Chapter 3

SOME GROWTH PROPERTIES OF WRONSKIANS

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3.1 Introduction, Definitions and Notations.

For any two transcendental entire functions f and g defined in the open complex plane $\mathbb{C}$, Clunie [7] proved that

$$\lim_{r\to\infty}\frac{T(r,fog)}{T(r,f)}=\lim_{r\to\infty}\frac{T(r,fog)}{T(r,g)}=\infty.$$

Singh [43] studied some comparative growth properties of $\log T(r,fog)$ and $T(r,f)$. He [43] also raised the question of investigating the comparative growth of $\log T(r,fog)$ and $T(r,g)$ which he was unable to solve. Lahiri [27] proved some results on the comparative growth of $\log T(r,fog)$ and $T(r,g)$.

Some mathematicians like H.X.Yi [51] and many more studied the comparative growth of a meromorphic function and its derivatives.

Since the natural extension of a derivative is a differential polynomial, in this chapter we prove our results for a special type of linear differential polynomials viz., the wronskians. In the chapter we also establish some newly developed results based on the comparative growth properties of composite entire or meromorphic functions and wronskians generated by one of the factors.

In the sequel we use the following notations: (i) $\log^{[k]} x = \log(\log^{[k-1]} x)$ for k=1,2,3,... and $\log^{[0]} x = x$ and (ii) $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ for k=1,2,3,... and $\exp^{[0]} x = x$.

The following definitions are well known.

The results of this chapter have been published in **International Journal of Mathematical Analysis,** see **[10]**.

Definition 3.1.1. *The order ρ_f and lower order λ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that,

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 3.1.2. *The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of a meromorphic function f is defined as*

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

If f is entire, then

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 3.1.3. *[31] Let f be a meromorphic function of order zero. Then the quantities ρ_f^*, λ_f^* and $\bar{\rho}_f^*$, $\bar{\lambda}_f^*$ are defined in the following way*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}$$

and

$$\bar{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}, \quad \bar{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}.$$

If f is entire then clearly

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}$$

and

$$\bar{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}, \quad \bar{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}.$$

Definition 3.1.4. *The type σ_f of a meromorphic function f is defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}},\ 0 < \rho_f < \infty.$$

When f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}},\ 0 < \rho_f < \infty.$$

Definition 3.1.5. *A meromorphic function $a = a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$.*

Definition 3.1.6. *Let $a_1, a_2, ..., a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ..., a_k, f)$ the Wronskian determinant of $a_1, a_2, ..., a_k, f$ i.e.,*

$$\mathrm{L(f)} = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 3.1.7. *If $a \in \mathbb{C} \cup \{\infty\}$, the quantity*

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

is called the Nevanlinna deficiency of the value 'a'.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C}\cup\{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) \leq 2$ (cf.[22], p.43). If in particular $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximum deficiency sum.

3.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 3.2.1. *[7] If f and g be two entire functions then for all sufficiently large values of r,*

$$M(r, fog) \geq M(\frac{1}{8}M(\frac{r}{2}, g) - |g(0)|, f).$$

Lemma 3.2.2. *[1] Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, fog) \leq \{1 + o(1)\}\frac{T(r, g)}{\log M(r, g)}T(M(r, g), f).$$

Lemma 3.2.3. *[4] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, fog) \geq T(\exp(r^{\mu}), f)).$$

Lemma 3.2.4. *[34] Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$\lim_{r \to \infty} \frac{T(r, L(f))}{T(r, f)} = 1 + k - k\delta(\infty; f).$$

Lemma 3.2.5. *If f be a transcendental meromorphic function with the maximum deficiency sum, then the order and lower order of $L(f)$ are same as those of f. Also the type of $L(f)$ is $\{1 + k - k\delta(\infty; f)\}$ times that of f when f is of finite positive order.*

Proof. By Lemma 3.2.4,

$$\lim_{r \to \infty} \frac{\log T(r, L(f))}{\log T(r, f)}$$

exists and is equal to 1. So

$$\begin{aligned} \rho_{L(f)} &= \limsup_{r \to \infty} \frac{\log T(r, L(f))}{\log r} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} . \lim_{r \to \infty} \frac{\log T(r, L(f))}{\log T(r, f)} \\ &= \rho_f . 1 = \rho_f. \end{aligned}$$

