Chapter 5

ON ENTIRE FUNCTIONS OF *L*-BOUNDED INDEX AND OF NON UNIFORM *L*-BOUNDED INDEX

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5.1 Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. Lepson [23] introduced the concept of a function of bounded index. Gross [19] weakened the suppositions and introduced the idea of functions of non uniform bounded index. Later Lakshminarasimhan [24] worked on the functions of L-bounded index where $L = L(r)$ is a positive continuous function increasing slowly in the sense of 'Karamata' i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. In order to extend some of the properties of entire functions of bounded index to L-bounded index , the notions of L-order and L-type for entire functions are very much essential. Somasundaram and Thamizharasi [44] introduced the notions of L-order and L-type for entire functions. With the help of these notions we establish some theorems on the comparative growth properties of composite entire functions whose left or right factors are either of L-bounded index or of non uniform L-bounded index. In the chapter we also prove a few results based on the relationship between a slowly changing function $L(r)$ and $T(r, fog)$ where g is either an entire function of L-bounded index or of non uniform L-bounded index.

In the sequel we use the following notations:

(i) $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$ and

The results of this chapter have been published in **Wesleyan Journal of Research,** see **[11]** and **Journal of Mathematics** see {**[12]**, **[13]**}

(ii) $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\exp^{[0]} x = x$.

To start our chapter we require the following definitions.

Definition 5.1.1. *[24] An entire function f is said to be of L-bounded index if there exists a positive integer M such that*

$$\max_{0 \le i \le M} \{\frac{L(i+2)}{i!} \left| f^{(i)} \right| \} \ge \frac{L(j+2)}{j!} \left| f^{(j)} \right|$$

for all $z \in \mathbb{C}$ and $j = 0, 1, 2, 3, \ldots$

The least value N_L of the integer M for which the above inequality holds is called the L-index of f.

Definition 5.1.2. *[29] An entire function f is said to be of non uniform L-bounded index if there exists N and $N_j (j = 0, 1, 2, 3, \ldots)$ such that*

$$\frac{L(j+2)}{j!} \left| f^{(j)} \right| \ge \sum_{i=0}^{N} \frac{L(i+2)}{i!} \left| f^{(i)} \right|$$

for $|z| > N_j$ and $j = 0, 1, 2, 3, \ldots$

The least integer N for which the above inequality holds is called the non uniform L-index of f and is denoted by N_L^*.

From the definitions it is clear that if f is of L-bounded index then it is of non uniform L-bounded index.

Definition 5.1.3. *[38] A positive continuous function $L(r)$ is called a 'slowly changing function' if for $\varepsilon > 0$,*

$$\frac{1}{k^{\varepsilon}} \le \frac{L(kr)}{L(r)} \le k^{\varepsilon}$$

for $r \ge r(\varepsilon)$ and uniformly for $k(\ge 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = 0.$$

Definition 5.1.4. *The order ρ_f and lower order λ_f of an entire function f is defined as*

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 5.1.5. *The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of an entire function f is defined as*

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \quad and \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 5.1.6. *The type σ_f of an entire function f is defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Definition 5.1.7. *[44] The L-order ρ_f^L and L-lower order λ_f^L of an entire function f is defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \quad and \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

If f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]} \quad and \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}.$$

Definition 5.1.8. *[44] The L-type σ_f^L of an entire function f with L-order ρ_f^L is defined as*

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log M(r,f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

If f is meromorphic, then

$$\sigma_f^L = \limsup_{r\to\infty} \frac{T(r,f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

The more generalised concept of L-order and L-lower order of entire and meromorphic functions are L^*-order and L^*-lower order. In order to prove our results we also require the following definitions:

Definition 5.1.9. *The L^*-order, L^*-lower order and L^*- type of a meromorphic function f respectively denoted by $\rho_f^{L^*}$, $\lambda_f^{L^*}$, and $\sigma_f^{L^*}$ are defined as follows :*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]}$$

and

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}$$

and

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

Definition 5.1.10. *Let 'a' be a complex number finite or infinite. The Nevanlinna deficiency $\delta(a;f)$ of 'a' with respect to an entire function f is defined as*

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}.$$

Definition 5.1.11. *[31] Let f be an entire function of order zero. Then the quantities ρ_f^*, λ_f^* and $\bar{\rho}_f^*$, $\bar{\lambda}_f^*$ are defined as follows*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}$$

and

$$\bar{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}, \quad \bar{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}.$$

For meromorphic f, one can easily verify that

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}$$

and

$$\bar{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}, \quad \bar{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}.$$

Let f and g be two entire functions and $F(r) \equiv M(r,f) = \max\{|f(z)| : |z| = r\}$, $G(r) \equiv M(r,g) = \max\{|g(z)| : |z| = r\}$. If f is non constant then $F(r)$ is strictly increasing and continuous and its inverse $F^{-1} : (|f(0)|, \infty) \to (0,\infty)$ exists and is such that

$$\lim_{s\to\infty} F^{-1}(s) = \infty.$$

Bernal [2] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$, as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : F(r) < G(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.\end{aligned}$$

Similarly one may define the relative lower order of f with respect to g, denoted by $\lambda_g(f)$ in the following manner

$$\begin{aligned}\lambda_g(f) &= \sup\{\mu' > 0 : F(r) > G(r^{\mu'}) \text{ for all } r > r_0(\mu') > 0\} \\ &= \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.\end{aligned}$$

The above two definitions coincide with the classical definitions of order and lower order if $g(z) = \exp z$ [46].

In the chapter we introduce the definition of relative L−order and relative L−lower order of f with respect to g where f and g are both entire functions and study some of their properties.

Definition 5.1.12. *Two entire functions f and g are said to be asymtotically equivalent if there exists l, $0 < l < \infty$ such that*

$$\frac{F(r)}{G(r)} \to l \quad \text{as } r \to \infty,$$

and in this case we write $f \sim g$.

If $f \sim g$, then clearly $g \sim f$.

Definition 5.1.13. *The relative L−order $\rho_g^L(f)$ and relative L−lower order $\lambda_g^L(f)$ of an entire function f with respect to an entire function g are defined as*

$$\begin{aligned}\rho_g^L(f) &= \inf\{\mu > 0 : F(r) < G([rL(r)]^{\mu}) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]}\end{aligned}$$

and

$$\begin{aligned}\lambda_g^L(f) &= \sup\{\mu' > 0 : F(r) > G([rL(r)]^{\mu'}) \text{ for all } r > r_0(\mu') > 0\} \\ &= \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]}.\end{aligned}$$

Throughout the chapter we shall assume; f, g, h etc. to be non-constant and if they are entire then $F(r)$, $G(r)$, $H(r)$ etc. denote respectively their maximum modulus on $|z| = r$.

Let f and g be two non constant meromorphic functions defined in the open complex plane $\mathbb{C}$ and let $a \in \mathbb{C} \cup \{\infty\}$. If $f - a$ and $g - a$ have the same zeros CM (counting multiplicities) and IM (ignoring multiplicities) then we say that f and g share the value a CM or IM respectively. Similarly f, g share ∞ CM or IM means that $\frac{1}{f}, \frac{1}{g}$ share 0 CM or IM respectively.

I. Lahiri [33] initiated the idea of weighted sharing of values for meromorphic functions. He [33] introduced the following two definitions:

Definition 5.1.14. *[33] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$.*

Definition 5.1.15. *[33] Let k be a nonnegative integer or infinity. If for $a \in \mathbb{C} \cup \{\infty\}$, $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

It is evident that if f, g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity $m\,(\leq k)$ if and only if it is a zero of $g - a$ of multiplicity $m\,(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m\,(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n\,(> k)$ where m is not necessarily equal to n. Also f, g share (a, k) means that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p \leq k$. Again we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively. If f and g are two non-constant entire functions then f, g share (∞, ∞).

