Chapter 4

SOME COMPARATIVE GROWTH PROPERTIES OF DIFFERENTIAL POLYNOMIALS

Chapter 4

SOME COMPARATIVE GROWTH PROPERTIES OF DIFFERENTIAL POLYNOMIALS

4.1 Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. In the sequel we use the following notations: (i) $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$ and (ii) $\exp^{[k]} z = \exp(\exp^{[k-1]} z)$ for $k = 1, 2, 3...$and $\exp^{[0]} z = z$.

We recall the following definitions:

Definition 4.1.1. *The order ρ_f and lower order λ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that,

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 4.1.2. *The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function f is defined as follows*

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} \quad and \quad \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

The results of this chapter have been published in **International Journal of Mathematical Sciences and Engineering Applications**, see **[14]**.

If f is entire, then

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \quad \text{and} \quad \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 4.1.3. *The type σ_f of a meromorphic function f is defined as follows*

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

When f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

Definition 4.1.4. *A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function f of finite lower order λ_f if*

(i)$\lambda_f(r)$ is non-negative and continuous for $r \geq r_0$, say

(ii)$\lambda_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda_f'(r+0)$ and $\lambda_f'(r-0)$ exist,

(iii)$\lim\limits_{r\to\infty} \lambda_f(r) = \lambda_f$,

(iv)$\lim\limits_{r\to\infty} r\lambda_f'(r)\log r = 0$ *and*

(v)$\liminf\limits_{r\to\infty} \dfrac{T(r,f)}{r^{\lambda_f(r)}} = 1.$

Definition 4.1.5. *Let 'a' be a complex number, finite or infinite. The Nevanlinna deficiency and Valiron deficiency of 'a' with respect to a meromorphic function f are defined as*

$$\begin{aligned}\delta(a;f) &= 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)} \\ \text{and } \Delta(a;f) &= 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}.\end{aligned}$$

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also let n_{0j}, n_{1j}, ..., n_{kj} $(k \geq 1)$ be non-negative integers such that for each j, $\sum\limits_{i=0}^{k} n_{ij} \geq 1$. We call

$$M_j[f] = A_j(f)^{n_{0j}}(f^{(1)})^{n_{1j}}...(f^{(k)})^{n_{kj}}$$

where $T(r, A_j) = S(r, f)$, to be a differential monomial generated by f. The numbers $\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$ and $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$ are called the degree and weight of $M_j[f]$ ([8]) respectively. The expression $P[f] = \sum_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max_{1\le j\le s} \gamma_{M_j}$ and $\Gamma_P = \max_{1\le j\le s} \Gamma_{M_j}$ are respectively called the degree and weight of $P[f]$ ([8]). Also we call the numbers $\underline{\gamma}_P = \min_{1\le j\le s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma}_P = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial.

H.X.Yi [52] worked on the value distribution of differential polynomials. In the chapter we prove some new results depending on the comparative growth properties of composite entire or meromorphic functions and differential polynomials generated by one of the factors which improve some earlier theorems. Throughout the chapter we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e. for which $n_{0j} = 0$ for $j = 1, 2, ..., s$. We consider only those $P[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other.

The following definitions are also well known.

Definition 4.1.6. *The quantity $\Theta(a; f)$ of a meromorphic function f is defined as follows*

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

Definition 4.1.7. *{[30], [51]} For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \le r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\ge p$ is counted exactly p times; and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define*

$$\delta_p(a; f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

Definition 4.1.8. *[5] $P[f]$ is said to be admissible if*

(i) $P[f]$ *is homogeneous, or*
(ii) $P[f]$ *is non homogeneous and* $m(r, f) = S(r, f)$.

4.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 4.2.1. *[1]If* f *is meromorphic and* g *is entire then for all sufficiently large values of* r,

$$T(r, fog) \leq \{1 + o(1)\}\frac{T(r, g)}{\log M(r, g)}T(M(r, g), f).$$

Lemma 4.2.2. *[4]Let* f *be meromorphic and* g *be entire and suppose that* $0 < \mu < \rho_g \leq \infty$. *Then for a sequence of values of* r *tending to infinity,*

$$T(r, fog) \geq T(\exp(r^{\mu}), f)).$$

Lemma 4.2.3. *[5]If* f *be a meromorphic function of finite order or of non zero lower order and* $\sum\limits_{a \neq \infty} \Theta(a; f) = 2$. *Also let* $P_0[f]$ *be admissible then*