In a similar manner, $\lambda_{L(f)} = \lambda_f$. Again

$$\begin{aligned} \sigma_{L(f)} &= \limsup_{r \to \infty} \frac{T(r, L(f))}{r^{\rho_{L(f)}}} = \lim_{r \to \infty} \frac{T(r, L(f))}{T(r, f)} . \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}} \\ &= \{1 + k - k\delta(\infty; f)\}.\sigma_f. \end{aligned}$$

This proves the lemma. ■

Lemma 3.2.6. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then the hyper order (hyper lower order) of $L(f)$ and f are equal.*

Proof. In view of Lemma 3.2.4,

$$\lim_{r\to\infty} \frac{\log^{[2]} T(r, L(f))}{\log^{[2]} T(r, f)}$$

exists and is equal to 1. So

$$\begin{aligned}\bar{\rho}_{L(f)} &= \limsup_{r\to\infty} \frac{\log^{[2]} T(r, L(f))}{\log r} = \limsup_{r\to\infty} \frac{\log^{[2]} T(r, f)}{\log r} . \lim_{r\to\infty} \frac{\log^{[2]} T(r, L(f))}{\log^{[2]} T(r, f)} \\ &= \bar{\rho}_f .1 = \bar{\rho}_f .\end{aligned}$$

In a similar manner, $\bar{\lambda}_{L(f)} = \bar{\lambda}_f$.

This proves the lemma. ■

Lemma 3.2.7. *Let f be meromorphic and g be transcendental entire such that $\rho_f = 0$ and $\rho_g < \infty$. Then*

$$\rho_{fog} \leq \rho_f^* . \rho_g .$$

Proof. In view of Lemma 3.2.2 and the inequality $T(r, g) \leq \log^+ M(r, g)$, we get that

$$\begin{aligned}\rho_{fog} &= \limsup_{r\to\infty} \frac{\log T(r, fog)}{\log r} \leq \limsup_{r\to\infty} \frac{\log T(M(r, g), f) + o(1)}{\log r} \\ &\leq \limsup_{r\to\infty} \frac{\log T(M(r, g), f)}{\log^{[2]} M(r, g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log r} = \rho_f^* . \rho_g .\end{aligned}$$

This proves the lemma. ■

Remark 3.2.1. *The sign '$\leq$' in Lemma 3.2.7 cannot be removed by '$<$' only as we see in the following example.*

Example 3.2.1. *Let $f = z$ and $g = \exp z$.*
Then $\rho_{fog} = 1$, $\rho_g = 1$ and $\rho_f = 0$. So

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r} = \limsup_{r\to\infty} \frac{\log^{[2]} r}{\log^{[2]} r} = 1.$$

Therefore $\rho_{fog} = \rho_f^* . \rho_g$.

3.3 Theorems.

In this section we present the main results of the chapter.

Theorem 3.3.1. *Let f be transcendental meromorphic and g be entire satisfying the following conditions:(i) ρ_f and ρ_g are both finite, (ii) ρ_f is positive and (iii) $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$. Then for $p'>0$ and each $\alpha\in(-\infty,\infty)$,*

$$\liminf_{r\to\infty}\frac{\{\log T(r,fog)\}^{1+\alpha}}{\log T(\exp(r^{p'}),L(f))}=0 \quad \text{if } p'>(1+\alpha)\rho_g.$$

Proof. If $1+\alpha\leq 0$, the theorem is trivial. So we take $1+\alpha>0$. Since $T(r,g)\leq\log^{+}M(r,g)$, by Lemma 3.2.2, we get for all sufficiently large values of r,

$$\begin{aligned}
T(r,fog) &\leq \{1+o(1)\}T(M(r,g),f)\\
\text{i.e.,}\log T(r,fog) &\leq \log\{1+o(1)\}+\log T(M(r,g),f)\\
\text{i.e.,}\log T(r,fog) &\leq o(1)+(\rho_f+\varepsilon)\log M(r,g)\\
\text{i.e.,}\log T(r,fog) &\leq o(1)+(\rho_f+\varepsilon)r^{(\rho_g+\varepsilon)}\\
\text{i.e.,}\log T(r,fog) &\leq r^{(\rho_g+\varepsilon)}\{(\rho_f+\varepsilon)+o(1)\}\\
\text{i.e.,}\{\log T(r,fog)\}^{1+\alpha} &\leq r^{(\rho_g+\varepsilon)(1+\alpha)}\{(\rho_f+\varepsilon)+o(1)\}^{1+\alpha}. \quad (3.1)
\end{aligned}$$