In the chapter we also establish some results on the relationship between L-orders (L-lower orders, L-types) and L^*-orders (L^*-lower orders, L^*-types) of two entire functions based on the idea of sharing of values of them.

Banerjee and Dutta [6] introduced the following definition based on the idea of relative sharing of values of two meromorphic functions with respect to another meromorphic function.

Definition 5.1.16. *[6] Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We say that f, g share a CM (IM) relatively with respect to a meromorphic function h, provided the functions F and G share a CM (IM) where $F = \frac{f}{h}$ and $G = \frac{g}{h}$.*

The purpose of this definition of relative sharing of values of two meromorphic functions f and g in the chapter is to study some properties of f and g by that of F and G constructed with the help of a suitably chosen meromorphic function h.

5.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 5.2.1. *[7] If f and g are entire functions then for all sufficiently large values of r,*

$$M(r, fog) \leq M(M(r, g), f).$$

Lemma 5.2.2. *[25] Let g be an entire function with $\lambda_g < \infty$. Also let a_i $(i = 1, 2, ..., n; n \leq \infty)$ are entire functions satisfying (i) $T(r, a_i) = o\{T(r, g)\}$ and (ii) $\sum_{i=1}^{n} \delta(a_i; g) = 1$. Then*

$$\lim_{r\to\infty} \frac{T(r, g)}{\log M(r, g)} = \frac{1}{\pi}.$$

Lemma 5.2.3. *[24] If f be an entire function of L-bounded index with L-index N_L, then*

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{[rL(r)]} \leq \frac{(N_L + 1)\ L(N_L + 2)}{L(N_L + 3)}.$$

Lemma 5.2.4. *[44] If an entire function f be of L-bounded index then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]} \leq 1.$$

Lemma 5.2.5. *[29] Let f have at most a finite number of zeros. Then f is of non uniform L-bounded index if and only if f is of L-bounded index.*

Lemma 5.2.6. *[29] Let f be an entire function of non uniform L-bounded index then the order of f cannot exceed unity.*

Lemma 5.2.7. *[1] If f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, fog) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

Lemma 5.2.8. *[36] Let f and g be two entire functions. If*

$$M(r, g) > \frac{(2 + \varepsilon)}{\varepsilon} |g(0)| \quad \text{for any } \varepsilon > 0,$$

then

$$T(r, fog) \leq (1 + \varepsilon) T(M(r, g), f)$$

for all $r > 0$.

Lemma 5.2.9. *[20] Let f be meromorphic and g be transcendental entire. If $\rho_{fog} < \infty$ then $\rho_f = 0$.*

Lemma 5.2.10. *[2] Let f be entire and $\alpha > 1$, $0 < \beta < \alpha$. Then $F(\alpha r) > \beta F(r)$ for all large r.*

Lemma 5.2.11. *[6] Let f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) where $(k_2 - 1)(k_1 k_2 - 1) > (1 + k_2)^2$. Then*

$$T(r, f) \leq (2 + O(1)) T(r, g)$$

and

$$T(r, g) \leq (2 + O(1)) T(r, f).$$

Lemma 5.2.12. *{[21], [6]} If f and g share three values IM then*

$$\frac{1}{3} T(r, g)(1 + o(1)) \leq T(r, f) \leq 3T(r, g)(1 + o(1))$$

as $r \to \infty$ possibly outside a set E of finite linear measure.

5.3 Theorems.

In this section we present the main results of the chapter.

Theorem 5.3.1. *Let f and g be two entire functions such that $0 < \lambda_{fog}^L \leq \rho_{fog}^L < \infty$ and $0 < \lambda_g^L \leq \rho_g^L < \infty$. Then for any positive number A,*

$$\begin{aligned} \frac{\lambda_{fog}^L}{A\rho_g^L} &\leq \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \leq \frac{\lambda_{fog}^L}{A\lambda_g^L} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \leq \frac{\rho_{fog}^L}{A\lambda_g^L}. \end{aligned}$$

Proof. From the definition of L-order and L-lower order we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[2]} M(r, fog) \geq (\lambda_{fog}^L - \varepsilon) \log[rL(r)] \tag{5.1}$$

and

$$\log^{[2]} M(r^A, g) \leq A(\rho_g^L + \varepsilon) \log[rL(r)]. \tag{5.2}$$

Now from (5.1) and (5.2) it follows for all sufficiently large values of r,

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \geq \frac{(\lambda_{fog}^L - \varepsilon)}{A(\rho_g^L + \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \geq \frac{\lambda_{fog}^L}{A\rho_g^L}. \tag{5.3}$$

Again for a sequence of values of r tending to infinity

$$\log^{[2]} M(r, fog) \leq (\lambda_{fog}^L + \varepsilon) \log[rL(r)] \tag{5.4}$$

and for all large values of r,

$$\log^{[2]} M(r^A, g) \geq A(\lambda_g^L - \varepsilon) \log[rL(r)]. \tag{5.5}$$

So combining (5.4) and (5.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \leq \frac{\lambda_{fog}^L + \varepsilon}{A(\lambda_g^L - \varepsilon)}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that,

$$\liminf_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,g)}\leq\frac{\lambda_{fog}^L}{A\lambda_g^L}. \tag{5.6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[2]}M(r^A,g)\leq A(\lambda_g^L+\varepsilon)\log[rL(r)]. \tag{5.7}$$

Now from (5.1) and (5.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,g)}\geq\frac{\lambda_{fog}^L-\varepsilon}{A(\lambda_g^L+\varepsilon)}.$$

Choosing $\varepsilon\to 0$ we get

$$\limsup_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,g)}\geq\frac{\lambda_{fog}^L}{A\lambda_g^L}. \tag{5.8}$$

Also for all large values of r,

$$\log^{[2]}M(r,fog)\leq(\rho_{fog}^L+\varepsilon)\log[rL(r)]. \tag{5.9}$$

So from (5.5) and (5.9) it follows for all sufficiently large values of r,

$$\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,g)}\leq\frac{\rho_{fog}^L+\varepsilon}{A(\lambda_g^L-\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain

$$\limsup_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,g)}\leq\frac{\rho_{fog}^L}{A\lambda_g^L}. \tag{5.10}$$

Thus the theorem follows from (5.3), (5.6), (5.8) and (5.10). ■

Remark 5.3.1. *The sign '$\leq$' cannot be replaced by '$<$' only in Theorem 5.3.1 as we see in the following example.*

Example 5.3.1. *Let $f=z$, $g=\exp z$, $A=1$ and $L(r)=\frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. So*

$$\begin{aligned}\rho_{fog}^L &= \limsup_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log[rL(r)]}\\ &= \limsup_{r\to\infty}\frac{\log^{[2]}\exp r}{\log[r\frac{1}{p}\exp(\frac{1}{r})]}\\ &= \limsup_{r\to\infty}\frac{\log r}{\log r+\log\frac{1}{p}+\frac{1}{r}}=1.\end{aligned}$$

Similarly, $\lambda^L_{fog} = 1$. *Again*

$$\begin{aligned}\rho^L_g &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[rL(r)]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log[r\frac{1}{p}\exp(\frac{1}{r})]} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \log\frac{1}{p} + \frac{1}{r}} = 1.\end{aligned}$$

Similarly, $\lambda^L_g = 1$. *So*

$$\rho^L_{fog} = \lambda^L_{fog} = \rho^L_g = \lambda^L_g = 1.$$

Also

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1 \\ \text{and} \quad \liminf_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} &= \liminf_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1.\end{aligned}$$

Threrefore

$$\begin{aligned}\frac{\lambda^L_{fog}}{A\rho^L_g} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} = \frac{\lambda^L_{fog}}{A\lambda^L_g} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} = \frac{\rho^L_{fog}}{A\lambda^L_g}.\end{aligned}$$

Theorem 5.3.2. *If f and g be two entire functions with $0 < \lambda^{L^*}_{fog} \leq \rho^{L^*}_{fog} < \infty$ and $0 < \lambda^{L^*}_g \leq \rho^{L^*}_g < \infty$ then for any positive number A,*

$$\begin{aligned}\frac{\lambda^{L^*}_{fog}}{A\rho^{L^*}_g} &\leq \liminf_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} \leq \frac{\lambda^{L^*}_{fog}}{A\lambda^{L^*}_g} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,g)} \leq \frac{\rho^{L^*}_{fog}}{A\lambda^{L^*}_g}.\end{aligned}$$

We omit the proof of Theorem 5.3.2 because it can be carried out in the line of Theorem 5.3.1.