$$\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \Gamma_{P_0}.$$

Lemma 4.2.4. *[30] Let* f *be either of finite order or of non zero lower order such that* $\Theta(\infty; f) = \sum\limits_{a \neq \infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum\limits_{a \neq \infty} \delta(a; f) = 1$. *Then for homogeneous* $P_0[f]$,

$$\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma_{P_0}.$$

Lemma 4.2.5. *Let* f *be a meromorphic function of finite order or of non zero lower order. If* $\sum\limits_{a \neq \infty} \Theta(a; f) = 2$, *then the order (lower order) of homogeneous* $P_0[f]$ *is same as that of* f. *Also the type of* $P_0[f]$ *is* Γ_{P_0} *times that of* f *when* f *is of finite positive order.*

Proof. By Lemma 4.2.3,

$$\lim_{r\to\infty} \frac{\log T(r, P_0[f])}{\log T(r, f)}$$

exists and is equal to 1. So,

$$\begin{aligned}\rho_{P_0[f]} &= \limsup_{r\to\infty} \frac{\log T(r, P_0[f])}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} . \lim_{r\to\infty} \frac{\log T(r, P_0[f])}{\log T(r, f)} \\ &= \rho_f . 1 = \rho_f .\end{aligned}$$

In a similar manner, $\lambda_{P_0[f]} = \lambda_f$.

Again by Lemma 4.2.3,

$$\begin{aligned}\sigma_{P_0[f]} &= \limsup_{r\to\infty} \frac{T(r, P_0[f])}{r^{\rho_{P_0[f]}}} \\ &= \lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r, f)} . \limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho_f}} \\ &= \Gamma_{P_0} . \sigma_f .\end{aligned}$$

This proves the lemma. ■

Lemma 4.2.6. *Let f be a meromorphic function of finite order or of non zero lower order such that $\Theta(\infty; f) = \sum\limits_{a\neq\infty} \delta_p(a; f) = 1$. Then the order (lower order) of homogeneous $P_0[f]$ and f are same. Also the type of $P_0[f]$ is γ_{P_0} times that of f when f is of finite positive order.*

Proof. In view of Lemma 4.2.4,

$$\lim_{r\to\infty} \frac{\log T(r, P_0[f])}{\log T(r, f)}$$

exists and is equal to 1. So,

$$\begin{aligned}\rho_{P_0[f]} &= \limsup_{r\to\infty} \frac{\log T(r, P_0[f])}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} . \lim_{r\to\infty} \frac{\log T(r, P_0[f])}{\log T(r, f)} \\ &= \rho_f . 1 = \rho_f .\end{aligned}$$

In a similar manner, $\lambda_{P_0[f]} = \lambda_f$.

Again by Lemma 4.2.4,

$$\begin{aligned}\sigma_{P_0[f]} &= \limsup_{r\to\infty} \frac{T(r, P_0[f])}{r^{\rho_{P_0[f]}}} \\ &= \lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r, f)} . \limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho_f}} \\ &= \gamma_{P_0}.\sigma_f. \ \{\text{cf. } [30]\}\end{aligned}$$

This proves the lemma. ■

In a similar manner we can state the following lemma without proof.

Lemma 4.2.7. *Let f be a meromorphic function of finite order or of non zero lower order such that $\delta(\infty; f) = \sum\limits_{a\neq\infty} \delta(a; f) = 1$. Then for every homogeneous $P_0[f]$ the order (lower order) of $P_0[f]$ and is same as that of f. Also the type of $P_0[f]$ is γ_{P_0} times that of f when f is of finite positive order.*

Lemma 4.2.8. *Let f be a meromorphic function of finite order or of non zero lower order and $\sum\limits_{a\neq\infty} \Theta(a; f) = 2$. Then the hyper order (hyper lower order) of $P_0[f]$ and f are equal.*

Proof. By Lemma 4.2.3,

$$\lim_{r\to\infty} \frac{\log^{[2]} T(r, P_0[f])}{\log^{[2]} T(r, f)}$$

exists and is equal to 1. So,

$$\begin{aligned}\bar{\rho}_{P_0[f]} &= \limsup_{r\to\infty} \frac{\log^{[2]} T(r, P_0[f])}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log^{[2]} T(r, f)}{\log r} . \lim_{r\to\infty} \frac{\log^{[2]} T(r, P_0[f])}{\log^{[2]} T(r, f)} \\ &= \bar{\rho}_f.1 = \bar{\rho}_f.\end{aligned}$$

In a similar manner, $\bar{\lambda}_{P_0[f]} = \bar{\lambda}_f$.

