)f is *ps-ro* continuous.

(b) Inverse image of each *ps-ro* open fuzzy sets on Y under f is *ps-ro* open on X.

(c) For each fuzzy point x_α on X and each *ps-ro* open *nbd.* V of $f(x_\alpha)$, there exists a *ps-ro* open fuzzy set U on X, such that $x_\alpha \leq U$ and $f(U) \leq V$.

(d)For each *ps-ro* closed fuzzy set F on Y, $f^{-1}(F)$ is *ps-ro* closed on X.

(e) For each fuzzy point x_α on X, the inverse image under f of every *ps-ro nbd.* of $f(x_\alpha)$ on Y is a *ps-ro nbd.* of x_α on X.

(f)For all fuzzy set A on X, $f(ps\text{-}cl(A)) \leq ps\text{-}cl(f(A))$.

(g)For all fuzzy set B on Y, $ps\text{-}cl(f^{-1}(B)) \leq f^{-1}(ps\text{-}cl(B))$.

(h)For all fuzzy set B on Y, $f^{-1}(ps\text{-}int(B)) \leq ps\text{-}int(f^{-1}(B))$.

Proof. $(a) \Rightarrow (b)$ Let f be *ps-ro* continuous, and μ be any *ps-ro* open fuzzy set on Y. Then $\mu = \vee\mu_i$, where μ_i is pseudo regular open fuzzy set on Y, for each i. Now, $f^{-1}(\mu) = f^{-1}(\vee_i\mu_i) = \vee_i f^{-1}(\mu_i)$. Since f is *ps-ro* continuous, $f^{-1}(\mu_i)$ is *ps-ro* open fuzzy set and consequently, $f^{-1}(\mu)$ is *ps-ro* open on X.

$(b) \Rightarrow (c)$ Let V be any *ps-ro* open *nbd.* of $f(x_\alpha)$ on Y. Then there is a *ps-ro* open fuzzy set V_1 on Y such that $f(x_\alpha) \leq V_1 \leq V$. By (b) $f^{-1}(V_1)$ is *ps-ro* open fuzzy set on X. Again, $x_\alpha \leq f^{-1}(V_1) \leq f^{-1}(V)$. So, $f^{-1}(V)$ is a *ps-ro nbd.* of x_α, such that $f(f^{-1}(V)) \leq V$, as desired.

$(c) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y and $x_\alpha \leq f^{-1}(V)$. Then $f(x_\alpha) \leq V$ and so by (c), there exists *ps-ro* open fuzzy set U on X such that $x_\alpha \leq U$ and $f(U) \leq V$. Hence, $x_\alpha \leq U \leq f^{-1}(V)$. i.e., $f^{-1}(V)$ is a *ps-ro nbd.* of each of the fuzzy points contained in it. Thus $f^{-1}(V)$ is *ps-ro* open fuzzy set on X.

$(b) \Leftrightarrow (d)$ Obvious.

$(b) \Rightarrow (e)$ Suppose, W is a *ps-ro* open *nbd.* of $f(x_\alpha)$. Then there exists a *ps-ro* open fuzzy set U on Y such that $f(x_\alpha) \leq U \leq W$. Then $x_\alpha \leq f^{-1}(U) \leq f^{-1}(W)$. By (b), $f^{-1}(U)$ is *ps-ro* open fuzzy set on X and hence the result is obtained.

$(e) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y. If $x_\alpha \leq f^{-1}(V)$ then $f(x_\alpha) \leq V$ and so $f^{-1}(V)$ is a *ps-ro nbd.* of x_α.

$(d) \Rightarrow (f)$ *ps-cl*$(f(A))$ being a *ps-ro* closed fuzzy set on Y, $f^{-1}($*ps-cl*$(f(A)))$ is *ps-ro* closed fuzzy set on X. Again,

$f(A) \leq$ *ps-cl*$(f(A))$

$\Rightarrow A \leq f^{-1}($*ps-cl*$(f(A)))$.

As *ps-cl*(A) is the smallest *ps-ro* closed fuzzy set on X containing A, *ps-cl*$(A) \leq f^{-1}($*ps-cl*$(f(A)))$. Hence,

$f($*ps-cl*$(A)) \leq ff^{-1}($*ps-cl*$(f(A))) \leq$ *ps-cl*$(f(A))$.