Remark 5.3.2. *The sign '$\leq$' cannot be replaced by '$<$' only in Theorem 5.3.2 which is evident from the following example.*

Example 5.3.2. *If $f = z$, $g = \exp z$, $A = 1$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number then*

$$\begin{aligned}\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log[re^{L(r)}]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log r + L(r)} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \frac{1}{p}\exp(\frac{1}{r})} = 1.\end{aligned}$$

Similarly, $\lambda_{fog}^{L^*} = 1$. Again

$$\begin{aligned}\rho_{g}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log[re^{L(r)}]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log r + L(r)} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \frac{1}{p}\exp(\frac{1}{r})} = 1.\end{aligned}$$

Similarly, $\lambda_{g}^{L^*} = 1$. So

$$\rho_{fog}^{L^*} = \lambda_{fog}^{L^*} = \rho_{g}^{L^*} = \lambda_{g}^{L^*} = 1.$$

Also

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1 \\ \text{and} \quad \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} &= \liminf_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1.\end{aligned}$$

Threrefore

$$\begin{aligned}\frac{\lambda_{fog}^{L^*}}{A\rho_{g}^{L^*}} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} = \frac{\lambda_{fog}^{L^*}}{A\lambda_{g}^{L^*}} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} = \frac{\rho_{fog}^{L^*}}{A\lambda_{g}^{L^*}}.\end{aligned}$$

Theorem 5.3.3. *Let f and g be two entire functions such that $0 < \lambda_{fog}^{L} \le \rho_{fog}^{L} < \infty$ and $0 < \rho_{g}^{L} < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \le \frac{\rho_{fog}^{L}}{A\rho_{g}^{L}} \le \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)}.$$

Proof. From the definition of L-order we get for a sequence of values of r tending to infinity,

$$\log^{[2]} M(r^A, g) \ge A(\rho_{g}^{L} - \varepsilon) \log[rL(r)]. \tag{5.11}$$

Now from (5.9) and (5.11) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \le \frac{\rho_{fog}^{L} + \varepsilon}{A(\rho_{g}^{L} - \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \le \frac{\rho_{fog}^{L}}{A\rho_{g}^{L}}. \tag{5.12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[2]} M(r, fog) \ge (\rho_{fog}^{L} - \varepsilon) \log[rL(r)]. \tag{5.13}$$

So combining (5.2) and (5.13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \ge \frac{\rho_{fog}^{L} - \varepsilon}{A(\rho_{g}^{L} + \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \ge \frac{\rho_{fog}^{L}}{A\rho_{g}^{L}}. \tag{5.14}$$

Thus the theorem follows from (5.12) and (5.14). ■

Remark 5.3.3. *The sign '$\le$' cannot be replaced by '$<$' only in Theorem 5.3.3 as we see in the following example.*

Example 5.3.3. *Let $f = z$, $g = \exp z$, $A = 1$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$, where p is any positive real number. Then*

$$\begin{aligned} \rho^L_{fog} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log[rL(r)]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log[r\frac{1}{p}\exp(\frac{1}{r})]} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \log\frac{1}{p} + \frac{1}{r}} = 1. \end{aligned}$$

Similarly, $\lambda^L_{fog} = 1$. Again

$$\begin{aligned} \rho^L_{g} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log[rL(r)]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log[r\frac{1}{p}\exp(\frac{1}{r})]} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \log\frac{1}{p} + \frac{1}{r}} = 1. \end{aligned}$$

Similarly, $\lambda^L_g = 1$. So

$$\rho^L_{fog} = \lambda^L_{fog} = \rho^L_g = \lambda^L_g = 1.$$

Also

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1 \\ \text{and} \quad \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} &= \liminf_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1. \end{aligned}$$

Threrefore

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} = \frac{\rho^L_{fog}}{A\rho^L_g} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)}.$$

In the line of the Theorem 5.3.3 we may prove the following theorem.

Theorem 5.3.4. *If f and g be two entire functions such that $0 < \lambda_{fog}^{L^*} \leq \rho_{fog}^{L^*} < \infty$ and $0 < \rho_g^{L^*} < \infty$, then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \leq \frac{\rho_{fog}^{L^*}}{A\rho_g^{L^*}} \leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)}.$$

Remark 5.3.4. *The sign '$\leq$' cannot be replaced by '$<$' only in Theorem 5.3.4 as we see in the following example.*

Example 5.3.4. *If $f = z$, $g = \exp z$, $A = 1$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number then*

$$\rho_{fog}^{L^*} = \lambda_{fog}^{L^*} = \rho_g^{L^*} = \lambda_g^{L^*} = 1.$$

Also

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} = \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1$$

$$\text{and} \quad \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} = \liminf_{r\to\infty} \frac{\log^{[2]} \exp r}{\log^{[2]} \exp r} = 1.$$

Threrefore,

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} = \frac{\rho_{fog}^{L^*}}{A\rho_g^{L^*}} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)}.$$

Theorem 5.3.5. *Let f and g be two entire functions satisfying (i) $0 < \rho_g^L < \infty$, (ii) $0 < \sigma_g^L < \infty$, (iii) $\rho_{fog}^L = \rho_g^L$ and (iv) $0 < \sigma_{fog}^L < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log M(r, fog)}{\log M(r, g)} \leq \frac{\sigma_{fog}^L}{\sigma_g^L} \leq \limsup_{r\to\infty} \frac{\log M(r, fog)}{\log M(r, g)}.$$

Proof. From the definition of L-type of a composite entire function we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log M(r, fog) \leq (\sigma_{fog}^L + \varepsilon)[rL(r)]^{\rho_{fog}^L}. \tag{5.15}$$

Also for a sequence of values of r tending to infinity,

$$\log M(r, g) \geq (\sigma_g^L - \varepsilon)[rL(r)]^{\rho_g^L}. \tag{5.16}$$

As $\rho_{fog}^L = \rho_g^L$ from (5.15) and (5.16) it follows for a sequence of values of r tending to infinity,

$$\frac{\log M(r, fog)}{\log M(r, g)} \leq \frac{\sigma_{fog}^L + \varepsilon}{\sigma_g^L - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r \to \infty} \frac{\log M(r, fog)}{\log M(r, g)} \leq \frac{\sigma_{fog}^L}{\sigma_g^L}. \qquad (5.17)$$

Again for a sequence of values of r tending to infinity,

$$\log M(r, fog) \geq (\sigma_{fog}^L - \varepsilon)[rL(r)]^{\rho_{fog}^L} \qquad (5.18)$$

and for all sufficiently large values of r,

$$\log M(r, g) \leq (\sigma_g^L + \varepsilon)[rL(r)]^{\rho_g^L}. \qquad (5.19)$$