This proves the lemma. ■

Remark 4.2.1. *The conclusion of Lemma 4.2.8 can also be deduced under the hypothesis $\Theta(\infty; f) = \sum\limits_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum\limits_{a\neq\infty} \delta(a; f) = 1$ instead of $\sum\limits_{a\neq\infty} \Theta(a; f) = 2$.*

Lemma 4.2.9. *For a meromorphic function f of finite lower order, lower proximate order exists.*

The lemma can be proved in the line of Theorem 1 [26] and so the proof is omitted.

Lemma 4.2.10. *Let f be a meromorphic function of finite lower order λ_f. Then for $\delta(>0)$ the function $r^{\lambda_f+\delta-\lambda_f(r)}$ is ultimately an increasing function of r.*

Proof. Since

$$\frac{d}{dr}r^{\lambda_f+\delta-\lambda_f(r)} = \{\lambda_f+\delta-\lambda_f(r)-r\lambda_f'(r)\log r\}r^{\lambda_f+\delta-\lambda_f(r)-1} > 0$$

for all sufficiently large values of r, the lemma follows. ■

Lemma 4.2.11. *[25] Let f be an entire function of finite lower order. If there exists entire functions a_i $(i=1,2,...,n; n \leq \infty)$ satisfying $T(r,a_i)=o\{T(r,f)\}$ and $\sum_{i=1}^{n}\delta(a_i;f)=1$, then*

$$\lim_{r\to\infty}\frac{T(r,f)}{\log M(r,f)}=\frac{1}{\pi}.$$

4.3 Theorems.

In this section we present the main results of the chapter.

Theorem 4.3.1. *Let f be a meromorphic function and g be an entire function satisfying*
(i)λ_f, λ_g are both finite and
(ii) $\sum_{a\neq\infty}\Theta(a;g)=2$. Then

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{T(r,P_0[g])}\leq\frac{3.\rho_f.2^{\lambda_g}}{\Gamma_{P_0}}.$$

Proof. If $\rho_f=\infty$, the result is obvoius. So we suppose that $\rho_f<\infty$. Since $T(r,g)\leq\log^+ M(r,g)$,in view of Lemma 4.2.1, we get for all sufficiently large values of r,

$$\begin{aligned} T(r, fog) &\leq \{1+o(1)\}T(M(r,g),f) \\ i.e., \log T(r, fog) &\leq \log\{1+o(1)\} + \log T(M(r,g),f) \\ i.e., \log T(r, fog) &\leq o(1) + (\rho_f + \varepsilon)\log M(r,g) \\ i.e., \liminf_{r\to\infty} \frac{\log T(r, fog)}{T(r,g)} &\leq (\rho_f+\varepsilon)\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}. \end{aligned}$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{T(r,g)} \leq \rho_f . \liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}. \tag{4.1}$$

As by condition (v) of Definition 4.1.4

$$\liminf_{r\to\infty}\frac{T(r,g)}{r^{\lambda_g(r)}} = 1,$$

so for given $\varepsilon(0<\varepsilon<1)$ we get for a sequence of values of r tending to infinity,

$$T(r,g) \leq (1+\varepsilon)r^{\lambda_g(r)} \tag{4.2}$$

and for all sufficiently large values of r,

$$T(r,g) > (1-\varepsilon)r^{\lambda_g(r)}. \tag{4.3}$$

Since $\log M(r,g) \leq 3T(2r,g)$ {cf. [22]}, we have by (4.2), for a sequence of values of r tending to infinity,

$$\log M(r,g) \leq 3T(2r,g) \leq 3(1+\varepsilon)(2r)^{\lambda_g(2r)}. \tag{4.4}$$

Combining (4.3) and (4.4) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)}.\frac{(2r)^{\lambda_g(2r)}}{r^{\lambda_g(r)}}.$$

Now for any $\delta>0$, for a sequence of values of r tending to infinity,

$$\frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)}.\frac{(2r)^{\lambda_g+\delta}}{(2r)^{\lambda_g+\delta-\lambda_g(2r)}}.\frac{1}{r^{\lambda_g(r)}}$$

$$i.e., \ \frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)}.2^{\lambda_g+\delta} \tag{4.5}$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is ultimately an increasing function of r by Lemma 4.2.10. Since $\varepsilon(>0)$ and $\delta(>0)$ are arbitrary,it follows from (4.5) that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}\leq 3.2^{\lambda_g}. \tag{4.6}$$

