Chapter 6

GENERALISED GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS

Chapter 6

GENERALISED GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS

6.1 Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The following definitions are well-known. In the sequel we use the following notations: (i) $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$ (ii) $\exp^{[t]} x = \exp(\exp^{[t-1]} x)$ for $t = 1, 2, 3, ...$ and $\exp^{[0]} x = x$.

Definition 6.1.1. *The order ρ_f and lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \textit{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

Definition 6.1.2. *[31]Let the order ρ_f of an entire function f be zero. Then the quantities ρ_f^* and λ_f^* are defined as*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r} \quad \textit{and} \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r}.$$

In the line of Definition 6.1.1 and Definition 6.1.2 we may give the following definitions respectively.

Some portion of the results of this chapter have been published in **International Journal of Pure and Applied Mathematics,** see [15] and the remaining portion have been accepted for publication and to appear in **International Journal of Contemporary Mathematical Sciences,** see [18] .

Definition 6.1.3. *The t-th generalised order* ${}^{(t)}\rho_f$ *and t-th generalised lower order* ${}^{(t)}\lambda_f$ *of an entire function* f *are defined in the following way*

$${}^{(t)}\rho_f = \limsup_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log r} \quad \text{and} \quad {}^{(t)}\lambda_f = \liminf_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log r}$$

where $t = 2, 3, ...$

Definition 6.1.4. *Let the t-th generalised order* ${}^{(t)}\rho_f$ *of an entire function* f *be zero. Then the quantities* ${}^{(t)}\rho_f^*$ *and* ${}^{(t)}\lambda_f^*$ *are defined in the following way*

$${}^{(t)}\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log^{[2]} r} \quad \text{and} \quad {}^{(t)}\lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log^{[2]} r}$$

where $t = 2, 3, ...$

Let $L = L(r)$ be a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Singh and Barker [38] defined it in the following way :

Definition 6.1.5. *[38] A positive continuous function* $L(r)$ *is called a 'slowly changing function' if for* $\varepsilon > 0$,

$$\frac{1}{k^\varepsilon} \le \frac{L(kr)}{L(r)} \le k^\varepsilon$$

for $r \ge r(\varepsilon)$ *and uniformly for* $k(\ge 1)$.

If further, $L(r)$ *is differentiable, the above condition is equivalent to*

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0.$$

Somasundaram and Thamizharasi [44] introduced the notions of L-order and L-type for entire functions defined in the open complex plane $\mathbb{C}$. The more generalised concept for L-order and L-type of entire functions are L^*-order and L^*-type respectively. Their definitions are as follows:

Definition 6.1.6. *[44] The* L^**-order* $\rho_f^{L^*}$ *and* L^**-lower order* $\lambda_f^{L^*}$ *of an entire function* f *are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}.$$

Definition 6.1.7. *[44] The L^*-type $\sigma_f^{L^*}$ of an entire function f is defined as follows*

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}},\ 0 < \rho_f^{L^*} < \infty.$$

Definition 6.1.8. *Let the L^*-order $\rho_f^{L^*}$ of an entire function f be zero. Then the quantities $\left(\rho_f^{L^*}\right)^*$ and $\left(\lambda_f^{L^*}\right)^*$ are defined in the following way*

$$\left(\rho_f^{L^*}\right)^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]}[re^{L(r)}]} \quad \text{and} \quad \left(\lambda_f^{L^*}\right)^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]}[re^{L(r)}]}$$

In the line of Definition 6.1.6, Definition 6.1.7 and Definition 6.1.8 we may introduce the following three definitions.

Definition 6.1.9. *The t-th generalised L^*-order ${}^{(t)}\rho_f^{L^*}$ and t-th L^*-lower order ${}^{(t)}\lambda_f^{L^*}$ of an entire function f are defined in the following way*

$${}^{(t)}\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log[re^{L(r)}]} \quad \text{and} \quad {}^{(t)}\lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log[re^{L(r)}]}$$

where $t = 2, 3, ...$

Definition 6.1.10. *The t-th generalised L^*-type ${}^{(t)}\sigma_f^{L^*}$ of an entire function f is defined as*

$${}^{(t)}\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[t-1]} M(r,f)}{[re^{L(r)}]^{{}^{(t)}\rho_f^{L^*}}},\ \text{for } t = 2, 3, ... \text{ and } 0 < {}^{(t)}\rho_f^{L^*} < \infty.$$

Definition 6.1.11. *Let the t-th generalised L^*-order ${}^{(t)}\rho_f^{L^*}$ of an entire function f be zero. Then the quantities $\left({}^{(t)}\rho_f^{L^*}\right)^*$ and $\left({}^{(t)}\lambda_f^{L^*}\right)^*$ are defined in the following way*

$$\left({}^{(t)}\rho_f^{L^*}\right)^* = \limsup_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log^{[2]}[re^{L(r)}]} \quad \text{and} \quad \left({}^{(t)}\lambda_f^{L^*}\right)^* = \liminf_{r\to\infty} \frac{\log^{[t]} M(r,f)}{\log^{[2]}[re^{L(r)}]}$$

where $t = 2, 3, ...$

In the chapter we intend to establish some results on the comparative growth properties of composite entire functions using generalised L^*-order (generalised L^*-lower order) and generalised L^*-type.

Let f and g be two entire functions and $F(r) \equiv M(r,f) = \max\{|f(z)| : |z| = r\}$, $G(r) \equiv M(r,g) = \max\{|g(z)| : |z| = r\}$. If f is non constant then $F(r)$ is strictly increasing and continuous and its inverse $F^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that

$$\lim_{s\to\infty} F^{-1}(s) = \infty.$$

Bernal [2] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$, as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : F(r) < G(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.\end{aligned}$$

Similarly one may define the relative lower order of f with respect to g, denoted by $\lambda_g(f)$ in the following manner

$$\begin{aligned}\lambda_g(f) &= \sup\{\mu' > 0 : F(r) > G(r^{\mu'}) \text{ for all } r > r_0(\mu') > 0\} \\ &= \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.\end{aligned}$$

The above two definitions coincide with the classical definitions of order and lower order if $g(z) = \exp z$ [46].

In the chapter we also introduce the definition of generelised relative L^*-order and generelised relative L^*-lower order of f with respect to g where f and g are both entire functions and study some of their properties.

Definition 6.1.12. *If $t \geq 1$ is a positive integer, then the t-th generalised relative L^*-order and t-th generalised relative L^*-lower order of an entire function f with respect to an entire function g, denoted respectively by ${}^{(t)}\rho_g^{L^*}(f)$ and ${}^{(t)}\lambda_g^{L^*}(f)$ are defined by*

$$\begin{aligned}&{}^{(t)}\rho_g^{L^*}(f) \\ &= \inf\{\mu_0 > 0 : F(r) < G(\exp^{[t-1]}[re^{L(r)}]^{\mu_0}) \textit{ for all } r > r_0(\mu_0) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(r)}{\log[re^{L(r)}]}\end{aligned}$$

and

$$
\begin{aligned}
&{}^{(t)}\lambda_g^{L^*}(f) \\
= \;& \sup\{\mu_0' > 0 : F(r) > G(\exp^{[t-1]}[re^{L(r)}]^{\mu_0'}) \textit{ for all } r > r_0(\mu_0') > 0\} \\
= \;& \liminf_{r\to\infty} \frac{\log^{[t]} G^{-1}F(r)}{\log[re^{L(r)}]} \textit{ for } t = 1, 2, 3, ...
\end{aligned}
$$

Definition 6.1.13. *Two entire functions f and g are said to be asymtotically equivalent if there exists l, $0 < l < \infty$ such that*

$$\frac{F(r)}{G(r)} \to l \quad \textit{as } r \to \infty,$$

and in this case we write $f \sim g$.

If $f \sim g$, then clearly $g \sim f$.

Throughout the chapter we shall assume f, g, h etc. to be non-constant and if they are entire then $F(r)$, $G(r)$, $H(r)$ etc. denote respectively their maximum modulus on $|z| = r$.