By condition (iii) we obtain from (5.18) and (5.19) for a sequence of values of r tending to infinity,

$$\frac{\log M(r, fog)}{\log M(r, g)} \geq \frac{\sigma_{fog}^L - \varepsilon}{\sigma_g^L + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, we get that

$$\limsup_{r \to \infty} \frac{\log M(r, fog)}{\log M(r, g)} \geq \frac{\sigma_{fog}^L}{\sigma_g^L}. \qquad (5.20)$$

Thus the theorem follows from (5.17) and (5.20). ■

Remark 5.3.5. *The sign '$\leq$' in Theorem 5.3.5 cannot be replaced by '$<$' only as we verify in the following example.*

Example 5.3.5. *Let $f = z$, $g = \exp z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. So*

$$\begin{aligned}
\rho_{fog}^L &= \limsup_{r \to \infty} \frac{\log^{[2]} M(r, fog)}{\log[rL(r)]} \\
&= \limsup_{r \to \infty} \frac{\log^{[2]} \exp r}{\log[r\frac{1}{p}\exp(\frac{1}{r})]} \\
&= \limsup_{r \to \infty} \frac{\log r}{\log r + \log\frac{1}{p} + \frac{1}{r}} = 1.
\end{aligned}$$

and

$$\begin{aligned}\rho_g^L &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[rL(r)]} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} \exp r}{\log[r\frac{1}{p}\exp(\frac{1}{r})]} \\ &= \limsup_{r\to\infty} \frac{\log r}{\log r + \log\frac{1}{p} + \frac{1}{r}} = 1.\end{aligned}$$

Thus

$$\rho_{fog}^L = \rho_g^L = 1.$$

Also

$$\begin{aligned}\sigma_{fog}^L &= \limsup_{r\to\infty} \frac{\log M(r,fog)}{[rL(r)]^{\rho_{fog}^L}} \\ &= \limsup_{r\to\infty} \frac{\log(\exp r)}{[r\frac{1}{p}\exp(\frac{1}{r})]} \\ &= \limsup_{r\to\infty} \frac{r}{[r\frac{1}{p}\exp(\frac{1}{r})]} = p.\end{aligned}$$

and

$$\begin{aligned}\sigma_g^L &= \limsup_{r\to\infty} \frac{\log M(r,g)}{[rL(r)]^{\rho_g^L}} \\ &= \limsup_{r\to\infty} \frac{\log(\exp r)}{[r\frac{1}{p}\exp(\frac{1}{r})]} \\ &= \limsup_{r\to\infty} \frac{r}{[r\frac{1}{p}\exp(\frac{1}{r})]} = p.\end{aligned}$$

Again

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log M(r,fog)}{\log M(r,g)} &= \limsup_{r\to\infty} \frac{\log(\exp r)}{\log(\exp r)} = 1 \\ \text{and} \quad \liminf_{r\to\infty} \frac{\log M(r,fog)}{\log M(r,g)} &= \liminf_{r\to\infty} \frac{\log(\exp r)}{\log(\exp r)} = 1.\end{aligned}$$

Hence

$$\liminf_{r\to\infty} \frac{\log M(r,fog)}{\log M(r,g)} = \frac{\sigma_{fog}^L}{\sigma_g^L} = 1 = \limsup_{r\to\infty} \frac{\log M(r,fog)}{\log M(r,g)}.$$

Theorem 5.3.6. *If f and g be two entire functions satisfying (i) $0 < \rho_g^{L^*} < \infty$, (ii) $0 < \sigma_g^{L^*} < \infty$, (iii) $\rho_{fog}^{L^*} = \rho_g^{L^*}$ and (iv) $0 < \sigma_{fog}^{L^*} < \infty$ then*

$$\liminf_{r\to\infty} \frac{\log M(r,fog)}{\log M(r,g)} \le \frac{\sigma_{fog}^{L^*}}{\sigma_g^{L^*}} \le \limsup_{r\to\infty} \frac{\log M(r,fog)}{\log M(r,g)}.$$

The proof of Theorem 5.3.6 is omitted because it can be carried out in the line of Theorem 5.3.5.

Remark 5.3.6. *The sign '$\le$' in Theorem 5.3.6 cannot be replaced by '$<$' only as we see in the following example.*

Example 5.3.6. *If $f = z$, $g = \exp z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number then*

$$\rho_{fog}^{L^*} = \rho_g^{L^*} = 1 \quad \text{and} \quad \sigma_{fog}^{L^*} = \sigma_g^{L^*} = \frac{1}{e^{\frac{1}{p}}}.$$

Hence

$$\liminf_{r\to\infty} \frac{\log M(r,fog)}{\log M(r,g)} = \frac{\sigma_{fog}^{L^*}}{\sigma_g^{L^*}} = 1 = \limsup_{r\to\infty} \frac{\log M(r,fog)}{\log M(r,g)}.$$

Theorem 5.3.7. *Let f and g be any two entire functions both of L-bounded index such that ρ_f, ρ_g are both finite and $\bar{\lambda}_f$ is positive. Also let $\bar{\lambda}_f > \rho_g$. Then*

$$\lim_{r\to\infty} \frac{\log T(r,fog)\{\log T(\log r,fog) + \log^{[2]} M(\log r,g)\}}{\{\log[rL(r)]\}^2} = 0.$$

Proof. From Lemma 5.2.7 we see that

$$T(r,fog)\log M(r,g) \le \{1+o(1)\}T(r,g)T(M(r,g),f).$$

which gives for all sufficiently large values of r,

$$\begin{aligned} &\log T(r,fog) + \log^{[2]} M(r,g) \\ \le\ & \log\{1+o(1)\} + (\rho_g+\varepsilon)\log r + (\rho_f+\varepsilon)r^{(\rho_g+\varepsilon)} \\ =\ & r^{(\rho_g+\varepsilon)}\{\rho_f+\varepsilon+o(1)\}, \end{aligned}$$

$$\begin{aligned} &\text{i.e.,} \log T(\log r,fog) + \log^{[2]} M(\log r,g) \\ \le\ & (\log r)^{(\rho_g+\varepsilon)}\{\rho_f+\varepsilon+o(1)\}. \end{aligned} \tag{5.21}$$

Also for all sufficiently large values of r, we obtain that

$$\log^{[2]} M(r,g) \geq (\lambda_g - \varepsilon)\log r. \tag{5.22}$$

Since g is of L-bounded index so it is of non uniform L-bounded index. Also by Lemma 5.2.6, $\rho_g < 1$. Therefore we can choose $\varepsilon(>0)$ in such a way that

$$\rho_g + \varepsilon < 1. \tag{5.23}$$

Thus from (5.21), (5.22) and (5.23) it follows that,

$$\lim_{r\to\infty} \frac{\{\log T(\log r, fog) + \log^{[2]} M(\log r, g)\}}{\log^{[2]} M(r,g)} = 0. \tag{5.24}$$

Now by (5.24) and Lemma 5.2.4 we obtain that

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\{\log T(\log r, fog) + \log^{[2]} M(\log r, g)\}}{\log[rL(r)]} \\
= &\lim_{r\to\infty} \frac{\{\log T(\log r, fog) + \log^{[2]} M(\log r, g)\}}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[rL(r)]} \\
\leq &\ 0.1 = 0.
\end{aligned}$$

$$\text{i.e., } \lim_{r\to\infty} \frac{\{\log T(\log r, fog) + \log^{[2]} M(\log r, g)\}}{\log[rL(r)]} = 0. \tag{5.25}$$

For $\varepsilon = 1$ we obtain from Lemma 5.2.8 that for all large values of r,

$$\begin{aligned}
T(r, fog) &\leq 2T(M(r,g), f) \\
\text{i.e.,} \log T(r, fog) &\leq \log T(M(r,g), f) + O(1) \\
\text{i.e.,} \log T(r, fog) &\leq (\rho_f + \varepsilon)\log M(r,g) + O(1) \\
\text{i.e.,} \log T(r, fog) &\leq (\rho_f + \varepsilon) r^{(\rho_g + \varepsilon)} + O(1). \qquad (5.26)
\end{aligned}$$