Thus from (4.1) and (4.6) we obtain that

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{T(r,g)}\leq 3.\rho_f.2^{\lambda_g}. \tag{4.7}$$

Now in view of Lemma 4.2.3 and (4.7) we get

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log T(r,fog)}{T(r,P_0[g])} &= \liminf_{r\to\infty}\frac{\log T(r,fog)}{T(r,g)}.\lim_{r\to\infty}\frac{T(r,g)}{T(r,P_0[g])}\\ &\leq \frac{3.\rho_f.2^{\lambda_g}}{\Gamma_{P_0}}.\end{aligned}$$

This proves the theorem. ■

Remark 4.3.1. *The conclusion of Theorem 4.3.1 can also be drawn under the hypothesis* $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ *or* $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)$ $=1$ *instead of* $\sum\limits_{a\neq\infty}\Theta(a;f)=2$.

Theorem 4.3.2. *Let f be meromorphic and g be entire such that* $\rho_f<\infty$, $\lambda_g<\infty$ *and* $\delta(\infty;g)=\sum\limits_{a\neq\infty}\delta(a;g)=1$. *Then*

$$\liminf_{r\to\infty}\frac{\log^{[2]}T(r,fog)}{\log T(r,P_0[g])}\leq 1.$$

Proof. Since $T(r,g)\leq\log^{+}M(r,g)$,in view of Lemma 4.2.1, we get for all sufficiently large values of r,

$$\begin{aligned}\log T(r,fog) &\leq \log T(M(r,g),f)+\log\{1+o(1)\}\\ i.e., \log T(r,fog) &\leq (\rho_f+\varepsilon)\log M(r,g)+o(1)\\ i.e., \log^{[2]}T(r,fog) &\leq \log^{[2]}M(r,g)+O(1). \qquad (4.8)\end{aligned}$$

It is well known that for any entire function g, $\log M(r,g)\leq 3T(2r,g)$ {cf. [22]}.

Then for $0 < \varepsilon < 1$ and $\delta(> 0)$, for a sequence of values of r tending to infinity it follows from (4.5) that

$$\log^{[2]} M(r,g) \leq \log T(r,g) + O(1). \tag{4.9}$$

Now combining (4.8) and (4.9) we obtain for a sequence of values of r tending to infinity,

$$\begin{aligned} \log^{[2]} T(r,fog) &\leq \log T(r,g) + O(1) \\ i.e., \frac{\log^{[2]} T(r,fog)}{\log T(r,g)} &\leq 1. \end{aligned} \tag{4.10}$$

As by Lemma 4.2.4,

$$\lim_{r\to\infty} \frac{\log T(r,g)}{\log T(r,P_0[g])}$$

exists and is equal to 1, then from (4.10) we get that

$$\begin{aligned} & \liminf_{r\to\infty} \frac{\log^{[2]} T(r,fog)}{\log T(r,P_0[g])} \\ = \; & \liminf_{r\to\infty} \frac{\log^{[2]} T(r,fog)}{\log T(r,g)} . \lim_{r\to\infty} \frac{\log T(r,g)}{\log T(r,P_0[g])} \\ \leq \; & 1.1 = 1. \end{aligned}$$

Thus the theorem is established. ■

Remark 4.3.2. *The condition $\rho_f < \infty$ is essential in Theorem 4.3.2 which is evident from the following example.*

Example 4.3.1. *Let $f = \exp^{[2]} z$ and $g = \exp z$.*
Then

$$fog = \exp^{[3]} z \text{ and } \delta(\infty; g) = \sum_{a\neq\infty} \delta(a;g) = 1.$$

Also let $s = 1$, $A_1 = 1$ and

$$\begin{aligned} n_{i1} &= 1, \text{ for } i = 1. \\ &= 0, \text{ for } i \neq 1 \end{aligned}$$

Then $P_0[g] = \exp z$.