6.2 Lemmas.

In this section we present some lemmas which will be needed in sequel.

Lemma 6.2.1. *[7] If f and g are two entire functions, then for all sufficiently large values of r,*

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \le M(r, fog) \le M(M(r, g), f).$$

Lemma 6.2.2. *[42] Let f be entire and g be a transcendental entire function of finite lower order. Then for any $\delta > 0$*

$$M\left(r^{1+\delta}, fog\right) \ge M(M(r, g), f) \quad (r \ge r_0).$$

Lemma 6.2.3. *[20] Let f be meromorphic and g be transcendental entire. If $\rho_{fog} < \infty$ then $\rho_f = 0$.*

Lemma 6.2.4. *Let f be entire and g be transcendental entire. If $\rho_{fog} < \infty$ then*

$${}^{(t)}\rho_f = 0 \quad \textit{and} \quad {}^{(t)}\rho_f^{L^*} = 0, \quad \textit{for } t = 2, 3, ...$$

Proof. Since $\rho_{fog} < \infty$, in view of Lemma 6.2.3 we have $\rho_f = 0$.
Again ${}^{(t)}\rho_f \le \rho_f = 0$ implies that ${}^{(t)}\rho_f = 0$ where $t = 2, 3, ...$
Also ${}^{(t)}\rho_f^{L^*} \le \rho_f^{L^*} \le \rho_f = 0$. Hence ${}^{(t)}\rho_f^{L^*} = 0$ where $t = 2, 3, ...$
This proves the lemma. ■

Lemma 6.2.5. *Let f be entire and g be transcendental entire such that ${}^{(t)}\rho_f = 0$ and $\rho_g < \infty$. Then*

$$ {}^{(t)}\lambda_f^*.\rho_g^{L^*} \le {}^{(t)}\rho_{fog}^{L^*} \le {}^{(t)}\rho_f^*.\rho_g^{L^*}, \quad \textit{for } t = 2, 3, ... $$

Proof. In view of Lemma 6.2.1 we get

$$ \begin{aligned} {}^{(t)}\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log[re^{L(r)}]} \\ &\le \limsup_{r\to\infty} \frac{\log^{[t]} M\left(M\left(r,g\right),f\right)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[re^{L(r)}]} \\ &= {}^{(t)}\rho_f^*.\rho_g^{L^*}. \end{aligned} $$

Also from Lemma 6.2.2 and since δ (> 0) is arbitrary, it follows that

$$ \begin{aligned} {}^{(t)}\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[t]} M(r^{1+\delta}, fog)}{\log[re^{L(r)}]^{1+\delta}} \\ &\ge \liminf_{r\to\infty} \frac{\log^{[t]} M\left(M\left(r,g\right),f\right)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[re^{L(r)}]} \\ &= {}^{(t)}\lambda_f^*.\rho_g^{L^*}. \end{aligned} $$

Combining the above two inequalities we obtain

$$ {}^{(t)}\lambda_f^*.\rho_g^{L^*} \le {}^{(t)}\rho_{fog}^{L^*} \le {}^{(t)}\rho_f^*.\rho_g^{L^*}, \quad \text{for } t = 2, 3, ... $$

This proves the lemma. ■

Lemma 6.2.6. *Let f be entire and g be transcendental entire with $\rho_g = 0$. Also let $\rho_{fog} = 0$. Then*

$$ {}^{(t)}\rho_f^*.\left(\lambda_g^{L^*}\right)^* \le \left({}^{(t)}\rho_{fog}^{L^*}\right)^* \le {}^{(t)}\rho_f^*.\left(\rho_g^{L^*}\right)^*, \quad \textit{for } t = 2, 3, ... $$

Proof. Since $\rho_{fog} = 0$ $(< \infty)$, in view of Lemma 6.2.4 we have ${}^{(t)}\rho_f = 0$.
Also ${}^{(t)}\rho_{fog}^{L^*} \le \rho_{fog}^{L^*} \le \rho_{fog} = 0$ implies that ${}^{(t)}\rho_{fog}^{L^*} = 0$ for $t = 2, 3, ...$

Now in view of Lemma 6.2.1 we get

$$\begin{aligned}\left({}^{(t)}\rho^{L^*}_{fog}\right)^* &= \limsup_{r\to\infty}\frac{\log^{[t]} M(r, fog)}{\log^{[2]}[re^{L(r)}]}\\ &\leq \limsup_{r\to\infty}\frac{\log^{[t]} M\left(M\left(r,g\right),f\right)}{\log^{[2]} M(r,g)}.\limsup_{r\to\infty}\frac{\log^{[2]} M(r,g)}{\log^{[2]}[re^{L(r)}]}\\ &= {}^{(t)}\rho^*_f.\left(\rho^{L^*}_g\right)^*.\end{aligned}$$

Also from Lemma 6.2.2 since δ (>0) is arbitrary, it follows that

$$\begin{aligned}\left({}^{(t)}\rho^{L^*}_{fog}\right)^* &= \limsup_{r\to\infty}\frac{\log^{[t]} M(r^{1+\delta}, fog)}{\log^{[2]}[re^{L(r)}]^{1+\delta}}\\ &\geq \limsup_{r\to\infty}\frac{\log^{[t]} M\left(M\left(r,g\right),f\right)}{\log^{[2]} M(r,g)}.\liminf_{r\to\infty}\frac{\log^{[2]} M(r,g)}{\log^{[2]}[re^{L(r)}]}\\ &= {}^{(t)}\rho^*_f.\left(\lambda^{L^*}_g\right)^*.\end{aligned}$$

Combining the above two inequalities we obtain that

$${}^{(t)}\rho^*_f.\left(\lambda^{L^*}_g\right)^* \leq \left({}^{(t)}\rho^{L^*}_{fog}\right)^* \leq {}^{(t)}\rho^*_f.\left(\rho^{L^*}_g\right)^*, \quad \text{for } t=2,3,...$$

Thus the lemma is established. ■

Lemma 6.2.7. *Let f be entire and g be transcendental entire such that ${}^{(t)}\lambda_f = 0$ and $\lambda_g < \infty$, then*

$${}^{(t)}\lambda^{L^*}_{fog} \geq {}^{(t)}\lambda^*_f.\lambda^{L^*}_g, \quad \textit{for } t=2,3,...$$

Proof. In view of Lemma 6.2.2 and since $\delta(>0)$ is arbitrary we obtain

$$\begin{aligned}{}^{(t)}\lambda^{L^*}_{fog} &= \liminf_{r\to\infty}\frac{\log^{[t]} M(r^{1+\delta}, fog)}{\log[re^{L(r)}]^{1+\delta}}\\ &\geq \liminf_{r\to\infty}\frac{\log^{[t]} M\left(M\left(r,g\right),f\right)}{\log^{[2]} M(r,g)}.\liminf_{r\to\infty}\frac{\log^{[2]} M(r,g)}{\log[re^{L(r)}]}\\ &= {}^{(t)}\lambda^*_f.\lambda^{L^*}_g.\end{aligned}$$

where $t=2,3,...$

This proves the lemma. ■

Lemma 6.2.8. *Let f be entire and g be transcendental entire with $\rho_{fog}=0$ and $\rho_g=0$. Then*

$$\left({}^{(t)}\lambda_{fog}^{L^*}\right)^* \geq {}^{(t)}\lambda_f^*.\left(\lambda_g^{L^*}\right)^*, \quad for\ t=2,3,...$$

Proof. By Lemma 6.2.2 and as $\delta(>0)$ is arbitrary we get that

$$\begin{aligned}\left({}^{(t)}\lambda_{fog}^{L^*}\right)^* &= \liminf_{r\to\infty}\frac{\log^{[t]}M(r^{1+\delta},fog)}{\log^{[2]}[re^{L(r)}]^{1+\delta}}\\ &\geq \liminf_{r\to\infty}\frac{\log^{[t]}M\left(M\left(r,g\right),f\right)}{\log^{[2]}M(r,g)}.\liminf_{r\to\infty}\frac{\log^{[2]}M(r,g)}{\log^{[2]}[re^{L(r)}]}\\ &= {}^{(t)}\lambda_f^*.\left(\lambda_g^{L^*}\right)^*.\end{aligned}$$

where $t=2,3,...$

Thus the lemma is established. ■

Lemma 6.2.9. *[2] Let f be entire and $\alpha>1$, $0<\beta<\alpha$. Then $F(\alpha r)>\beta F(r)$ for all large r.*

Lemma 6.2.10. *[2] Let f be an entire function. If $\alpha>1$, $0<\beta<\alpha$, $s>1$, $0<\mu<\lambda$ and n is a positive integer, then*
(a) $F(\alpha r)>\beta F(r)$
(b) $\lim_{r\to\infty}\frac{F(r^s)}{r^nF(r)}=\infty=\lim_{r\to\infty}\frac{F(r^\lambda)}{r^nF(r^\mu)}$, for transcendental f.