Again in view of Lemma 3.2.5 we have for a sequence of values of r tending to infinity and for $\varepsilon>0$,

$$\log T(\exp(r^{p'}),L(f))>(\rho_{L(f)}-\varepsilon)\log(\exp(r^{p'}))=(\rho_f-\varepsilon)r^{p'}. \quad (3.2)$$

Now combining (3.1) and (3.2) we obtain for a sequence of values of r tending to infinity,

$$\frac{\{\log T(r,fog)\}^{1+\alpha}}{\log T(\exp(r^{p'}),L(f))}\leq\frac{r^{(\rho_g+\varepsilon)(1+\alpha)}\{(\rho_f+\varepsilon)+o(1)\}^{1+\alpha}}{(\rho_f-\varepsilon)r^{p'}}$$

from which the theorem follows because we can choose ε such that

$$0<\varepsilon<\min\{\rho_f,\frac{p'}{1+\alpha}-\rho_g\}.$$

This proves the theorem. ■

Remark 3.3.1. *The condition $p' > (1+\alpha)\rho_g$ is essential in Theorem 3.3.1 as we see in the next example.*

Example 3.3.1. *Let $f = \exp z$, $g = \exp z$, $\alpha = 0$ and $p' = 1$. Then*

$$\rho_f = 1 = \rho_g \text{ and } \sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$$

Also let

$$L(f) = \begin{vmatrix} a_1 & f \\ a_1' & f' \end{vmatrix}.$$

Then taking $a_1 = 1$ we get

$$L(f) = \begin{vmatrix} 1 & e^z \\ 0 & e^z \end{vmatrix} = e^z.$$

Now

$$\begin{aligned} \log T(r, fog) &= \log T(r, \exp^{[2]} z)) \\ &\sim \log\left\{\frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\right\} \quad (r\to\infty) \\ &\sim r - \frac{1}{2}\log r + O(1) \quad (r\to\infty). \end{aligned}$$

Therefore,

$$\begin{aligned} \liminf_{r\to\infty} \frac{\{\log T(r, fog)\}^{1+\alpha}}{\log T(\exp(r^{p'}), L(f))} &= \liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(\exp r, \exp z)} \\ &= \liminf_{r\to\infty} \frac{r - \frac{1}{2}\log r + O(1)}{\log\left\{\frac{\exp r}{\pi}\right\}} \\ &= \liminf_{r\to\infty} \frac{r - \frac{1}{2}\log r + O(1)}{r + O(1)} \\ &= 1. \end{aligned}$$

Theorem 3.3.2. *If f be meromorphic and g be transcendental entire such that $\rho_g < \infty$, $\rho_{fog} = \infty$ and $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$, then for every $A > 0$,*

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, L(g))} = \infty.$$

Proof. If possible, let there exists a constant β such that for all sufficiently large values of r, we have

$$\log T(r, fog) \leq \beta \log T(r^A, L(g)). \tag{3.3}$$

In view of Lemma 3.2.5, for all sufficiently large values of r we get that

$$\begin{aligned} \log T(r^A, L(g)) &\leq (\rho_{L(g)} + \varepsilon) A \log r \\ \text{i.e.,} \log T(r^A, L(g)) &\leq (\rho_g + \varepsilon) A \log r. \end{aligned} \tag{3.4}$$