Now we have

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log^{[2]}(\exp^{[2]} r)}{\log r} = \infty$$

$$\text{and } \lambda_g = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log r} = \liminf_{r\to\infty} \frac{\log^{[2]}(\exp r)}{\log r} = 1.$$

Again from the inequality $T(r,f) \leq \log^+ M(r,f) \leq 3T(2r,f)$ *{cf. p.18, [22]} we obtain that*

$$T(r, P_0[g]) \leq \log M(r, P_0[g]) = \log(\exp r)$$

$$\text{i.e., } \log T(r, P_0[g]) \leq \log r$$

and

$$T(r, fog) \geq \frac{1}{3}\log M(\frac{r}{2}, fog) = \frac{1}{3}\exp^{[2]}(\frac{r}{2})$$

$$\text{i.e., } \log^{[2]} T(r, fog) \geq \frac{r}{2} + O(1).$$

Combining the above two inequalities we have

$$\frac{\log^{[2]} T(r, fog)}{\log T(r, P_0[g])} \geq \frac{\frac{r}{2} + O(1)}{\log r}.$$

Therefore

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, P_0[g])} = \infty.$$

Remark 4.3.3. *If we replace* $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$ *by* $\sum_{a\neq\infty} \Theta(a; g) = 2$ *or* $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ *and the other conditions remain same then also Theorem 4.3.2 is valid.*

Theorem 4.3.3. *Let f and g be two entire functions such that* $\rho_g < \lambda_f \leq \rho_f < \infty$ *and* $\sum_{a\neq\infty} \Theta(a; f) = \sum_{a\neq\infty} \Theta(a; g) = 2$*.Also there exist entire functions a_i $(i = 1, 2, ..., n; n \leq \infty)$ with*

(i) $T(r, a_i) = o\{T(r, g)\}$ *as* $r \to \infty$ *for* $i = 1, 2, ..., n$ *and*

(ii) $\sum_{i=1}^{n} \delta(a_i; g) = 1$*.Then*

$$\lim_{r\to\infty} \frac{\{\log T(r, fog)\}^2}{T(r, P_0[f])T(r, P_0[g])} = 0.$$

Proof. In view of the inequality $T(r,g) \leq \log^{+} M(r,g)$ and Lemma 4.2.1 we obtain for all sufficiently large values of r,

$$\begin{aligned} T(r,fog) &\leq \{1+o(1)\}T(M(r,g),f) \\ i.e., \log T(r,fog) &\leq \log\{1+o(1)\} + \log T(M(r,g),f) \\ i.e., \log T(r,fog) &\leq o(1) + (\rho_f+\varepsilon)\log M(r,g) \\ i.e., \log T(r,fog) &\leq o(1) + (\rho_f+\varepsilon) r^{(\rho_g+\varepsilon)}. \end{aligned} \tag{4.11}$$

Again in view of Lemma 4.2.5, we get for all sufficiently large values of r,

$$\begin{aligned} \log T(r,P_0[f]) &> (\lambda_{P_0[f]}-\varepsilon)\log r \\ i.e., \log T(r,P_0[f]) &> (\lambda_f-\varepsilon)\log r \\ i.e., \ T(r,P_0[f]) &> r^{\lambda_f-\varepsilon}. \end{aligned} \tag{4.12}$$

Now combining (4.11) and (4.12) it follows for all sufficiently large values of r,

$$\frac{\log T(r,fog)}{T(r,P_0[f])} \leq \frac{o(1)+(\rho_f+\varepsilon)r^{(\rho_g+\varepsilon)}}{r^{\lambda_f-\varepsilon}}. \tag{4.13}$$

Since $\rho_g < \lambda_f$, we can choose $\varepsilon(>0)$ in such a way that

$$\rho_g+\varepsilon < \lambda_f-\varepsilon. \tag{4.14}$$

So in view of (4.13) and (4.14) it follows that

$$\lim_{r\to\infty}\frac{\log T(r,fog)}{T(r,P_0[f])} = 0. \tag{4.15}$$

Again from Lemma 4.2.3, Lemma 4.2.5 and Lemma 4.2.11 we get for all sufficiently large values of r,

$$\begin{aligned} \frac{\log T(r,fog)}{T(r,P_0[g])} &\leq \frac{o(1)+(\rho_f+\varepsilon)\log M(r,g)}{T(r,P_0[g])} \\ i.e., \limsup_{r\to\infty}\frac{\log T(r,fog)}{T(r,P_0[g])} &\leq (\rho_f+\varepsilon)\limsup_{r\to\infty}\frac{\log M(r,g)}{T(r,P_0[g])} \\ i.e., \limsup_{r\to\infty}\frac{\log T(r,fog)}{T(r,P_0[g])} &\leq (\rho_f+\varepsilon)\limsup_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}.\lim_{r\to\infty}\frac{T(r,g)}{T(r,P_0[g])} \\ i.e., \limsup_{r\to\infty}\frac{\log T(r,fog)}{T(r,P_0[g])} &\leq (\rho_f+\varepsilon).\pi.\frac{1}{\Gamma_{P_0}}. \end{aligned}$$