Lemma 6.2.11. *Let P be a polynomial and f be a transcendental entire function. Then for any entire function g, ${}^{(t)}\rho_g^{L^*}(Pf)={}^{(t)}\rho_g^{L^*}(f)$ where $t=1,2,3,...$*

Proof. Let $P(z)$ be a polynomial of degree m. Then we can always choose a positive number α, $0<\alpha<1$ and choose a positive integer $n(>m)$ such that

$$2\alpha<|P(z)|<r^n \qquad (*)$$

hold on $|z|=r$ for all large r.

So by the first part of Lemma 6.2.10 we get that

$$\begin{aligned}F(r) &= F(\frac{1}{\alpha}.\alpha r)>\frac{1}{2\alpha}F(\alpha r)\\ i.e., F(\alpha r) &< 2\alpha F(r).\end{aligned}$$

Let $h(z) = P(z)f(z)$. Then by the second part of Lemma 6.2.10 and (*) it follows for $s > 1$ that

$$\begin{aligned} F(\alpha r) &< 2\alpha F(r) < H(r) < r^n F(r) < F(r^s) \\ i.e., F(\alpha r) &< H(r) < F(r^s). \end{aligned}$$

This gives that

$$\begin{aligned} &\limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(\alpha r)}{\log[\alpha r e^{L(r)}]} \cdot \frac{\log[\alpha r e^{L(r)}]}{\log[r e^{L(r)}]} \\ \leq\ &\limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}H(r)}{\log[r e^{L(r)}]} \\ \leq\ &\limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(r^s)}{\log[r e^{L(r)}]^s} \cdot \frac{\log[r e^{L(r)}]^s}{\log[r e^{L(r)}]} \end{aligned}$$

$$i.e., {}^{(t)}\rho_g^{L^*}(f) \leq {}^{(t)}\rho_g^{L^*}(h) \leq s \cdot {}^{(t)}\rho_g^{L^*}(f).$$

Since $s > 1$ is arbitrary, letting $s \to 1 + 0$, we obtain that

$${}^{(t)}\rho_g^{L^*}(Pf) = {}^{(t)}\rho_g^{L^*}(f).$$

This proves the lemma. ■

6.3 Theorems.

In this section we present the main results of the chapter.

Theorem 6.3.1. *Let f and g be two entire functions such that $0 < {}^{(t)}\lambda_{fog}^{L^*} \leq {}^{(t)}\rho_{fog}^{L^*} < \infty$ and $0 < {}^{(t)}\lambda_g^{L^*} \leq {}^{(t)}\rho_g^{L^*} < \infty$. Then for any positive number A,*

$$\begin{aligned} \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A\,{}^{(t)}\rho_g^{L^*}} &\leq \liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A\,{}^{(t)}\lambda_g^{L^*}} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A\,{}^{(t)}\lambda_g^{L^*}}, \end{aligned}$$

where $t = 2, 3, \ldots$

Proof. From the definition of t-th generalised L^*-order and t-th generalised L^*-lower order we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[t]} M(r, fog) \geq ({}^{(t)}\lambda_{fog}^{L^*} - \varepsilon) \log[re^{L(r)}] \tag{6.1}$$

and

$$\log^{[t]} M(r^A, g) \leq A({}^{(t)}\rho_g^{L^*} + \varepsilon) \log[re^{L(r)}]. \tag{6.2}$$

Now from (6.1) and (6.2) it follows for all sufficiently large values of r,

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{({}^{(t)}\lambda_{fog}^{L^*} - \varepsilon)}{A({}^{(t)}\rho_g^{L^*} + \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A^{(t)}\rho_g^{L^*}}. \tag{6.3}$$

Again for a sequence of values of r tending to infinity

$$\log^{[t]} M(r, fog) \leq ({}^{(t)}\lambda_{fog}^{L^*} + \varepsilon) \log[re^{L(r)}] \tag{6.4}$$

and for all large values of r,

$$\log^{[t]} M(r^A, g) \geq A({}^{(t)}\lambda_g^{L^*} - \varepsilon) \log[re^{L(r)}]. \tag{6.5}$$

So combining (6.4) and (6.5) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\lambda_{fog}^{L^*} + \varepsilon}{A({}^{(t)}\lambda_g^{L^*} - \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that,

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A^{(t)}\lambda_g^{L^*}}. \tag{6.6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r^A, g) \leq A({}^{(t)}\lambda_g^{L^*} + \varepsilon) \log[re^{L(r)}]. \tag{6.7}$$

Now from (6.1) and (6.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\lambda_{fog}^{L^*} - \varepsilon}{A({}^{(t)}\lambda_g^{L^*} + \varepsilon)}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A^{(t)}\lambda_g^{L^*}}. \tag{6.8}$$

Also for all large values of r,

$$\log^{[t]} M(r, fog) \leq ({}^{(t)}\rho_{fog}^{L^*} + \varepsilon)\log[re^{L(r)}]. \tag{6.9}$$

So from (6.5) and (6.9) it follows for all sufficiently large values of r,

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*} + \varepsilon}{A({}^{(t)}\lambda_g^{L^*} - \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A^{(t)}\lambda_g^{L^*}}. \tag{6.10}$$

Thus the theorem follows from (6.3), (6.6), (6.8) and (6.10). ■

Remark 6.3.1. *The sign '$\leq$' cannnot be replaced by '$<$' only in Theorem 6.3.1 as we see in the following example.*

Example 6.3.1. *Let $f = z$, $g = \exp z$, $A = 1$, $t = 2$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. Then*

$$\begin{aligned} {}^{(2)}\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log[re^{L(r)}]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log r + L(r)} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \frac{1}{p}\exp(\frac{1}{r})} = 1. \end{aligned}$$

Similarly, ${}^{(2)}\lambda_{fog}^{L^*} = 1$. *Again*

$$\begin{aligned} {}^{(2)}\rho_g^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[re^{L(r)}]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log r + L(r)} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \frac{1}{p}\exp(\frac{1}{r})} = 1. \end{aligned}$$

Similarly, ${}^{(2)}\lambda_g^{L^*} = 1$. *So*

$${}^{(2)}\rho_{fog}^{L^*} = {}^{(2)}\lambda_{fog}^{L^*} = {}^{(2)}\rho_g^{L^*} = {}^{(2)}\lambda_g^{L^*} = 1.$$

Also

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1 \\ \text{and} \quad \liminf_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} &= \liminf_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1. \end{aligned}$$

Threrefore,

$$\begin{aligned} \frac{{}^{(2)}\lambda_{fog}^{L^*}}{A\,{}^{(2)}\rho_g^{L^*}} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} = \frac{{}^{(2)}\lambda_{fog}^{L^*}}{A\,{}^{(2)}\lambda_g^{L^*}} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} = \frac{{}^{(2)}\rho_{fog}^{L^*}}{A\,{}^{(2)}\lambda_g^{L^*}}. \end{aligned}$$