Now combining (3.3) and (3.4) we obtain for all sufficiently large values of r,

$$\begin{aligned} \log T(r, fog) &\leq \beta(\rho_g + \varepsilon) A \log r \\ \text{i.e., } \rho_{fog} &\leq \beta A(\rho_g + \varepsilon), \end{aligned}$$

which contradicts the condition $\rho_{fog} = \infty$. So for a sequence of values of r tending to infinity, it follows that

$$\log T(r, fog) > \beta \log T(r^A, L(g)),$$

from which the theorem follows. ∎

Corollary 3.3.1. *Under the assumptions of Theorem 3.3.2,*

$$\limsup_{r \to \infty} \frac{T(r, fog)}{T(r^A, L(g))} = \infty.$$

Proof. By Theorem 3.3.2 we obtain for all sufficiently large values of r and for $K_1 > 1$,

$$\begin{aligned} \log T(r, fog) &> K_1 \log T(r^A, L(g)) \\ \text{i.e., } T(r, fog) &> \{T(r^A, L(g))\}^{K_1}, \end{aligned}$$

from which the corollary follows. ∎

Remark 3.3.2. *The condition $\rho_{fog} = \infty$ is necessary in Theorem 3.3.2 and Corollary 3.3.1 which is evident from the following example.*

Example 3.3.2. *Let $f = z$, $g = \exp z$ and $A = 1$.*
Then

$$\rho_g = 1 < \infty, \ \rho_{fog} = 1 < \infty \ \textit{and} \ \sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2.$$

Also let

$$L(g) = \begin{vmatrix} a_1 & g \\ a_1' & g' \end{vmatrix}.$$

Then considering $a_1 = 1$, *we obtain*

$$L(g) = \begin{vmatrix} 1 & e^z \\ 0 & e^z \end{vmatrix} = e^z.$$

Now

$$\begin{aligned} T(r, fog) &= T(r, \exp z) = \frac{r}{\pi} \\ \text{and } T(r^A, L(g)) &= T(r, \exp z) = \frac{r}{\pi}. \end{aligned}$$

Therefore

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, L(g))} = \limsup_{r\to\infty} \frac{\log r + O(1)}{\log r + O(1)} = 1$$

and

$$\limsup_{r\to\infty} \frac{T(r, fog)}{T(r^A, L(g))} = \limsup_{r\to\infty} \frac{\left(\frac{r}{\pi}\right)}{\left(\frac{r}{\pi}\right)} = 1.$$

Remark 3.3.3. *If we take* $\rho_f < \infty$ *and* $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ *instead of* $\rho_g < \infty$ *and* $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$ *respectively, then-Theorem 3.3.2 and Corollary 3.3.1 remain valid with* $L(g)$ *replaced by* $L(f)$ *in the denominator as we see in the following theorem and corollary.*

Theorem 3.3.3. *If* f *be transcendental meromorphic and* g *be entire such that* $\rho_f < \infty$, $\rho_{fog} = \infty$ *and* $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, *then for every* $A > 0$,

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, L(f))} = \infty.$$

Proof. If possible, let there exists a constant γ such that for all sufficiently large values of r, we have

$$\log T(r, fog) \leq \gamma \log T(r^A, L(f)).$$

In view of Lemma 3.2.5, for all sufficiently large values of r we get that

$$\log T(r^A, L(f)) \leq (\rho_f + \varepsilon) A \log r.$$

Now combining the above two inequalities, we get for all sufficiently large values of r,

$$\begin{aligned} \log T(r, fog) &\leq \gamma(\rho_f + \varepsilon) A \log r \\ \text{i.e., } \rho_{fog} &\leq \gamma A(\rho_f + \varepsilon), \end{aligned}$$

which contradicts the condition $\rho_{fog} = \infty$. So for a sequence of values of r tending to infinity, it follows that

$$\log T(r, fog) > \gamma \log T(r^A, L(f)),$$

from which the theorem follows. ■

Corollary 3.3.2. *Under the assumptions of Theorem 3.3.3,*

$$\limsup_{r \to \infty} \frac{T(r, fog)}{T(r^A, L(f))} = \infty.$$

Proof. In view of Theorem 3.3.3 we obtain for all sufficiently large values of r and for $K_2 > 1$,

$$\begin{aligned} \log T(r, fog) &> K_2 \log T(r^A, L(f)) \\ \text{i.e., } T(r, fog) &> \{T(r^A, L(f))\}^{K_2}, \end{aligned}$$

from which the corollary follows. ■

Remark 3.3.4. *The condition $\rho_{fog} = \infty$ is necessary in Theorem 3.3.3 and Corollary 3.3.2 which is evident from the following example.*