Since $\varepsilon(>0)$ is arbitrary it follows from above that

$$\limsup_{r\to\infty} \frac{\log T(r,fog)}{T(r,P_0[g])} \le \rho_f.\pi.\frac{1}{\Gamma_{P_0}}. \tag{4.16}$$

Combining (4.15) and (4.16) we obtain that

$$\begin{aligned} &\limsup_{r\to\infty} \frac{\{\log T(r,fog)\}^2}{T(r,P_0[f])T(r,P_0[g])} \\ = \;& \lim_{r\to\infty} \frac{\log T(r,fog)}{T(r,P_0[f])}.\limsup_{r\to\infty} \frac{\log T(r,fog)}{T(r,P_0[g])} \\ \le \;& 0.\frac{\pi.\rho_f}{\Gamma_{P_0}} = 0, \end{aligned}$$

$$i.e., \lim_{r\to\infty} \frac{\{\log T(r,fog)\}^2}{T(r,P_0[f])T(r,P_0[g])} = 0.$$

This proves the theorem. ■

Remark 4.3.4. *Theorem 4.3.3 is still valid under the condition* $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1 = \Theta(\infty;g) = \sum_{a\neq\infty} \delta_p(a;g)$ *or* $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1 = \delta(\infty;g) = \sum_{a\neq\infty} \delta(a;g)$ *instead of* $\sum_{a\neq\infty} \Theta(a;f) = \sum_{a\neq\infty} \Theta(a;g) = 2$.

Theorem 4.3.4. *If f and g be two entire functions satisfying the following conditions (i)*$\lambda_f > 0$ *(ii)*$\overline{\rho}_f < \infty$ *(iii)*$0 < \lambda_g \le \rho_g$ *and (iv)*$\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$, *then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r,P_0[f])} \ge \max\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\}.$$

Proof. We know that for $r > 0$ {cf. [37]} and for all sufficiently large values of r,

$$T(r,fog) \ge \frac{1}{3}\log M\{\frac{1}{8}M(\frac{r}{4},g) + o(1), f\}. \tag{4.17}$$

Since λ_f and λ_g are the lower orders of f and g respectively then for given $\varepsilon(>0)$ and for all sufficiently large values of r we obtain that

$$\log M(r,f) > r^{\lambda_f-\varepsilon} \quad \text{and} \quad \log M(r,g) > r^{\lambda_g-\varepsilon}$$

where $0 < \varepsilon < \min\{\lambda_f, \lambda_g\}$.

So from (4.17) we have for all sufficiently large values of r,

$$\begin{aligned} T(r, fog) &\geq \frac{1}{3}\{\frac{1}{8}M(\frac{r}{4}, g) + o(1)\}^{\lambda_f - \varepsilon} \\ i.e., T(r, fog) &\geq \frac{1}{3}\{\frac{1}{9}M(\frac{r}{4}, g)\}^{\lambda_f - \varepsilon} \\ i.e., \log T(r, fog) &\geq O(1) + (\lambda_f - \varepsilon)\log M(\frac{r}{4}, g) \\ i.e., \log T(r, fog) &\geq O(1) + (\lambda_f - \varepsilon)(\frac{r}{4})^{\lambda_g - \varepsilon} \\ i.e., \log^{[2]} T(r, fog) &\geq O(1) + (\lambda_g - \varepsilon)\log r. \end{aligned} \tag{4.18}$$

Again in view of Remark 4.2.1, we get for a sequence of values of r tending to infinity,

$$\begin{aligned} \log^{[2]} T(r, P_0[f]) &\leq (\overline{\lambda}_{P_0[f]} + \varepsilon)\log r \\ i.e., \log^{[2]} T(r, P_0[f]) &\leq (\overline{\lambda}_f + \varepsilon)\log r. \end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19), it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \geq \frac{O(1) + (\lambda_g - \varepsilon)\log r}{(\overline{\lambda}_f + \varepsilon)\log r}.$$

Since $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \geq \frac{\lambda_g}{\overline{\lambda}_f}. \tag{4.20}$$

Again from (4.17) we get for a sequence of values of r tending to infinity,

$$\begin{aligned} \log T(r, fog) &\geq O(1) + (\lambda_f - \varepsilon)(\frac{r}{4})^{\rho_g - \varepsilon} \\ i.e., \log^{[2]} T(r, fog) &\geq O(1) + (\rho_g - \varepsilon)\log r. \end{aligned} \tag{4.21}$$