Theorem 6.3.2. *Let f be entire and g be transcendental entire satisfying the following conditions (i) ${}^{(t)}\rho_f = 0$ and $\rho_g < \infty$, (ii) $0 < {}^{(t)}\lambda_{fog}^{L^*} \le {}^{(t)}\rho_{fog}^{L^*} < \infty$ and (iii) $0 < {}^{(t)}\lambda_g^{L^*} \le {}^{(t)}\rho_g^{L^*} < \infty$. Then for any positive number A,*

$$\begin{aligned} \frac{{}^{(t)}\lambda_f^{*}.\lambda_g^{L^*}}{A\,{}^{(t)}\rho_g^{L^*}} &\le \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A\,{}^{(t)}\rho_g^{L^*}} \le \liminf_{r\to\infty} \frac{\log^{[t]} M(r,fog)}{\log^{[t]} M(r^A,g)} \le \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A\,{}^{(t)}\lambda_g^{L^*}} \\ &\le \limsup_{r\to\infty} \frac{\log^{[t]} M(r,fog)}{\log^{[t]} M(r^A,g)} \le \frac{{}^{(t)}\rho_{fog}^{L^*}}{A\,{}^{(t)}\lambda_g^{L^*}} \le \frac{{}^{(t)}\rho_f^{*}.\rho_g^{L^*}}{A\,{}^{(t)}\lambda_g^{L^*}}. \end{aligned}$$

where $t = 2, 3, \ldots$

Proof. In view of Lemma 6.2.7 and the second part of Lemma 6.2.5, Theorem 6.3.2 follows from Theorem 6.3.1. ■

Theorem 6.3.3. *Let f and g be two entire functions such that $0 < {}^{(t)}\lambda_{fog}^{L^*} \leq {}^{(t)}\rho_{fog}^{L^*} < \infty$ and $0 < {}^{(t)}\rho_g^{L^*} < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A\,{}^{(t)}\rho_g^{L^*}} \leq \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)},$$

where $t = 2, 3, \ldots$

Proof. From the definition of t-th generalised L^*-order we get for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r^A, g) \geq A({}^{(t)}\rho_g^{L^*} - \varepsilon) \log[re^{L(r)}]. \tag{6.11}$$

Now from (6.9) and (6.11) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*} + \varepsilon}{A({}^{(t)}\rho_g^{L^*} - \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A\,{}^{(t)}\rho_g^{L^*}}. \tag{6.12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r, fog) \geq ({}^{(t)}\rho_{fog}^{L^*} - \varepsilon) \log[re^{L(r)}]. \tag{6.13}$$

So combining (6.2) and (6.13) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\rho_{fog}^{L^*} - \varepsilon}{A({}^{(t)}\rho_g^{L^*} + \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A\,{}^{(t)}\rho_g^{L^*}}. \tag{6.14}$$

Thus the theorem follows from (6.12) and (6.14). ■

Remark 6.3.2. *The sign '$\leq$' cannnot be replaced by '$<$' only in Theorem 6.3.3 which is evident from the following example.*

Example 6.3.2. *If $f = z$, $g = \exp z$, $A = 1$, $t = 2$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number, then*

$$\begin{aligned} {}^{(2)}\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log[re^{L(r)}]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log r + L(r)} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \frac{1}{p}\exp(\frac{1}{r})} = 1. \end{aligned}$$

Similarly, ${}^{(2)}\lambda_{fog}^{L^*} = 1$. *Again*

$$\begin{aligned} {}^{(2)}\rho_{g}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log[re^{L(r)}]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log r + L(r)} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \frac{1}{p}\exp(\frac{1}{r})} = 1. \end{aligned}$$

Similarly, ${}^{(2)}\lambda_{g}^{L^*} = 1$. *So*

$${}^{(2)}\rho_{fog}^{L^*} =^{(2)} \lambda_{fog}^{L^*} =^{(2)} \rho_{g}^{L^*} =^{(2)} \lambda_{g}^{L^*} = 1.$$

Also

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1 \\ \text{and} \quad \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} &= \liminf_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1. \end{aligned}$$

Threrefore,

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} = \frac{{}^{(2)}\rho_{fog}^{L^*}}{A^{(2)}\rho_{g}^{L^*}} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)}.$$

Theorem 6.3.4. *Let f be entire and g be transcendental entire satisfying the following conditions (i)* ${}^{(t)}\rho_f = 0$ *and* $\rho_g < \infty$, *(ii)* $0 < {}^{(t)}\lambda_{fog}^{L^*} \leq {}^{(t)}\rho_{fog}^{L^*} < \infty$ *and (iii)* $0 < {}^{(t)}\rho_g^{L^*} < \infty$. *Then for any positive number A,*

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\lambda_f^* . \rho_g^{L^*}}{A {}^{(t)}\rho_g^{L^*}}$$

$$\text{and} \quad \liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_f^* . \rho_g^{L^*}}{A {}^{(t)}\rho_g^{L^*}}$$

where $t = 2, 3, \ldots$

Proof. In view of Lemma 6.2.5 we obtain from Theorem 6.3.3

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \geq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A {}^{(t)}\rho_g^{L^*}} \geq \frac{{}^{(t)}\lambda_f^* . \rho_g^{L^*}}{A {}^{(t)}\rho_g^{L^*}}$$

and

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A {}^{(t)}\rho_g^{L^*}} \leq \frac{{}^{(t)}\rho_f^* . \rho_g^{L^*}}{A {}^{(t)}\rho_g^{L^*}}$$

for $t = 2, 3, \ldots$

This proves the theorem. ■

The following theorem is a natural consequence of Theorem 6.3.1 and Theorem 6.3.3.

Theorem 6.3.5. *Let f and g be two entire functions such that* $0 < {}^{(t)}\lambda_{fog}^{L^*} \leq {}^{(t)}\rho_{fog}^{L^*} < \infty$ *and* $0 < {}^{(t)}\lambda_g^{L^*} \leq {}^{(t)}\rho_g^{L^*} < \infty$. *Then for any positive number A,*

$$\begin{aligned} \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A {}^{(t)}\rho_g^{L^*}} &\leq \liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \\ &\leq \min\left\{ \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A {}^{(t)}\lambda_g^{L^*}}, \frac{{}^{(t)}\rho_{fog}^{L^*}}{A {}^{(t)}\rho_g^{L^*}} \right\} \leq \max\left\{ \frac{{}^{(t)}\lambda_{fog}^{L^*}}{A {}^{(t)}\lambda_g^{L^*}}, \frac{{}^{(t)}\rho_{fog}^{L^*}}{A {}^{(t)}\rho_g^{L^*}} \right\} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, g)} \leq \frac{{}^{(t)}\rho_{fog}^{L^*}}{A {}^{(t)}\lambda_g^{L^*}}, \end{aligned}$$

where $t = 2, 3, \ldots$

Theorem 6.3.6. *Let f and g be two entire functions satisfying (i) $0 < {}^{(t)}\rho_g^{L^*} < \infty$, (ii) $0 < {}^{(t)}\sigma_g^{L^*} < \infty$, (iii) ${}^{(t)}\rho_{fog}^{L^*} = {}^{(t)}\rho_g^{L^*}$ and (iv) $0 < {}^{(t)}\sigma_{fog}^{L^*} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[t-1]} M(r, fog)}{\log^{[t-1]} M(r, g)} \leq \frac{{}^{(t)}\sigma_{fog}^{L^*}}{{}^{(t)}\sigma_g^{L^*}} \leq \limsup_{r\to\infty} \frac{\log^{[t-1]} M(r, fog)}{\log^{[t-1]} M(r, g)},$$

where $t = 2, 3, ...$

Proof. From the definition of t-th generalised L^*–type of a composite entire function we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[t-1]} M(r, fog) \leq ({}^{(t)}\sigma_{fog}^{L^*} + \varepsilon)[re^{L(r)}]^{{}^{(t)}\rho_{fog}^{L^*}}. \tag{6.15}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[t-1]} M(r, g) \geq ({}^{(t)}\sigma_g^{L^*} - \varepsilon)[re^{L(r)}]^{{}^{(t)}\rho_g^{L^*}}. \tag{6.16}$$