Again for all sufficiently large values of r,

$$\begin{aligned}
\log^{[3]} M(r,f) &\geq (\bar{\lambda}_f - \varepsilon)\log r \\
\text{i.e.,} \log^{[2]} M(r,f) &\geq r^{(\bar{\lambda}_f - \varepsilon)}. \qquad (5.27)
\end{aligned}$$

Since $\bar{\lambda}_f > \rho_g$ we can choose $\varepsilon(>0)$ such that

$$\bar{\lambda}_f - \varepsilon > \rho_g + \varepsilon. \tag{5.28}$$

Now from (5.26), (5.27) and (5.28) we get

$$\lim_{r\to\infty}\frac{\log T(r,fog)}{\log^{[2]}M(r,f)}=0. \tag{5.29}$$

From (5.29) and Lemma 5.2.4 it follows that

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log[rL(r)]} &= \lim_{r\to\infty}\frac{\log T(r,fog)}{\log^{[2]}M(r,f)}.\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log[rL(r)]}\\ &\leq 0.1=0.\end{aligned}$$

$$\text{i.e., }\lim_{r\to\infty}\frac{\log T(r,fog)}{\log[rL(r)]}=0. \tag{5.30}$$

Thus the theorem follows from (5.25) and (5.30). ■

Theorem 5.3.8. *Let f be a non-constant entire function such that $\rho_f<\infty$ and g be entire of L-bounded index with L-index N_L. Also suppose that there exist entire functions $a_i(i=1,2,...n;n\leq\infty)$ such that (i) $T(r,a_i)=o\{T(r,g)\}$ as $r\to\infty$ for $i=1,2,...n$ and (ii) $\sum_{i=1}^{n}\delta(a_i;g)=1$. Then*

$$\limsup_{r\to\infty}\frac{\log T(r,fog).\log^{[2]}T(r,fog)}{[rL(r)].\log[rL(r)]}\leq\rho_f.\frac{(N_L+1)\ L(N_L+2)}{L(N_L+3)}.$$

Proof. Since g is of L-bounded index therefore it is of non uniform L-bounded index. Hence by Lemma 5.2.6, $\rho_g\leq 1$ i.e.,$\lambda_g<\infty$.

Since for an entire function f, $T(r,f)\leq\log^{+}M(r,f)$ by Lemma 5.2.1 we obtain for all sufficiently large values of r and given $\varepsilon(>0)$,

$$\begin{aligned}T(r,fog) &\leq \log M(M(r,g),f)\leq\{M(r,g)\}^{\rho_f+\varepsilon}\\ \text{i.e., }\log T(r,fog) &\leq (\rho_f+\varepsilon)\log M(r,g)\\ \text{i.e., }\limsup_{r\to\infty}\frac{\log T(r,fog)}{T(r,g)} &\leq (\rho_f+\varepsilon)\limsup_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}. \end{aligned} \tag{5.31}$$

Since $\varepsilon(>0)$ is arbitrary by Lemma 5.2.2 it follows from (5.31) that

$$\limsup_{r\to\infty}\frac{\log T(r,fog)}{T(r,g)}\leq\pi\rho_f. \tag{5.32}$$

Now using (5.32) we obtain in view of Lemma 5.2.3

$$\begin{aligned}
&\limsup_{r\to\infty}\frac{\log T(r,fog)}{rL(r)}\\
=\ &\limsup_{r\to\infty}\frac{\log T(r,fog)}{T(r,g)}.\lim_{r\to\infty}\frac{T(r,g)}{\log M(r,g)}.\limsup_{r\to\infty}\frac{\log M(r,g)}{rL(r)}\\
\le\ &\pi.\rho_f.\frac{1}{\pi}.\frac{(N_L+1)\ L(N_L+2)}{L(N_L+3)}\\
=\ &\rho_f.\frac{(N_L+1)\ L(N_L+2)}{L(N_L+3)}. \qquad (5.33)
\end{aligned}$$

Again in view of the inequality $T(r,f)\le\log^{+}M(r,f)$ and by Lemma 5.2.1 it follows for all sufficiently large values of r and given $\varepsilon(>0)$,

$$\log^{[2]}T(r,fog)\le\log^{[2]}M(r,g)+O(1). \qquad (5.34)$$

Now from (5.34) we get for all sufficiently large values of r,

$$\frac{\log^{[2]}T(r,fog)}{\log^{[2]}M(r,g)}\le 1+\frac{O(1)}{\log^{[2]}M(r,g)}$$

$$\text{i.e., }\limsup_{r\to\infty}\frac{\log^{[2]}T(r,fog)}{\log^{[2]}M(r,g)}\le 1. \qquad (5.35)$$

So from (5.35) and Lemma 5.2.4 we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[2]}T(r,fog)}{\log[rL(r)]}\le 1. \qquad (5.36)$$

Thus the theorem follows from (5.33) and (5.36). ■

Corollary 5.3.1. *Let f be a non-constant entire function of finite order and g be an entire function of non uniform L-bounded index. Also let there exist entire functions $a_i(i=1,2,...n;n\le\infty)$ such that (i) $T(r,a_i)=o\{T(r,g)\}$ as $r\to\infty$ for $i=1,2,...n$,*

(ii) $\sum\limits_{i=1}^{n}\delta(a_i;g)=1$ and (iii) g has at most a finite number of zeros. Then

$$\limsup_{r\to\infty}\frac{\log T(r,fog).\log^{[2]}T(r,fog)}{[rL(r)].\log[rL(r)]}\le\rho_f.\frac{(N_L+1)\ L(N_L+2)}{L(N_L+3)}.$$

Proof. Since g has at most a finite number of zeros, by Lemma 5.2.5, g is of L-bounded index. Thus the corollary follows from Theorem 5.3.8. ■

Theorem 5.3.9. *Let f, g be both entire with $\rho_{fog} < \infty$, $\rho_f^* < \infty$ and $\lambda_g > 0$. Also suppose that there exist entire functions $a_i (i = 1, 2, ...n; n \leq \infty)$ such that (i) $T(r, a_i) = o\{T(r, g)\}$ as $r \to \infty$ for $i = 1, 2, ...n$ and (ii) $\sum_{i=1}^{n} \delta(a_i; g) = 1$.*

If g be of L- bounded index with L-index N_L then

$$\limsup_{r\to\infty} \frac{\{\log T(r, fog)\}^2}{\log[rL(r)]. \log T(r, g)} \leq \frac{\rho_{fog}.\rho_f^*}{\lambda_g}.$$

Proof. Since g be of L- bounded index, it is of non uniform L-bounded index. Hence by Lemma 5.2.6

$$\rho_g \leq 1 < \infty, \text{ i.e., } \lambda_g < \infty.$$

From the definition of order and lower order we have for arbitrary positive ε and for all large values of r,

$$\log T(r, fog) \leq (\rho_{fog} + \varepsilon) \log r \tag{5.37}$$

and

$$\log^{[2]} M(r, g) \geq (\lambda_g - \varepsilon) \log r. \tag{5.38}$$

Now from (5.37) and (5.38) it follows for all large values of r,

$$\frac{\log T(r, fog)}{\log^{[2]} M(r, g)} \leq \frac{(\rho_{fog} + \varepsilon)}{(\lambda_g - \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log^{[2]} M(r, g)} \leq \frac{\rho_{fog}}{\lambda_g}. \tag{5.39}$$

Now by using (5.39) and Lemma 5.2.4 it follows that

$$\begin{aligned} &\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log[rL(r)]} \\ \leq\ &\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log^{[2]} M(r, g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log[rL(r)]} \\ \leq\ &\frac{\rho_{fog}}{\lambda_g}.1 = \frac{\rho_{fog}}{\lambda_g}. \end{aligned} \tag{5.40}$$

Since $\rho_{fog} < \infty$ and $0 < \lambda_g < \infty$ i.e., g is transcendental, then by Lemma 5.2.9 we get $\rho_f = 0$.