Again in view of Remark 4.2.1, for all sufficiently large values of r, we have

$$\begin{aligned} \log^{[2]} T(r, P_0[f]) &\leq (\overline{\rho}_{P_0[f]} + \varepsilon)\log r \\ i.e., \log^{[2]} T(r, P_0[f]) &\leq (\overline{\rho}_f + \varepsilon)\log r. \end{aligned} \tag{4.22}$$

Now from (4.21) and (4.22) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \geq \frac{O(1) + (\rho_g - \varepsilon) \log r}{(\overline{\rho}_f + \varepsilon) \log r}.$$

As $\varepsilon (0 < \varepsilon < \rho_g)$ is arbitrary, we obtain from above that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \geq \frac{\rho_g}{\overline{\rho}_f}. \tag{4.23}$$

Therefore from (4.20) and (4.23) we get

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \geq \max\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\}.$$

Thus the theorem is established. ■

Remark 4.3.5. *The conclusion of Theorem 4.3.4 can also be deduced under the hypothesis* $\sum_{a \neq \infty} \Theta(a; f) = 2$ *or* $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ *instead of* $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$.

Theorem 4.3.5. *Let f be meromorphic and g be entire such that (i) $0 < \overline{\lambda}_f < \overline{\rho}_f$, (ii) $\rho_g < \infty$, (iii) $\rho_f < \infty$ and (iv) $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. Then*

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \leq \min\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\}.$$

Proof. In view of Lemma 4.2.1 and the inequality $T(r, g) \leq \log^+ M(r, g)$, we obtain for all sufficiently large values of r,

$$\log T(r, fog) \leq o(1) + (\rho_f + \varepsilon) \log M(r, g). \tag{4.24}$$

Also for a sequence of values of r tending to infinity,

$$\log M(r, g) \leq r^{\lambda_g + \varepsilon}. \tag{4.25}$$

Combining (4.24) and (4.25) it follows for a sequence of values of r tending to infinity,

$$\begin{aligned} \log T(r, fog) &\leq o(1) + (\rho_f + \varepsilon) r^{\lambda_g + \varepsilon} \\ i.e., \log T(r, fog) &\leq r^{\lambda_g + \varepsilon}\{o(1) + (\rho_f + \varepsilon)\} \\ i.e., \log^{[2]} T(r, fog) &\leq O(1) + (\lambda_g + \varepsilon) \log r. \end{aligned} \tag{4.26}$$

Again in view of Remark 4.2.1, we have for all sufficiently large values of r,

$$\begin{aligned} \log^{[2]} T(r, P_0[f]) &> (\overline{\lambda}_{P_0[f]} - \varepsilon) \log r \\ i.e., \log^{[2]} T(r, P_0[f]) &> (\overline{\lambda}_f - \varepsilon) \log r. \end{aligned} \tag{4.27}$$

Now from (4.26) and (4.27) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \leq \frac{O(1) + (\lambda_g + \varepsilon) \log r}{(\overline{\lambda}_f - \varepsilon) \log r}.$$

As $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \leq \frac{\lambda_g}{\overline{\lambda}_f}. \tag{4.28}$$

In view of Lemma 4.2.1 we obtain for all sufficiently large values of r,

$$\log^{[2]} T(r, fog) \leq O(1) + (\rho_g + \varepsilon) \log r. \tag{4.29}$$

Also by Remark 4.2.1, we have for a sequence of values of r tending to infinity,

$$\begin{aligned} \log^{[2]} T(r, P_0[f]) &> (\overline{\rho}_{P_0[f]} - \varepsilon) \log r \\ i.e., \log^{[2]} T(r, P_0[f]) &> (\overline{\rho}_f - \varepsilon) \log r. \end{aligned} \tag{4.30}$$

Combining (4.29) and (4.30) we have for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \leq \frac{O(1) + (\rho_g + \varepsilon) \log r}{(\overline{\rho}_f - \varepsilon) \log r}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \leq \frac{\rho_g}{\overline{\rho}_f}. \tag{4.31}$$

Now from (4.28) and (4.31) we get that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} \leq \min\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\}.$$

This proves the theorem. ■

Remark 4.3.6. *Theorem 4.3.5 remains true if we replace the condition* $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ *by* $\sum_{a\neq\infty} \Theta(a; f) = 2$ *or* $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ *and the other conditions remain same.*

The following theorem is a natural consequence of Theorem 4.3.4 and Theorem 4.3.5.