As ${}^{(t)}\rho_{fog}^{L^*} = {}^{(t)}\rho_g^{L^*}$ from (6.15) and (6.16) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[t-1]} M(r, fog)}{\log^{[t-1]} M(r, g)} \leq \frac{{}^{(t)}\sigma_{fog}^{L^*} + \varepsilon}{{}^{(t)}\sigma_g^{L^*} - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[t-1]} M(r, fog)}{\log^{[t-1]} M(r, g)} \leq \frac{{}^{(t)}\sigma_{fog}^{L^*}}{{}^{(t)}\sigma_g^{L^*}}. \tag{6.17}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[t-1]} M(r, fog) \geq ({}^{(t)}\sigma_{fog}^{L^*} - \varepsilon)[re^{L(r)}]^{{}^{(t)}\rho_{fog}^{L^*}} \tag{6.18}$$

and for all sufficiently large values of r,

$$\log^{[t-1]} M(r, g) \leq ({}^{(t)}\sigma_g^{L^*} + \varepsilon)[re^{L(r)}]^{{}^{(t)}\rho_g^{L^*}}. \tag{6.19}$$

By condition (iii) we obtain from (6.18) and (6.19) for a sequence of values of r tending to infinity that

$$\frac{\log^{[t-1]} M(r, fog)}{\log^{[t-1]} M(r, g)} \geq \frac{{}^{(t)}\sigma_{fog}^{L^*} - \varepsilon}{{}^{(t)}\sigma_g^{L^*} + \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, we get from above

$$\limsup_{r\to\infty} \frac{\log^{[t-1]} M(r, fog)}{\log^{[t-1]} M(r, g)} \geq \frac{{}^{(t)}\sigma_{fog}^{L^*}}{{}^{(t)}\sigma_g^{L^*}}. \tag{6.20}$$

Thus the theorem follows from (6.17) and (6.20). ■

Remark 6.3.3. *The sign '$\leq$' in Theorem 6.3.6 cannnot be replaced by '$<$' only which is evident from the following example.*

Example 6.3.3. *Let $f = z$, $g = \exp z$, $t = 2$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. So*

$$ {}^{(2)}\rho_{fog}^{L^*} = {}^{(2)}\rho_g^{L^*} = 1 \quad and \quad {}^{(2)}\sigma_{fog}^{L^*} = {}^{(2)}\sigma_g^{L^*} = \frac{1}{\exp(\frac{1}{p})}.$$

Also

$$\limsup_{r\to\infty} \frac{\log M(r, fog)}{\log M(r, g)} = 1 = \liminf_{r\to\infty} \frac{\log M(r, fog)}{\log M(r, g)}.$$

Hence

$$\liminf_{r\to\infty} \frac{\log M(r, fog)}{\log M(r, g)} = \frac{{}^{(2)}\sigma_{fog}^{L^*}}{{}^{(2)}\sigma_g^{L^*}} = 1 = \limsup_{r\to\infty} \frac{\log M(r, fog)}{\log M(r, g)}.$$

In the next few theorems we intend to establish some results on the comparative growth properties of composition of two entire functions f and g with respect to the left factor f where the t-th generalised L^*-order of f be zero i.e., the results involving $\left({}^{(t)}\rho_f^{L^*}\right)^*$ and $\left({}^{(t)}\lambda_f^{L^*}\right)^*$.

Theorem 6.3.7. *Let f be entire and g be transcendental entire functions such that $\rho_{fog} = 0$. Also let $0 < \left({}^{(t)}\lambda_{fog}^{L^*}\right)^* \leq \left({}^{(t)}\rho_{fog}^{L^*}\right)^* < \infty$ and $0 < \left({}^{(t)}\lambda_f^{L^*}\right)^* \leq \left({}^{(t)}\rho_f^{L^*}\right)^* < \infty$. Then for any positive number A,*

$$\begin{aligned} \frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*} &\leq \liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}, \end{aligned}$$

where $t = 2, 3, \ldots$

Proof. Since $\rho_{fog} = 0$ by Lemma 6.2.4, ${}^{(t)}\rho_f^{L^*} = 0$.

Again ${}^{(t)}\rho_{fog}^{L^*} \leq \rho_{fog}^{L^*} \leq \rho_{fog} = 0$ implies that ${}^{(t)}\rho_{fog}^{L^*} = 0$, where $t = 2, 3, ...$

From the definition of $\left({}^{(t)}\rho_f^{L^*}\right)^*$ and $\left({}^{(t)}\lambda_f^{L^*}\right)^*$ we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[t]} M(r, fog) \geq \left(\left({}^{(t)}\lambda_{fog}^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}] \tag{6.21}$$

and

$$\log^{[t]} M(r^A, f) \leq \left(\left({}^{(t)}\rho_f^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1). \tag{6.22}$$

Now from (6.21) and (6.22) it follows for all sufficiently large values of r,

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{\left(\left({}^{(t)}\lambda_{fog}^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}]}{\left(\left({}^{(t)}\rho_f^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1)}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*}. \tag{6.23}$$

Again for a sequence of values of r tending to infinity

$$\log^{[t]} M(r, fog) \leq \left(\left({}^{(t)}\lambda_{fog}^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}] \tag{6.24}$$

and for all large values of r,

$$\log^{[t]} M(r^A, f) \geq \left(\left({}^{(t)}\lambda_f^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1). \tag{6.25}$$

So combining (6.24) and (6.25) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left(\left({}^{(t)}\lambda_{fog}^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}]}{\left(\left({}^{(t)}\lambda_f^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1)}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that,

$$\liminf_{r\to\infty}\frac{\log^{[t]}M(r,fog)}{\log^{[t]}M(r^A,f)}\leq\frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}. \tag{6.26}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[t]}M(r^A,f)\leq\left(\left({}^{(t)}\lambda_f^{L^*}\right)^*+\varepsilon\right)\log^{[2]}[re^{L(r)}]+O(1). \tag{6.27}$$

Now from (6.21) and (6.27) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[t]}M(r,fog)}{\log^{[t]}M(r^A,f)}\geq\frac{\left(\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*-\varepsilon\right)\log^{[2]}[re^{L(r)}]}{\left(\left({}^{(t)}\lambda_f^{L^*}\right)^*+\varepsilon\right)\log^{[2]}[re^{L(r)}]+O(1)}.$$

Choosing $\varepsilon\to 0$ we get that

$$\limsup_{r\to\infty}\frac{\log^{[t]}M(r,fog)}{\log^{[t]}M(r^A,f)}\geq\frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}. \tag{6.28}$$

Also for all large values of r,

$$\log^{[t]}M(r,fog)\leq\left(\left({}^{(t)}\rho_{fog}^{L^*}\right)^*+\varepsilon\right)\log^{[2]}[re^{L(r)}]. \tag{6.29}$$

So from (6.25) and (6.29) it follows for all sufficiently large values of r,

$$\frac{\log^{[t]}M(r,fog)}{\log^{[t]}M(r^A,f)}\leq\frac{\left(\left({}^{(t)}\rho_{fog}^{L^*}\right)^*+\varepsilon\right)\log^{[2]}[re^{L(r)}]}{\left(\left({}^{(t)}\lambda_f^{L^*}\right)^*-\varepsilon\right)\log^{[2]}[re^{L(r)}]+O(1)}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[t]}M(r,fog)}{\log^{[t]}M(r^A,f)}\leq\frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}. \tag{6.30}$$