Now by Lemma 5.2.7 and the inequality $T(r, g) \leq \log^+ M(r, g)$, we get for all sufficiently large values of r,

$$\begin{aligned} T(r, fog) &\leq \{1 + o(1)\} T(M(r, g), f) \\ \text{i.e., } T(r, fog) &\leq \{1 + o(1)\}\{\log M(r, g)\}^{(\rho_f^* + \varepsilon)} \\ \text{i.e., } \log T(r, fog) &\leq o(1) + (\rho_f^* + \varepsilon) \log^{[2]} M(r, g) \\ \text{i.e., } \frac{\log T(r, fog)}{\log T(r, g)} &\leq \frac{o(1) + (\rho_f^* + \varepsilon) \log^{[2]} M(r, g)}{\log T(r, g)}. \end{aligned} \tag{5.41}$$

By Lemma 5.2.2, $\lim_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log T(r, g)}$ exists and equals to 1.

Now from (5.41) we obtain for all sufficiently large values of r,

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r, g)} \leq (\rho_f^* + \varepsilon). \limsup_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log T(r, g)} \tag{5.42}$$

As $\varepsilon(> 0)$ is arbitrary and

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log T(r, g)} = 1,$$

it follows from (5.42) that

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r, g)} \leq \rho_f^* \tag{5.43}$$

In view of (5.40) and (5.43) we obtain that

$$\begin{aligned} &\limsup_{r\to\infty} \frac{\{\log T(r, fog)\}^2}{\log[rL(r)]. \log T(r, g)} \\ \leq\ &\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log[rL(r)]}. \limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r, g)} \\ \leq\ &\frac{\rho_{fog}}{\lambda_g}.\rho_f^*. \end{aligned}$$

This proves the theorem. ■

Theorem 5.3.10. *Let f, g be both entire functions such that $\rho_{fog} < \infty$ and $\lambda_g > 0$. Also let $a_i (i = 1, 2, ...n; n \leq \infty)$ be entire functions* *such that (i)* $T(r,a_i)=o\{T(r,g)\}$ *as* $r\to\infty$ *for* $i=1,2,...n$ *and (ii)* $\sum_{i=1}^{n}\delta(a_i;g)=1$. *If* g *be of* $L-$ *bounded index with* $L-$*index* N_L *then*

$$\limsup_{r\to\infty}\frac{\log T(r,fog)\log^{[2]}T(r,fog)}{\log[rL(r)].\log T(r,g)}\leq\frac{\bar{\rho}_{fog}.\rho_f^*}{\lambda_g}.$$

Proof. Since $\rho_{fog}<\infty$, therefore $\bar{\rho}_{fog}$ is finite.So from the definition of order and lower order we get for arbitrary positive ε and for all large values of r,

$$\log^{[2]}T(r,fog)\leq(\bar{\rho}_{fog}+\varepsilon)\log r. \tag{5.44}$$

Thus from (5.38) and (5.44) it follows for all large values of r,

$$\frac{\log^{[2]}T(r,fog)}{\log^{[2]}M(r,g)}\leq\frac{(\bar{\rho}_{fog}+\varepsilon)}{(\lambda_g-\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[2]}T(r,fog)}{\log^{[2]}M(r,g)}\leq\frac{\bar{\rho}_{fog}}{\lambda_g} \tag{5.45}$$

Now in view of (5.45) and Lemma 5.2.4 it follows that

$$\begin{aligned}&\limsup_{r\to\infty}\frac{\log^{[2]}T(r,fog)}{\log[rL(r)]}\\ \leq\ &\limsup_{r\to\infty}\frac{\log^{[2]}T(r,fog)}{\log^{[2]}M(r,g)}.\limsup_{r\to\infty}\frac{\log^{[2]}M(r,g)}{\log[rL(r)]}\\ \leq\ &\frac{\bar{\rho}_{fog}}{\lambda_g}.1=\frac{\bar{\rho}_{fog}}{\lambda_g}.\end{aligned} \tag{5.46}$$

Combining (5.43) and (5.46) we get that

$$\begin{aligned}&\limsup_{r\to\infty}\frac{\log T(r,fog)\log^{[2]}T(r,fog)}{\log[rL(r)].\log T(r,g)}\\ \leq\ &\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(r,g)}.\limsup_{r\to\infty}\frac{\log^{[2]}T(r,fog)}{\log[rL(r)]}\\ \leq\ &\rho_f^*.\frac{\bar{\rho}_{fog}}{\lambda_g}=\frac{\bar{\rho}_{fog}\,\rho_f^*}{\lambda_g}.\end{aligned}$$

Thus the theorem is established. ∎

In the next few theorems we will show the equality of relative L-orders (relative L-lower orders) of two entire functions.

Theorem 5.3.11. *Let f, g, h be three entire functions such that $g \sim h$. Then*

$$\rho_g^L(f) = \rho_h^L(f) \quad \textit{and} \quad \lambda_g^L(f) = \lambda_h^L(f).$$

Proof. Let $\varepsilon(>0)$ be choosen arbitrary. By Lemma 5.2.10 it follows for all large values of r,

$$G(r) < (l+\varepsilon)H(r) < H(\alpha r) \tag{5.47}$$

where $0 < l < \infty$, $\alpha > \max\{1, l\}$ is such that $l + \varepsilon < \alpha$.

From (5.47) we obtain for all large r

$$\begin{aligned} r &< G^{-1}(H(\alpha r)) \\ \text{i.e., } \frac{1}{\alpha}H^{-1}(t) &< G^{-1}(t) \quad \text{where } t = H(\alpha r) \\ \text{i.e., } H^{-1}(r) &< \alpha G^{-1}(r). \end{aligned} \tag{5.48}$$

Now by (5.48) we get that

$$\begin{aligned} \rho_h^L(f) &= \limsup_{r\to\infty} \frac{\log[H^{-1}F(r)]}{\log[rL(r)]} \\ &\le \limsup_{r\to\infty} \frac{\log[\alpha G^{-1}F(r)]}{\log[rL(r)]} \\ &= \limsup_{r\to\infty} \frac{\log[G^{-1}F(r)]}{\log[rL(r)]} \\ &= \rho_g^L(f). \end{aligned} \tag{5.49}$$

Since $g \sim h$, therefore $h \sim g$ and in a similar manner one can verify that

$$\rho_g^L(f) \le \rho_h^L(f). \tag{5.50}$$

Thus from (5.49) and(5.50) it follows that

$$\rho_g^L(f) = \rho_h^L(f).$$

Again by (5.48) we obtain that

$$\begin{aligned}
\lambda_h^L(f) &= \liminf_{r\to\infty} \frac{\log[H^{-1}F(r)]}{\log[rL(r)]} \\
&\le \liminf_{r\to\infty} \frac{\log[\alpha G^{-1}F(r)]}{\log[rL(r)]} \\
&= \liminf_{r\to\infty} \frac{\log[G^{-1}F(r)]}{\log[rL(r)]} \\
&= \lambda_g^L(f). \qquad (5.51)
\end{aligned}$$

As $g \sim h$, so $h \sim g$ and in a like manner one can easily check that

$$\lambda_g^L(f) \le \lambda_h^L(f). \qquad (5.52)$$

Therefore in view of (5.51) and(5.52) it follows that

$$\lambda_g^L(f) = \lambda_h^L(f).$$

This proves the theorem. ■

Remark 5.3.7. *The converse of Theorem 5.3.7 is not true which is evident from the following example.*