Theorem 4.3.6. *Let f and g be two entire functions such that (i)$0 < \overline{\lambda}_f < \overline{\rho}_f < \infty$, (ii)$0 < \lambda_f \leq \rho_f < \infty$, (iii) $0 < \lambda_g \leq \rho_g < \infty$ and (iv)$\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$. Then*

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])} &\leq \min\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\} \\ &\leq \max\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\} \leq \limsup_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, P_0[f])}.\end{aligned}$$

Remark 4.3.7. *If we replace the condition* $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ *by* $\sum_{a\neq\infty} \Theta(a; f) = 2$ *or* $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ *in Theorem 4.3.6 and the other conditions are same then also the theorem is valid.*

Theorem 4.3.7. *Let f be meromorphic and g be entire such that $0 < \lambda_f \leq \rho_f < \infty$, $\rho_g > 0$ and $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), fog)}{\log^{[2]} T(\exp(r^{\mu}), P_0[f])} = \infty, \quad \text{where } 0 < \mu < \rho_g.$$

Proof. Let $0 < \mu' < \rho_g$. Then in view of Lemma 4.2.2 we get for a sequence of values of r tending to infinity,

$$\begin{aligned}\log T(r, fog) &\geq \log T(\exp(r^{\mu'}), f)) \\ i.e., \log T(r, fog) &\geq (\lambda_f - \varepsilon) \log\{\exp(r^{\mu'})\} \\ i.e., \log T(r, fog) &\geq (\lambda_f - \varepsilon) r^{\mu'} \\ i.e., \log^{[2]} T(r, fog) &\geq O(1) + \mu' \log r.\end{aligned}$$

So for a sequence of values of r tending to infinity,

$$\begin{aligned}\log^{[2]} T(\exp(r^{\rho_g}), fog) &\geq O(1) + \mu' \log\{\exp(r^{\rho_g})\} \\ i.e., \log^{[2]} T(\exp(r^{\rho_g}), fog) &\geq O(1) + \mu'.r^{\rho_g}. \qquad (4.32)\end{aligned}$$

Again in view of Lemma 4.2.7, we have for all sufficiently large values of r,

$$\begin{aligned}\log T(\exp(r^{\mu}), P_0[f]) &\leq (\rho_{P_0[f]} + \varepsilon)\log\{\exp(r^{\mu})\}\\ i.e., \log T(\exp(r^{\mu}), P_0[f]) &\leq (\rho_f + \varepsilon)r^{\mu}\\ i.e., \log^{[2]} T(\exp(r^{\mu}), P_0[f]) &\leq O(1) + \mu\log r. \qquad (4.33)\end{aligned}$$

Now combining (4.32) and (4.33) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), fog)}{\log^{[2]} T(\exp(r^{\mu}), P_0[f])} \geq \frac{O(1) + \mu' r^{\rho_g}}{O(1) + \mu\log r}$$

from which the theorem follows. ∎

Remark 4.3.8. *The condition $\rho_g > 0$ is necessary in Theorem 4.3.7 as we see in the following example.*

Example 4.3.2. *Let $f = \exp z$, $g = z$ and $\mu = 1 (> 0)$.*
Then

$$fog = \exp z \text{ and } \delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1.$$

Also let $s = 1$, $A_1 = 1$ and

$$\begin{aligned} n_{i1} &= 1, \text{ for } i = 1\\ &= 0, \text{ for } i \neq 1.\end{aligned}$$

Then $P_0[f] = \exp z$.
Now we have

$$\begin{aligned}\rho_f &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} = 1,\\ \lambda_f &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} = 1\\ \text{and } \rho_g &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log r} = 0.\end{aligned}$$

Also we get $T(r, f) = \dfrac{r}{\pi}$.
Therefore

$$T(\exp(r^{\rho_g}), fog) = \frac{e}{\pi}$$

and

$$T(\exp(r^{\mu}), P_0[f]) = \frac{\exp r}{\pi}.$$

So from the above two expressions we obtain that

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), fog)}{\log^{[2]} T(\exp(r^{\mu}), P_0[f])} = \frac{O(1)}{\log r + O(1)},$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), fog)}{\log^{[2]} T(\exp(r^{\mu}), P_0[f])} = 0.$$

Remark 4.3.9. *The conclusion of Theorem 4.3.7 can also be deduced under the hypothesis* $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ *or* $\sum_{a\neq\infty} \Theta(a; f) = 2$ *instead of* $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$.

– – – – –X – – – – –