$(f) \Rightarrow (d)$ For any *ps-ro* closed fuzzy set B on Y,

$f($*ps-cl*$(f^{-1}(B)))$

$\leq$ *ps-cl*$(f(f^{-1}(B)))$

$\leq$ *ps-cl*$(B) = B$.

Hence, *ps-cl*$(f^{-1}(B)) \leq f^{-1}(B) \leq$ *ps-cl*$(f^{-1}(B))$.

Thus, $f^{-1}(B)$ is *ps-ro* closed fuzzy set on X.

$(f) \Rightarrow (g)$ For any fuzzy set B on Y,

$f($*ps-cl*$(f^{-1}(B)))$

$\leq$ *ps-cl*$(f(f^{-1}(B)))$

$\leq$ *ps-cl*(B).

Hence, *ps-cl*$(f^{-1}(B)) \leq f^{-1}($*ps-cl*$(B))$.

$(g) \Rightarrow (f)$ Let $B = f(A)$ for some fuzzy set A on X. Then

$ps\text{-}cl(f^{-1}(B)) \leq f^{-1}(ps\text{-}cl(B))$

$\Rightarrow ps\text{-}cl(A) \leq ps\text{-}cl(f^{-1}(B)) \leq f^{-1}(ps\text{-}cl(f(A)))$.

So, $f(ps\text{-}cl(A)) \leq ps\text{-}cl(f(A))$.

$(b) \Rightarrow (h)$ For any fuzzy set B on Y, $f^{-1}(ps\text{-}int(B))$ is *ps-ro* open fuzzy set on X. Also, $f^{-1}(ps\text{-}int(B)) \leq f^{-1}(B)$.

So, $f^{-1}(ps\text{-}int(B)) \leq ps\text{-}int(f^{-1}(B))$.

$(h) \Rightarrow (b)$ Let B be any *ps-ro* open fuzzy set on Y. So, $ps\text{-}int(B) = B$. Now, $f^{-1}(ps\text{-}int(B)) \leq ps\text{-}int(f^{-1}(B))$

$\Rightarrow f^{-1}(B) \leq ps\text{-}int(f^{-1}(B)) \leq f^{-1}(B)$.

Hence, $f^{-1}(B)$ is *ps-ro* open fuzzy set on X.

We observe next that in terms of *ps-ro* open *q-nbds*. of fuzzy points also *ps-ro* continuity can be characterized.

Theorem 5.4.4 Let (X, τ) and (Y, σ) be two *fts*. A function $f : X \to Y$ is f is *ps-ro* continuous iff for every fuzzy point x_α on X and every *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$ there exists a *ps-ro* open fuzzy set U on X with $x_\alpha qU$ and $f(U) \leq V$.

Proof. Let f be *ps-ro* continuous and x_α a fuzzy point on X, V a *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$. So,

$V(f(x)) + \alpha > 1$

$\Rightarrow f^{-1}(V)(x) + \alpha > 1$

$\Rightarrow (f^{-1}(V))(x) + \alpha > 1$

$\Rightarrow x_\alpha q(f^{-1}(V))$.

Now, $ff^{-1}(V) \leq V$ is always true. Choosing $U = f^{-1}(V)$ we have, $f(U) \leq V$ with $x_\alpha qU$.

Conversely, let the condition hold. Let V be any *ps-ro* open fuzzy set on Y. To prove $f^{-1}(V)$ is *ps-ro* open fuzzy set on X, we shall prove $1 - f^{-1}(V)$ is *ps-ro* closed fuzzy set on X. Let x_α be any fuzzy point on X such that $x_\alpha > 1_X - f^{-1}(V)$. So,

$(1 - f^{-1}(V))(x) < \alpha$

$\Rightarrow f^{-1}(V)(x) + \alpha > 1$

$\Rightarrow V(f(x)) + \alpha > 1$

$\Rightarrow f(x_\alpha)qV$.