Example 5.3.7. *Let* $f = g = \exp z$, $h = \exp 2z$ *and* $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ *where* p *is any positive real number. Then*

$$F(r) = G(r) = \exp r \quad \textit{and} \quad H(r) = \exp 2r$$

so that

$$\frac{G(r)}{H(r)} = \frac{\exp r}{\exp 2r} \to 0 \quad \textit{as} \quad r \to \infty.$$

Hence g *not* $\sim h$. *But*

$$\begin{aligned}
\rho_h^L(f) &= \limsup_{r\to\infty} \frac{\log[H^{-1}F(r)]}{\log[rL(r)]} \\
&= \limsup_{r\to\infty} \frac{\log[\frac{1}{2}\log F(r)]}{\log[rL(r)]} \\
&= \limsup_{r\to\infty} \frac{\log^{[2]} F(r) + O(1)}{\log[rL(r)]} \\
&= \limsup_{r\to\infty} \frac{\log^{[2]}(\exp r) + O(1)}{\log[r\frac{1}{p}\exp(\frac{1}{r})]} \\
&= \limsup_{r\to\infty} \frac{\log r + O(1)}{\log r + \frac{1}{r} + O(1)} = 1.
\end{aligned}$$

and

$$\begin{aligned}\rho_g^L(f) &= \limsup_{r\to\infty}\frac{\log[G^{-1}F(r)]}{\log[rL(r)]}\\ &= \limsup_{r\to\infty}\frac{\log[\log F(r)]}{\log[rL(r)]}\\ &= \limsup_{r\to\infty}\frac{\log^{[2]}F(r)}{\log[rL(r)]}\\ &= \limsup_{r\to\infty}\frac{\log^{[2]}(\exp r)}{\log[r\frac{1}{p}\exp(\frac{1}{r})]}\\ &= \limsup_{r\to\infty}\frac{\log r+O(1)}{\log r+\frac{1}{r}+O(1)}=1.\end{aligned}$$

$$i.e.,\ \rho_g^L(f)=\rho_h^L(f).$$

Similarly, we have $\lambda_g^L(f)=\lambda_h^L(f)$.

Theorem 5.3.12. *Let f, g, h be three entire functions such that $g\sim h$. Then*

$$\rho_f^L(g)=\rho_f^L(h)\quad and\quad \lambda_f^L(g)=\lambda_f^L(h).$$

Proof. Since $g\sim h$, in view of Lemma 5.2.10 for $\varepsilon_1>0$,there exists $R_1>0$ such that

$$G(r)<(l+\varepsilon_1)H(r)<H(\beta r)\tag{5.53}$$

where $0<l<\infty$, $r>R_1$ and $\beta>\max\{1,\, l\}$ such that $l+\varepsilon_1<\beta$.

Now from (5.53) we obtain that

$$\begin{aligned}\rho_f^L(g) &= \limsup_{r\to\infty}\frac{\log F^{-1}G(r)}{\log[rL(r)]}\\ &\le \limsup_{r\to\infty}\frac{\log F^{-1}H(\beta r)}{\log[rL(r)]}.\end{aligned}\tag{5.54}$$

For $0<\varepsilon_2<1$ there exists $R_2>0$ such that for $r\geq R_2$

$$\log[rL(r)]>(1-\varepsilon_2)\log[\beta rL(r)].\tag{5.55}$$

So from (5.54) and (5.55) we get that

$$\begin{aligned}\rho_f^L(g) &\le \limsup_{r\to\infty}\frac{\log F^{-1}H(\beta r)}{(1-\varepsilon_2)\log[\beta rL(r)]}\\ &= \frac{1}{1-\varepsilon_2}\rho_f^L(h).\end{aligned}$$

Since $0 < \varepsilon_2 < 1$ is arbitrary,

$$\rho_f^L(g) \leq \rho_f^L(h). \tag{5.56}$$

As also $h \sim g$, we obtain in a like manner that

$$\rho_f^L(h) \leq \rho_f^L(g). \tag{5.57}$$

Combining (5.56) and (5.57) it follows that

$$\rho_f^L(g) = \rho_f^L(h).$$

Again in view of (5.53) we get

$$\begin{aligned} \lambda_f^L(g) &= \liminf_{r\to\infty} \frac{\log F^{-1}G(r)}{\log[rL(r)]} \\ &\leq \liminf_{r\to\infty} \frac{\log F^{-1}H(\beta r)}{\log[rL(r)]} \end{aligned} \tag{5.58}$$

So from (5.55) and (5.58) we obtain that

$$\begin{aligned} \lambda_f^L(g) &\leq \liminf_{r\to\infty} \frac{\log F^{-1}H(\beta r)}{(1-\varepsilon_2)\log[\beta rL(r)]} \\ &= \frac{1}{1-\varepsilon_2}\lambda_f^L(h). \end{aligned}$$

As $0 < \varepsilon_2 < 1$ is arbitrary,

$$\lambda_f^L(g) \leq \lambda_f^L(h). \tag{5.59}$$

Since also $h \sim g$, similarly we obtain that

$$\lambda_f^L(h) \leq \lambda_f^L(g). \tag{5.60}$$

Combining (5.59) and (5.60) it follows that

$$\lambda_f^L(g) = \lambda_f^L(h).$$

Thus the theorem is proved. ■

In the line of Theorem 5.3.11 and Theorem 5.3.12 we may prove the following theorem.

Theorem 5.3.13. *Let f, g, h, k be entire functions with $f \sim g$ and $h \sim k$. Then*

$$\rho_h^L(f) = \rho_k^L(f) = \rho_h^L(g) = \rho_k^L(g).$$

and

$$\lambda_h^L(f) = \lambda_k^L(f) = \lambda_h^L(g) = \lambda_k^L(g).$$

Proof. Since $h \sim k$, in view of Theorem 5.3.11 we get

$$\rho_h^L(f) = \rho_k^L(f), \quad \lambda_h^L(f) = \lambda_k^L(f) \tag{5.61}$$

$$\text{and} \quad \rho_h^L(g) = \rho_k^L(g), \quad \lambda_h^L(g) = \lambda_k^L(g). \tag{5.62}$$

Also as $f \sim g$, we obtain in view of Theorem 5.3.12

$$\rho_h^L(f) = \rho_h^L(g), \quad \lambda_h^L(f) = \lambda_h^L(g) \tag{5.63}$$

$$\text{and} \quad \rho_k^L(f) = \rho_k^L(g), \quad \lambda_k^L(f) = \lambda_k^L(g). \tag{5.64}$$

Now combining (5.61), (5.62), (5.63) and (5.64) it follows that

$$\rho_h^L(f) = \rho_k^L(f) = \rho_h^L(g) = \rho_k^L(g).$$

and

$$\lambda_h^L(f) = \lambda_k^L(f) = \lambda_h^L(g) = \lambda_k^L(g).$$

Thus the theorem is established. ■

With the notion of weighted sharing and relative sharing, in the following theorems we will investigate the relationship between L-orders (L-lower orders, L-types) and L^*-orders (L^*-lower orders, L^*-types) of two entire and meromorphic functions.