Example 3.3.3. *Let $f = \exp z$, $g = z$ and $A = 1$.*
Then

$$\rho_f = 1 < \infty, \ \rho_{fog} = 1 < \infty \text{ and } \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2.$$

Also let

$$L(f) = \begin{vmatrix} a_1 & f \\ a_1' & f' \end{vmatrix}.$$

Then taking $a_1 = 1$ *we get*

$$L(f) = e^z.$$

Now

$$\begin{aligned} T(r, fog) &= T(r, \exp z) = \frac{r}{\pi} \\ \text{and } T(r^A, L(f)) &= T(r, \exp z) = \frac{r}{\pi}. \end{aligned}$$

Therefore

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, L(f))} = \limsup_{r\to\infty} \frac{\log r + O(1)}{\log r + O(1)} = 1$$

and

$$\limsup_{r\to\infty} \frac{T(r, fog)}{T(r^A, L(f))} = 1.$$

Theorem 3.3.4. *Let* f *and* g *be two entire functions with* $\lambda_f > 0$ *and* $\rho_f < \lambda_g$. *Also let* f *be transcendental with* $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$. *Then*

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log M(r, L(f))} = \infty.$$

Proof. In view of Lemma 3.2.1,we have for all sufficiently large values of r,

$$\begin{aligned} M(r, fog) &\geq M(\frac{1}{16}M(\frac{r}{2}, g), f) \\ \text{i.e.,} \log^{[2]} M(r, fog) &\geq \log^{[2]} M(\frac{1}{16}M(\frac{r}{2}, g), f) \\ \text{i.e.,} \log^{[2]} M(r, fog) &\geq (\lambda_f - \varepsilon) \log(\frac{1}{16}M(\frac{r}{2}, g)) \\ \text{i.e.,} \log^{[2]} M(r, fog) &\geq (\lambda_f - \varepsilon) \log \frac{1}{16} + (\lambda_f - \varepsilon) \log M(\frac{r}{2}, g) \\ \text{i.e.,} \log^{[2]} M(r, fog) &\geq O(1) + (\lambda_f - \varepsilon)(\frac{r}{2})^{(\lambda_g - \varepsilon)}. \qquad (3.5) \end{aligned}$$

Again for all sufficiently large values of r, we get by Lemma 3.2.5 that

$$\log M(r, L(f)) \leq r^{(\rho_{L(f)} + \varepsilon)} = r^{(\rho_f + \varepsilon)}. \qquad (3.6)$$

Now combining (3.5) and (3.6) it follows for all sufficiently large values of r,

$$\frac{\log^{[2]} M(r, fog)}{\log M(r, L(f))} \geq \frac{O(1) + (\lambda_f - \varepsilon)(\frac{r}{2})^{(\lambda_g - \varepsilon)}}{r^{(\rho_f + \varepsilon)}}. \tag{3.7}$$

Since $\rho_f < \lambda_g$, we can choose $\varepsilon(> 0)$ in such a way that

$$\rho_f + \varepsilon < \lambda_g - \varepsilon. \tag{3.8}$$

Thus from (3.7) and (3.8) we obtain that

$$\liminf_{r \to \infty} \frac{\log^{[2]} M(r, fog)}{\log M(r, L(f))} = \infty,$$

from which the theorem follows. ■

Remark 3.3.5. *The condition $\rho_f < \lambda_g$ is necessary in Theorem 3.3.4 which is evident from the following two examples.*

Example 3.3.4. *Let $f = \exp z$ and $g = \exp z$.*
Then

$$\lambda_f = 1 > 0,\ \rho_f = 1 = \lambda_g \text{ and } \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2.$$

Also let

$$L(f) = \begin{vmatrix} a_1 & f \\ a_1' & f' \end{vmatrix}.$$

Then choosing $a_1 = 1$ we get

$$L(f) = e^z.$$

Again

$$\begin{aligned} M(r, fog) &= \exp^{[2]} r \\ \text{and } M(r, L(f)) &= \exp r. \end{aligned}$$