Thus the theorem follows from (6.23), (6.26), (6.28) and (6.30). ■

Theorem 6.3.8. *Let f be entire and g be transcendental entire satisfying the conditions (i) $\rho_{fog} = 0$ and $\rho_g = 0$, (ii) $0 < \left({}^{(t)}\lambda_{fog}^{L^*}\right)^* \leq \left({}^{(t)}\rho_{fog}^{L^*}\right)^* < \infty$ and (iii) $0 < \left({}^{(t)}\lambda_f^{L^*}\right)^* \leq \left({}^{(t)}\rho_f^{L^*}\right)^* < \infty$. Then for any positive number A,*

$$\frac{{}^{(t)}\lambda_f^* \left(\lambda_g^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*} \leq \frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*} \leq \liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*} \leq \frac{{}^{(t)}\rho_f^* \left(\rho_g^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}$$

where $t = 2, 3, ...$

Proof. In view of Lemma 6.2.8 and the second part of Lemma 6.2.6, Theorem 6.3.8 follows from Theorem 6.3.7. ∎

Theorem 6.3.9. *Let f be entire and g be transcendental entire functions such that $\rho_{fog} = 0$. Also let $0 < \left({}^{(t)}\lambda_{fog}^{L^*}\right)^* \leq \left({}^{(t)}\rho_{fog}^{L^*}\right)^* < \infty$ and $0 < \left({}^{(t)}\rho_f^{L^*}\right)^* < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*} \leq \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)},$$

where $t = 2, 3, ...$

Proof. In view of Lemma 6.2.4, $\rho_{fog} = 0$ $(< \infty)$ implies that ${}^{(t)}\rho_f^{L^*} = 0$.

Again ${}^{(t)}\rho_{fog}^{L^*} \leq \rho_{fog} = 0$ implies that ${}^{(t)}\rho_{fog}^{L^*} = 0$ for $t = 2, 3, ...$

From the definition of $\left({}^{(t)}\rho_f^{L^*}\right)^*$ we get for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r^A, f) \geq \left(\left({}^{(t)}\rho_f^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1). \tag{6.31}$$

Now from (6.29) and (6.31) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left(\left({}^{(t)}\rho_{fog}^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}]}{\left(\left({}^{(t)}\rho_f^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1)}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*}. \tag{6.32}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r, fog) \geq \left(\left({}^{(t)}\rho_{fog}^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}]. \tag{6.33}$$

So combining (6.22) and (6.33) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{\left(\left({}^{(t)}\rho_{fog}^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}]}{\left(\left({}^{(t)}\rho_f^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1)}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*}. \tag{6.34}$$

Thus the theorem follows from (6.32) and (6.34). ■

Theorem 6.3.10. *Let f be entire and g be transcendental entire functions satisfying the conditions (i) $\rho_{fog} = 0$ and $\rho_g = 0$, (ii) $0 < \left({}^{(t)}\lambda_{fog}^{L^*}\right)^* \leq \left({}^{(t)}\rho_{fog}^{L^*}\right)^* < \infty$ and (iii) $0 < \left({}^{(t)}\rho_f^{L^*}\right)^* < \infty$. Then for any positive number A,*

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{{}^{(t)}\rho_f^* . \left(\lambda_g^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*}$$

$$\text{and} \quad \liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{{}^{(t)}\rho_f^* . \left(\rho_g^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*}$$

where $t = 2, 3, \ldots$

Proof. In view of Lemma 6.2.6 we obtain from Theorem 6.3.9

$$\limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*} \geq \frac{{}^{(t)}\rho_f^* . \left(\lambda_g^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*}$$

and

$$\liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*} \leq \frac{{}^{(t)}\rho_f^* . \left(\rho_g^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*}$$

where $t = 2, 3, ...$ ■

The following theorem is a natural consequence of Theorem 6.3.7 and Theorem 6.3.9.

Theorem 6.3.11. *Let f be entire and g be transcendental entire functions such that $\rho_{fog} = 0$. Also let $0 < \left({}^{(t)}\lambda_{fog}^{L^*}\right)^* \leq \left({}^{(t)}\rho_{fog}^{L^*}\right)^* < \infty$ and $0 < \left({}^{(t)}\lambda_{fog}^{L^*}\right)^* \leq \left({}^{(t)}\rho_f^{L^*}\right)^* < \infty$. Then for any positive number A,*

$$\begin{aligned}
\frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*} &\leq \liminf_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \\
&\leq \min\left\{ \frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}, \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*} \right\} \\
&\leq \max\left\{ \frac{\left({}^{(t)}\lambda_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}, \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\rho_f^{L^*}\right)^*} \right\} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[t]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\left({}^{(t)}\rho_{fog}^{L^*}\right)^*}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}
\end{aligned}$$

where $t = 2, 3, ...$

Theorem 6.3.12. *Let f be entire and g be a transcendental entire function such that $\rho_{fog} < \infty$, $0 <^{(t)} \lambda_{fog}^{L^*} \leq^{(t)} \rho_{fog}^{L^*} < \infty$ and $0 < \left({}^{(t)}\lambda_f^{L^*}\right)^* \leq \left({}^{(t)}\rho_f^{L^*}\right)^* < \infty$. Then for any positive number A,*

$$\begin{aligned}
\frac{1}{\left({}^{(t)}\rho_f^{L^*}\right)^*} &= \liminf_{r\to\infty} \frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} = \frac{1}{\left({}^{(t)}\lambda_f^{L^*}\right)^*},
\end{aligned}$$

where $t = 2, 3, ...$

Proof. In view of Lemma 6.2.4, $\rho_{fog} < \infty$ implies ${}^{(t)}\rho_f^{L^*} = 0$ for $t = 2, 3, \ldots$

From the definition of t-th generalised L^*-lower order, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\begin{aligned} \log^{[t]} M(r, fog) &\geq \left({}^{(t)}\lambda_{fog}^{L^*} - \varepsilon\right) \log[re^{L(r)}] \\ \text{i.e., } \log^{[t+1]} M(r, fog) &\geq \log^{[2]}[re^{L(r)}] + O(1). \end{aligned} \tag{6.35}$$

Now in view of Definition 6.1.11, we obtain for all sufficiently large values of r,

$$\log^{[t]} M(r^A, f) \leq \left(\left({}^{(t)}\rho_f^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1). \tag{6.36}$$

Now from (6.35) and (6.36) it follows for all sufficiently large values of r,

$$\frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{\log^{[2]}[re^{L(r)}] + O(1)}{\left(\left({}^{(t)}\rho_f^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1)}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{1}{\left({}^{(t)}\rho_f^{L^*}\right)^*}. \tag{6.37}$$

Again for all sufficiently large values of r,

$$\begin{aligned} \log^{[t]} M(r, fog) &\leq \left({}^{(t)}\rho_{fog}^{L^*} + \varepsilon\right) \log[re^{L(r)}] \\ \text{i.e., } \log^{[t+1]} M(r, fog) &\leq \log^{[2]}[re^{L(r)}] + O(1). \end{aligned} \tag{6.38}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r^A, f) \geq \left(\left({}^{(t)}\rho_f^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1). \tag{6.39}$$

So combining (6.38) and (6.39), we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\log^{[2]}[re^{L(r)}] + O(1)}{\left(\left({}^{(t)}\rho_f^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1)}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that,

$$\liminf_{r\to\infty} \frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{1}{\left({}^{(t)}\rho_f^{L^*}\right)^*}. \tag{6.40}$$

Now from (6.37) and (6.40) we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} = \frac{1}{\left({}^{(t)}\rho_f^{L^*}\right)^*}. \tag{6.41}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[t]} M(r^A, f) \leq \left(\left({}^{(t)}\lambda_f^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1). \tag{6.42}$$

So from (6.35) and (6.42) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{\log^{[2]}[re^{L(r)}] + O(1)}{\left(\left({}^{(t)}\lambda_f^{L^*}\right)^* + \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1)}.$$

Choosing $\varepsilon \longrightarrow 0$ we get that

$$\limsup_{r\to\infty} \frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} \geq \frac{1}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}. \tag{6.43}$$