Theorem 5.3.14. *Let f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) such that $(k_2 - 1)(k_1 k_2 - 1) > (1 + k_2)^2$. If ρ_f^L and ρ_g^L be the respective L-orders of f and g then $\rho_f^L = \rho_g^L$.*

Proof. In view of Lemma 5.2.11, we obtain that

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log [rL(r)]} \le \limsup_{r\to\infty} \frac{\log T(r,g)}{\log [rL(r)]} = \rho_g^L$$

and similarly

$$\rho_g^L \le \rho_f^L.$$

Hence

$$\rho_f^L = \rho_g^L.$$

This proves the theorem. ■

Theorem 5.3.15. *Let f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) such that $(k_2 - 1)(k_1 k_2 - 1) > (1 + k_2)^2$. If λ_f^L and λ_g^L be the L-lower orders respectively of f and g then $\lambda_f^L = \lambda_g^L$.*

Proof. In view of Lemma 5.2.11, we obtain that

$$\lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log [rL(r)]} \leq \liminf_{r\to\infty} \frac{\log T(r, g)}{\log [rL(r)]} = \lambda_g^L$$

and similarly

$$\lambda_g^L \leq \lambda_f^L.$$

Hence

$$\lambda_f^L = \lambda_g^L.$$

Thus the theorem is established. ■

Theorem 5.3.16. *Let f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) such that $(k_2 - 1)(k_1 k_2 - 1) > (1 + k_2)^2$. If σ_f^L and σ_g^L be the L-types respectively of f and g then*

$$\frac{1}{(2 + O(1))} \sigma_f^L \leq \sigma_g^L \leq (2 + O(1)) \sigma_f^L.$$

Proof. In view of Lemma 5.2.11, we obtain that

$$\sigma_f^L = \limsup_{r\to\infty} \frac{T(r, f)}{[rL(r)]^{\rho_f^L}} \leq (2 + O(1)) \limsup_{r\to\infty} \frac{T(r, g)}{[rL(r)]^{\rho_g^L}}$$

$$\text{i.e., } \sigma_f^L \leq (2 + O(1)) \sigma_g^L$$

and similarly

$$\sigma_g^L \leq (2 + O(1)) \sigma_f^L.$$

Combining the above two inequalities we get

$$\frac{1}{(2 + O(1))} \sigma_f^L \leq \sigma_g^L \leq (2 + O(1)) \sigma_f^L.$$

This proves the theorem. ■

Theorem 5.3.17. *If f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) with $(k_2 - 1)(k_1 k_2 - 1) > (1 + k_2)^2$ then $\rho_f^{L^*} = \rho_g^{L^*}$ where $\rho_f^{L^*}, \rho_g^{L^*}$ denote respectively the L^*-orders of f and g.*

Proof. By Lemma 5.2.11, we get that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \leq \limsup_{r\to\infty} \frac{\log T(r,g)}{\log\left[re^{L(r)}\right]} = \rho_g^{L^*}$$

in a similar manner

$$\rho_g^{L^*} \leq \rho_f^{L^*}.$$

Therefore

$$\rho_f^{L^*} = \rho_g^{L^*}.$$

Thus the theorem is established. ∎

Theorem 5.3.18. *If f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) with $(k_2-1)(k_1k_2-1) > (1+k_2)^2$, then $\lambda_f^{L^*} = \lambda_g^{L^*}$, where $\lambda_f^{L^*}, \lambda_g^{L^*}$ denote respectively the L^*-lower orders of f and g.*

The proof of the theorem can be carried out in the line of Theorem 5.3.15.

In the line of Theorem 5.3.16 we may state the following theorem without proof.

Theorem 5.3.19. *Let f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) such that $(k_2-1)(k_1k_2-1) > (1+k_2)^2$. If $\sigma_f^{L^*}$ and $\sigma_g^{L^*}$ be the L^*-types respectively of f and g then*

$$\frac{1}{(2+O(1))}\sigma_f^{L^*} \leq \sigma_g^{L^*} \leq (2+O(1))\sigma_f^{L^*}.$$

Theorem 5.3.20. *Let f and g be two non-constant meromorphic functions. If there is a function h with $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$ such that F, G share a_1, a_2, a_3 IM then $\rho_f^L = \rho_g^L$ and $\lambda_f^L = \lambda_g^L$ where $F = \frac{f}{h}$ and $G = \frac{g}{h}$.*

Proof. As F, G share a_1, a_2, a_3 IM, in view of Lemma 5.2.12 we get that

$$\frac{1}{3}T(r,G)(1+o(1)) \leq T(r,F) \leq 3T(r,G)(1+o(1)).$$

From which we have

$$\rho_F^L = \rho_G^L. \tag{5.65}$$

Now $F = \frac{f}{h}$ gives that

$$\begin{aligned} T(r, F) &\leq T(r, f) + T(r, h) + O(1) \\ &\leq T(r, f)\,(1 + o(1)) + O(1). \end{aligned}$$

Hence

$$\rho_F^L \leq \rho_f^L.$$

Again from $f = hF$ we obtain that

$$\rho_f^L \leq \rho_F^L.$$

So

$$\rho_f^L = \rho_F^L. \tag{5.66}$$

Similarly from the relation $G = \frac{g}{h}$ we get that

$$\rho_G^L = \rho_g^L. \tag{5.67}$$

Combining (5.65), (5.66) and (5.67) we obtain

$$\rho_f^L = \rho_g^L.$$

Similarly we get that

$$\lambda_f^L = \lambda_g^L.$$

This proves the theorem. ∎

Remark 5.3.8. *The condition that 'F, G share a_1, a_2, a_3' in Theorem 5.3.20 is essential which is evident from the following example.*

Example 5.3.8. *Let us consider the functions $f(z) = \exp z$, $g(z) = \exp^{[2]} z$, $h(z) = z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. So*

$$F(z) = \frac{\exp z}{z} \quad \text{and} \quad G(z) = \frac{\exp^{[2]} z}{z}.$$

It is clear that F, G share only ∞. Also

$$T(r, h) = o\,(T(r, f)) \quad \text{and} \quad T(r, h) = o\,(T(r, g))\,.$$

But

$$\rho_f^L = \lambda_f^L = 1 \quad \text{and} \quad \rho_g^L = \lambda_g^L = \infty$$
$$\text{i.e., } \rho_f^L \neq \rho_g^L \text{ and } \lambda_f^L \neq \lambda_g^L.$$

Remark 5.3.9. *The condition '$T(r,h) = o(T(r,g))$' in Theorem 5.3.20 is necessary as we see in the following example.*

Example 5.3.9. *Let us choose the functions $f(z) = z^2$, $g(z) = z^2 \exp z$, $h(z) = z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. So*

$$F(z) = z \quad \textit{and} \quad G(z) = z \exp z.$$

It is obvious that F, G share 0, $2\pi i$, ∞. Also

$$T(r,h) = o(T(r,f)) \quad \textit{and} \quad T(r,h) \neq o(T(r,g)).$$

Here also

$$\rho_f^L \neq \rho_g^L \textit{ and } \lambda_f^L \neq \lambda_g^L.$$

In the line of Theorem 5.3.20 we may state the following theorem.

Theorem 5.3.21. *Let f and g be two non-constant meromorphic functions. If there is a function h with $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$ such that F, G share a_1, a_2, a_3 IM then $\rho_f^{L^*} = \rho_g^{L^*}$ and $\lambda_f^{L^*} = \lambda_g^{L^*}$ where $F = \frac{f}{h}$ and $G = \frac{g}{h}$.*

The proof is omitted.

Remark 5.3.10. *Let us consider the functions $f(z) = \exp z$, $g(z) = \exp^{[2]} z$, $h(z) = z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. Clearly F, G share only ∞. Also $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$ but $\rho_f^{L^*} \neq \rho_g^{L^*}$ and $\lambda_f^{L^*} \neq \lambda_g^{L^*}$.*

Remark 5.3.11. *Let us choose the functions $f(z) = z^2$, $g(z) = z^2 \exp z$, $h(z) = z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. Clearly F, G share 0, $2\pi i$, ∞. Also $T(r,h) = o(T(r,f))$ but $T(r,h) = o(T(r,g))$ does not hold. Here also $\rho_f^{L^*} \neq \rho_g^{L^*}$ and $\lambda_f^{L^*} \neq \lambda_g^{L^*}$.*

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