Therefore

$$\lim_{r \to \infty} \frac{\log^{[2]} M(r, fog)}{\log M(r, L(f))} = \lim_{r \to \infty} \frac{\log^{[2]}(\exp^{[2]} r)}{\log(\exp r)} = \lim_{r \to \infty} \frac{r}{r} = 1.$$

Example 3.3.5. *Let* $f = \exp z$ *and* $g = z$.
Then

$$\lambda_f = 1 > 0,\ \rho_f = 1 > 0 = \lambda_g \text{ and } \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2.$$

Also let

$$L(f) = \begin{vmatrix} a_1 & f \\ a_1' & f' \end{vmatrix}.$$

Then taking $a_1 = 1$ *we obtain*

$$L(f) = e^z.$$

Again

$$\begin{aligned} M(r, fog) &= \exp r \\ \text{and } M(r, L(f)) &= \exp r. \end{aligned}$$

Therefore

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log M(r, L(f))} = \lim_{r\to\infty} \frac{\log^{[2]}(\exp r)}{\log(\exp r)} = \lim_{r\to\infty} \frac{\log r}{r} = 0.$$

Theorem 3.3.5. *If* f *be a transcendental meromorphic function and* g *be entire with* $0 < \lambda_f \leq \rho_f < \infty$, $\rho_g < \infty$ *and* $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, *then*

$$\lim_{r\to\infty} \frac{T(r, fog)T(r, L(f))}{T(\exp(r^{p'}), L(f))} = 0, \text{ if } p' > \rho_g.$$

Proof. Since $T(r, g) \leq \log^+ M(r, g)$, for all sufficiently large values of r we get from Lemma 3.2.2,

$$\begin{aligned} T(r, fog) &\leq \{1 + o(1)\}T(M(r, g), f) \\ \text{i.e.,} T(r, fog) &\leq \{1 + o(1)\} \exp\{(\rho_f + \varepsilon)r^{(\rho_g+\varepsilon)}\}. \end{aligned} \quad (3.9)$$

Again by Lemma 3.2.5 we obtain for all sufficiently large values of r,

$$T(r, L(f)) \leq r^{(\rho_{L(f)}+\varepsilon)} = r^{(\rho_f+\varepsilon)}. \quad (3.10)$$

Now combining (3.9) and (3.10) it follows for all sufficiently large values of r,

$$T(r, fog)T(r, L(f)) \leq \{1 + o(1)\}r^{\rho_f+\varepsilon} \exp\{(\rho_f + \varepsilon)r^{\rho_g+\varepsilon}\}. \quad (3.11)$$

Also in view of Lemma 3.2.5 we have for all sufficiently large values of r,

$$\begin{aligned} \log T(\exp(r^{p'}), L(f)) &\geq (\lambda_{L(f)} - \varepsilon) \log\{\exp(r^{p'})\} \\ \text{i.e.,} T(\exp(r^{p'}), L(f)) &\geq \exp\{(\lambda_f - \varepsilon) r^{p'}\}. \end{aligned} \tag{3.12}$$

From (3.11) and (3.12) it follows for all sufficiently large values of r,

$$\frac{T(r, fog) T(r, L(f))}{T(\exp(r^{p'}), L(f))} \leq \frac{\{1 + o(1)\} r^{\rho_f + \varepsilon} \exp\{(\rho_f + \varepsilon) r^{\rho_g + \varepsilon}\}}{\exp\{(\lambda_f - \varepsilon) r^{p'}\}}. \tag{3.13}$$

As $p' > \rho_g$ so we can choose $\varepsilon (> 0)$ such that

$$p' > \rho_g + \varepsilon. \tag{3.14}$$

Thus the theorem follows from (3.13) and (3.14). ■

Theorem 3.3.6. *Let f be a transcendental meromorphic function and g be a transcendental entire function such that $0 < \lambda_f \leq \rho_f < \infty$ and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$. Then for every $A > 0$,*

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, L(f))} = \infty.$$

If further $\rho_g < \infty$ and $\sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$, then