Also for all sufficiently large values of r,

$$\log^{[t]} M(r^A, f) \geq \left(\left({}^{(t)}\lambda_f^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1). \tag{6.44}$$

Combining (6.38) and (6.44) we obtain for all sufficiently large values of r,

$$\frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{\log^{[2]}[re^{L(r)}] + O(1)}{\left(\left({}^{(t)}\lambda_f^{L^*}\right)^* - \varepsilon\right) \log^{[2]}[re^{L(r)}] + O(1)}.$$

As $\varepsilon(>0)$ is arbitrary, we get that

$$\limsup_{r\to\infty} \frac{\log^{[t+1]} M(r, fog)}{\log^{[t]} M(r^A, f)} \leq \frac{1}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}. \tag{6.45}$$

Now from (6.43) and (6.45) it follows that

$$\limsup_{r\to\infty}\frac{\log^{[t+1]}M(r,fog)}{\log^{[t]}M(r^A,f)}=\frac{1}{\left({}^{(t)}\lambda_f^{L^*}\right)^*} \tag{6.46}$$

Thus the theorem follows from (6.41) and (6.46). ∎

Remark 6.3.4. *In particular, if we take* $\left({}^{(t)}\rho_f^{L^*}\right)^*=\left({}^{(t)}\lambda_f^{L^*}\right)^*$ *in Theorem 6.3.12, then*

$$\lim_{r\to\infty}\frac{\log^{[t+1]}M(r,fog)}{\log^{[t]}M(r^A,f)}=\frac{1}{\left({}^{(t)}\rho_f^{L^*}\right)^*}=\frac{1}{\left({}^{(t)}\lambda_f^{L^*}\right)^*}\quad \textit{for } t=2,3,\ldots$$

Remark 6.3.5. *The sign '$\leq$' in Theorem 6.3.12 cannot be replaced by '$<$' only which is evident from the following example.*

Example 6.3.4. *If* $f=z$, $g=\exp z$, $t=2$ *and* $L(r)=\frac{1}{p}\exp(\frac{1}{r})$ *where* p *is any positive real number, then*

$$\begin{aligned}\rho_{fog} &= \limsup_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log r}\\ &= \limsup_{r\to\infty}\frac{\log^{[2]}\exp r}{\log r}=1.\end{aligned}$$

Also

$$\begin{aligned}\left({}^{(2)}\rho_f^{L^*}\right)^* &= \limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log^{[2]}[re^{L(r)}]}\\ &= \limsup_{r\to\infty}\frac{\log^{[2]}r}{\log\left[\log r+L(r)\right]}\\ &= \limsup_{r\to\infty}\frac{\log^{[2]}r}{\log\left[\log r\left(1+\frac{L(r)}{\log r}\right)\right]}\\ &= \limsup_{r\to\infty}\frac{\log^{[2]}r}{\log^{[2]}r+\log\left(1+\frac{\exp(\frac{1}{r})}{p\log r}\right)}\\ &= 1.\end{aligned}$$

Similarly, $\left({}^{(2)}\lambda_g^{L^*}\right)^*=1.$

Again

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log^{[3]}M(r,fog)}{\log^{[2]}M(r^A,f)} &= \limsup_{r\to\infty}\frac{\log^{[3]}\exp r}{\log^{[2]}r^A}\\ &= \limsup_{r\to\infty}\frac{\log^{[2]}r}{\log^{[2]}r+O(1)}\\ &= 1\end{aligned}$$

and similarly

$$\liminf_{r\to\infty}\frac{\log^{[3]}M(r,fog)}{\log^{[2]}M(r^A,f)}=1.$$

Therefore,

$$\begin{aligned}\frac{1}{\left({}^{(2)}\rho_f^{L^*}\right)^*} &= \liminf_{r\to\infty}\frac{\log^{[3]}M(r,fog)}{\log^{[2]}M(r^A,f)}\\ &= \limsup_{r\to\infty}\frac{\log^{[3]}M(r,fog)}{\log^{[2]}M(r^A,f)}=\frac{1}{\left({}^{(2)}\lambda_f^{L^*}\right)^*}.\end{aligned}$$

In the next few theorems we will show the equality of generalised relative L^*-orders (generalised relative L^*-lower orders) of two entire functions.

Theorem 6.3.13. *Let f, g, h be three entire functions such that $g\sim h$. Then*

$${}^{(t)}\rho_g^{L^*}(f)={}^{(t)}\rho_h^{L^*}(f)\quad\text{and}\quad {}^{(t)}\lambda_g^{L^*}(f)={}^{(t)}\lambda_h^{L^*}(f),$$

where $t=1,2,3,\ldots$

Proof. Let $\varepsilon(>0)$ be choosen arbitrary. By Lemma 6.2.9 it follows for all large values of r that

$$G(r)<(l+\varepsilon)H(r)<H(\alpha r) \tag{6.47}$$

where $0<l<\infty$, $\alpha>\max\{1,\ l\}$ is such that $l+\varepsilon<\alpha$.

From (6.47) we obtain for all large r that

$$\begin{aligned} r &< G^{-1}(H(\alpha r))\\ \text{i.e.,}\frac{1}{\alpha}H^{-1}(t) &< G^{-1}(t)\quad\text{where } t=H(\alpha r)\\ \text{i.e.,}H^{-1}(r) &< \alpha G^{-1}(r). \end{aligned} \tag{6.48}$$

Now by (6.48) we get that

$$\begin{aligned} {}^{(t)}\rho_h^{L^*}(f) &= \limsup_{r\to\infty} \frac{\log^{[t]} H^{-1}F(r)}{\log[re^{L(r)}]} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[t]}[\alpha G^{-1}F(r)]}{\log[re^{L(r)}]} \\ &= \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(r)}{\log[re^{L(r)}]} \\ &= {}^{(t)}\rho_g^{L^*}(f). \end{aligned} \tag{6.49}$$

Since $g \sim h$, therefore $h \sim g$ and in a similar manner one can verify that

$${}^{(t)}\rho_g^{L^*}(f) \leq {}^{(t)}\rho_h^{L^*}(f). \tag{6.50}$$

Thus from (6.49) and(6.50) it follows that

$${}^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\rho_h^{L^*}(f).$$

In a similar way one can show that

$${}^{(t)}\lambda_g^{L^*}(f) = {}^{(t)}\lambda_h^{L^*}(f).$$

This proves the theorem. ■

Remark 6.3.6. *The converse of Theorem 6.3.13 is not true which is evident from the following example.*

Example 6.3.5. *Let* $f = g = \exp z$, $h = \exp 2z$ *and* $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ *where* p *is any positive real number. Also let* $t = 1$.

Then $F(r) = G(r) = \exp r$ *and* $H(r) = \exp 2r$, *so that*

$$\frac{G(r)}{H(r)} = \frac{\exp r}{\exp 2r} \to 0 \text{ as } r \to \infty.$$

Hence g *not* $\sim h$

But

$${}^{(1)}\rho_g^{L^*}(f) = {}^{(1)}\rho_h^{L^*}(f) = 1 = {}^{(1)}\lambda_g^{L^*}(f) = {}^{(1)}\lambda_h^{L^*}(f).$$

Theorem 6.3.14. *Let* f, g, h *be three entire functions such that* $g \sim h$. *Then*

$${}^{(t)}\rho_f^{L^*}(g) = {}^{(t)}\rho_f^{L^*}(h) \quad \text{and} \quad {}^{(t)}\lambda_f^{L^*}(g) = {}^{(t)}\lambda_f^{L^*}(h),$$

where $t = 1, 2, 3, \ldots$

Proof. Since $g \sim h$, in view of Lemma 6.2.9 for $\varepsilon_1 > 0$,there exists $R_1 > 0$ such that

$$G(r) < (l + \varepsilon_1)H(r) < H(\beta r) \tag{6.51}$$

where $0 < l < \infty$, $r > R_1$ and $\beta > \max\{1, l\}$ such that $l + \varepsilon_1 < \beta$.