By given condition, there exists a *ps-ro* open fuzzy set on U such that $x_\alpha qU$ and $f(U) \leq V$. Now,

$U(t) + (1 - f^{-1}(V))(t)$

$= U(t) + 1 - V(f(t))$

$\leq V(f(t)) + 1 - V(f(t)) = 1, \forall\, t.$

Hence, $U \not{q} (1 - f^{-1}(V))$. Consequently, x_α is not a *ps*-cluster point of $1 - f^{-1}(V)$. This proves $1 - f^{-1}(V)$ is a *ps-ro* closed fuzzy set on X

Lemma 5.4.1 [5] Let Z, X, Y be fts and $f_1 : Z \to X$ and f_2 :

$Z \to Y$ be two functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1 U_2 are fuzzy sets on Z, X, Y respectively such that $f(B) \leq U_1 \times U_2$, then $f_1(B) \leq U_1$ and $f_2(B) \leq U_2$.

Theorem 5.4.5 Let X, Y, Z be fts. For any functions $f_1 : Z \to X$ and $f_2 : Z \to Y$, a function $f : Z \to X \times Y$ is defined as $f(x) = (f_1(x), f_2(x))$ for $x \in Z$, where $X \times Y$ is endowded with the product fuzzy topology. If f is *ps-ro* continuous then f_1 and f_2 are both *ps-ro* continuous.

Proof. Let U_1 be a *ps-ro q-nbd.* of $f_1(x_\alpha)$ on X, for any fuzzy point x_α on Z. Then $U_1 \times 1_Y$ is a *ps-ro q-nbd.* of $f(x_\alpha) = (f_1(x_\alpha), f_2(x_\alpha))$ on $X \times Y$. By *ps-ro* continuity of f, there exists *ps-ro q-nbd.* V of x_α on Z such that $f(V) \leq U_1 \times 1_Y$. Then $f(V)(t) \leq (U_1 \times 1_Y)(t) = U_1(t) \wedge 1_Y(t) = U_1(t), \forall\, t \in Z$. So, $f_1(V) \leq U_1$. Hence, f_1 is *ps-ro* continuous. Similarly, it can be shown that f_2 is also *ps-ro* continuous.

Lemma 5.4.2 [3] Let X, Y be fts and $g : X \to X \times Y$ be the graph of the function $f : X \to Y$. Then if A, B are fuzzy sets on X and Y respectively, $g^{-1}(A \times B) = A \wedge f^{-1}(B)$.

Theorem 5.4.6 Let $f : X \to Y$ be a function from a fts X to another fts Y and $g : X \to X \times Y$ be the graph of the function f. Then f is *ps-ro* continuous if g is so.

Proof. Let g be *ps-ro* continuous and B be *ps-ro* open fuzzy set on Y. By Lemma (5.4.2), $f^{-1}(B) = 1_X \wedge f^{-1}(B) = g^{-1}(1_X \times B)$. Now, as $1_X \times B$ is *ps-ro* open fuzzy set on $X \times Y$,$f^{-1}(B)$ becomes *ps-ro* open fuzzy set on X. Hence, f is *ps-ro* continuous.

Finally, we show that pseudo near compactness is preserved by *ps-ro* continuous function.

Theorem 5.4.7 In a fts X, the *ps-ro* continuous image of fuzzy pseudo nearly compact set is also so.

Proof. Suppose, A is fuzzy pseudo nearly compact set on X and $f : X \to Y$ is a *ps-ro* continuous function from a fts X into another fts Y. Let $B = f(A)$ and $\{U_\alpha : \alpha \in \Lambda\}$ be a *ps-ro* open cover of B in Y. By *ps-ro* continuity of f, each $f^{-1}(U_\alpha)$ is *ps-ro* open fuzzy set on X with $\bigvee f^{-1}(U_\alpha) = f^{-1}(\bigvee U_\alpha) \geq f^{-1}(B) \geq A$. As A is fuzzy pseudo nearly compact, there is a finite subset Λ_0 of Λ such that $\bigvee(f^{-1}(U_\alpha) : \alpha \in \Lambda_0) \geq A$. Consequently, $B = f(A) \leq f(\bigvee f^{-1}(U_\alpha) : \alpha \in \Lambda_0) \leq \bigvee(ff^{-1}(U_\alpha) : \alpha \in \Lambda_0) \leq \bigvee(U_\alpha : \alpha \in \Lambda_0)$, as desired.