Now from (6.51) we obtain that

$$\begin{aligned} {}^{(t)}\rho_f^{L^*}(g) &= \limsup_{r\to\infty} \frac{\log^{[t]} F^{-1}G(r)}{\log[re^{L(r)}]} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[t]} F^{-1}H(\beta r)}{\log[re^{L(r)}]}. \end{aligned} \tag{6.52}$$

For $0 < \varepsilon_2 < 1$ there exists $R_2 > 0$ such that for $r \geq R_2$

$$\log[re^{L(r)}] > (1 - \varepsilon_2) \log[\beta re^{L(r)}]. \tag{6.53}$$

So from (6.52) and (6.53) we get that

$$\begin{aligned} {}^{(t)}\rho_f^{L^*}(g) &\leq \limsup_{r\to\infty} \frac{\log^{[t]} F^{-1}H(\beta r)}{(1 - \varepsilon_2) \log[\beta re^{L(r)}]} \\ &= \frac{1}{1 - \varepsilon_2}\, {}^{(t)}\rho_f^{L^*}(h). \end{aligned}$$

Since $0 < \varepsilon_2 < 1$ is arbitrary,

$${}^{(t)}\rho_f^{L^*}(g) \leq {}^{(t)}\rho_f^{L^*}(h). \tag{6.54}$$

As also $h \sim g$, we obtain in a like manner that

$${}^{(t)}\rho_f^{L^*}(h) \leq {}^{(t)}\rho_f^{L^*}(g). \tag{6.55}$$

Combining (6.54) and (6.55) it follows that

$${}^{(t)}\rho_f^{L^*}(g) = {}^{(t)}\rho_f^{L^*}(h).$$

Similarly we can show that

$${}^{(t)}\lambda_f^{L^*}(g) = {}^{(t)}\lambda_f^{L^*}(h).$$

Thus the theorem is proved. ■

In the line of Theorem 6.3.13 and Theorem 6.3.14 we may state the following theorem without proof.

Theorem 6.3.15. *Let f, g, h, k be entire functions with $f \sim g$ and $h \sim k$. Then*

$$ {}^{(t)}\rho_h^{L^*}(f) = {}^{(t)}\rho_k^{L^*}(f) = {}^{(t)}\rho_h^{L^*}(g) = {}^{(t)}\rho_k^{L^*}(g). $$

and

$$ {}^{(t)}\lambda_h^{L^*}(f) = {}^{(t)}\lambda_k^{L^*}(f) = {}^{(t)}\lambda_h^{L^*}(g) = {}^{(t)}\lambda_k^{L^*}(g). $$

Theorem 6.3.16. *Let f, g and h be any three entire functions. If $F(r) \leq G(r)$ for all large r, then*

$$ {}^{(t)}\rho_h^{L^*}(f) \leq {}^{(t)}\rho_h^{L^*}(g), $$

where $t = 1, 2, 3, \ldots$

Proof. In view of the condition $F(r) \leq G(r)$ we get that

$$ \begin{aligned} {}^{(t)}\rho_h^{L^*}(f) &= \limsup_{r\to\infty} \frac{\log^{[t]} H^{-1}F(r)}{\log[re^{L(r)}]} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[t]} H^{-1}G(r)}{\log[re^{L(r)}]} \\ &= {}^{(t)}\rho_h^{L^*}(g). \end{aligned} $$

This proves the theorem. ■

The following theorem can be carried out in the line of Theorem 6.3.16 and so the proof is omitted.

Theorem 6.3.17. *Let f, g and h be any three entire functions. If $G(r) \leq H(r)$ for all large r, then*

$$ {}^{(t)}\rho_g^{L^*}(f) \geq {}^{(t)}\rho_h^{L^*}(f), $$

where $t = 1, 2, 3, \ldots$

In the next two theorems we will now investigate the relationship between the t-th generalised relative L^*-order of an entire function f and its derivative f' with respect to another entire function g and its derivative g' respectively.

Theorem 6.3.18. *Let f and g be any two transcendental entire functions. Then*

$$^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\rho_g^{L^*}(f')$$

where $t = 1, 2, 3, \ldots$ and f' denotes the derivative of f.

Proof. Let $\bar{F}(r) = \max_{|z|=r} |f'(z)|$. Without loss of any generality we may assume that $f(0) = 0$. Otherwise we set $f_1(z) = zf(z)$. Then $f_1(0) = 0$ and in view of Lemma 6.2.11 it follows that $^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\rho_g^{L^*}(f_1)$. We may write $f(z) = \int_0^z f'(t)\, dt$, where the line of integration is the segment from $z = 0$ to $z = re^{i\theta_0}$, $r > 0$. Let $z_1 = re^{i\theta_1}$ be such that $|f(z_1)| = \max_{|z|=r} |f(z)|$. Then

$$\begin{aligned} F(r) &= |f(z_1)| = \left| \int_0^{z_1} f'(t)\, dt \right| \\ &\leq r \max\{|f'(z)| : |z| = r\} \\ &= r\bar{F}(r). \end{aligned} \tag{6.56}$$

Let C denote the circle $|t - z_0| = r$, where z_0 is defined so that $|f'(z_0)| = \max_{|z|=r} |f'(z)|$. So

$$\begin{aligned} \bar{F}(r) &= \max_{|z|=r} |f'(z)| = |f'(z_0)| \\ &= \left| \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t - z_0)^2}\, dt \right| \\ &\leq \frac{1}{2\pi} \frac{F(2r)}{r^2} 2\pi r = \frac{F(2r)}{r}. \end{aligned} \tag{6.57}$$

From (6.56) and (6.57) we obtain that

$$\frac{F(r)}{r} \leq \bar{F}(r) \leq \frac{F(2r)}{r} \text{ for } r > 0. \tag{6.58}$$

Let $\sigma \in (0, 1)$. Since f is transcendental, from the second part of Lemma 6.2.10 it follows that

$$\lim_{r \to \infty} \frac{F(r^s)}{r^n F(r)} = \infty \ (s > 1).$$

We set $s = \dfrac{1}{\sigma}$ and so $F(r^s) > r^n F(r)$ for all large r. If we replace r by r^σ, then from above we get that

$$F(r^{s\sigma}) > r^{n\sigma} F(r^\sigma) \geq rF(r^\sigma),$$

where n is such that $n\sigma \geq 1$ i.e., $F(r) > rF(r^\sigma)$.

From (6.58) it follows that

$$\begin{aligned} F(r^\sigma) &< \frac{F(r)}{r} \leq \bar{F}(r) \leq \frac{F(2r)}{r} < F(2r), \; r > 1 \\ i.e., F(r^\sigma) &< \bar{F}(r) < F(2r) \end{aligned} \tag{6.59}$$

and so

$$\limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(r^\sigma)}{\log[re^{L(r)}]} \leq \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}\bar{F}(r)}{\log[re^{L(r)}]} \leq \limsup_{r\to\infty} \frac{\log^{[t]} G^{-1}F(2r)}{\log[re^{L(r)}]}$$

$$i.e., \sigma \cdot {}^{(t)}\rho_g^{L^*}(f) \leq {}^{(t)}\rho_g^{L^*}(f') \leq {}^{(t)}\rho_g^{L^*}(f'). \tag{6.60}$$

Letting $\sigma \to 1 - 0$, we obtain that

$${}^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\rho_g^{L^*}(f').$$

Thus the theorem is proved. ■

In the line of Theorem 6.3.18 we may state the following theorem without poof.

Theorem 6.3.19. *If f and g are any two transcendental entire functions, then*

$${}^{(t)}\rho_{g'}^{L^*}(f) = {}^{(t)}\rho_{g'}^{L^*}(f')$$

where $t = 1, 2, 3, ...$ and f', g' respectively denote the derivatives of f and g.

$- - - - -X - - - - -$