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, L(g))} = \infty.$$

Proof. Since $\lambda_f > 0$, $\lambda_{fog} = \infty$ {cf.[3]}. So it follows that for arbitrary large N and for all large values of r

$$\log T(r, fog) > AN \log r. \tag{3.15}$$

Again since $\rho_f < \infty$, for all large values of r we get by Lemma 3.2.5,

$$\log T(r^A, L(f)) < A(\rho_f + 1) \log r. \tag{3.16}$$

Now from (3.15) and (3.16) it follows for all large values of r that

$$\frac{\log T(r, fog)}{\log T(r^A, L(f))} > \frac{AN \log r}{A(\rho_f + 1) \log r}$$

and so

$$\lim_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, L(f))} = \infty.$$

Again since $\rho_g < \infty$ then for all large values of r we obtain by Lemma 3.2.5

$$\log T(r^A, L(g)) < A(\rho_g + 1) \log r. \tag{3.17}$$

Now from (3.15) and (3.17) it follows for all large values of r that

$$\frac{\log T(r, fog)}{\log T(r^A, L(g))} > \frac{AN \log r}{A(\rho_g + 1) \log r}. \tag{3.18}$$

Thus the theorem follows from (3.18). ■

Theorem 3.3.7. *Let f be a transcendental meromorphic function with $0 < \lambda_f \le \rho_f < \infty$ and $\sum_{a \ne \infty} \delta(a; f) + \delta(\infty; f) = 2$ and g be entire. Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), fog)}{\log T(\exp(r^{\mu}), L(f))} = \infty,$$

where $0 < \mu < \rho_g$.

Proof. Let $0 < \mu' < \rho_g$. Then in view of Lemma 3.2.3, we get for a sequence of values of r tending to infinity,

$$\begin{aligned} \log T(r, fog) &\ge \log T(\exp(r^{\mu'}), f)) \\ \text{i.e.,} \log T(r, fog) &\ge (\lambda_f - \varepsilon) \log\{\exp(r^{\mu'})\} \\ \text{i.e.,} \log T(r, fog) &\ge (\lambda_f - \varepsilon) r^{\mu'} \\ \text{i.e.,} \log^{[2]} T(r, fog) &\ge O(1) + \mu' \log r. \end{aligned}$$

So for a sequence of values of r tending to infinity,

$$\begin{aligned} \log^{[2]} T(\exp(r^{\rho_g}), fog) &\ge O(1) + \mu' \log\{\exp(r^{\rho_g})\} \\ \text{i.e.,} \log^{[2]} T(\exp(r^{\rho_g}), fog) &\ge O(1) + \mu' r^{\rho_g}. \end{aligned} \tag{3.19}$$

Again in view of Lemma 3.2.5, we obtain for all sufficiently large values of r,

$$\begin{aligned} \log T(\exp(r^{\mu}), L(f)) &\le (\rho_{L(f)} + \varepsilon) \log\{\exp(r^{\mu})\} \\ \text{i.e.,} \log T(\exp(r^{\mu}), L(f)) &\le (\rho_f + \varepsilon) r^{\mu}. \end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), fog)}{\log T(\exp(r^{\mu}), L(f))} \geq \frac{O(1) + \mu' r^{\rho_g}}{(\rho_f + \varepsilon) r^{\mu}}. \tag{3.21}$$

Since $\mu < \rho_g$,we get from (3.21) that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), fog)}{\log T(\exp(r^{\mu}), L(f))} = \infty.$$

This proves the theorem. ■

Remark 3.3.6. *The condition $\mu < \rho_g$ in Theorem 3.3.7 is essential as we see in the following example.*

Example 3.3.6. *Let $f = \exp z$, $g = z$ and $\mu = 1$.*
Then

$$\lambda_f = 1 = \rho_f, \ \rho_g = 0 \ and \ \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2.$$

Also let

$$L(f) = \begin{vmatrix} a_1 & f \\ a_1' & f' \end{vmatrix}.$$

Then considering $a_1 = 1$, we get

$$L(f) = e^z.$$

Also

$$T(r, \exp z) = \frac{r}{\pi}.$$

So

$$\log^{[2]} T(\exp(r^{\rho_g}), fog) = \log^{[2]} T(e, \exp z) = \log^{[2]}(\frac{e}{\pi}) = O(1)$$

and

$$\log T(\exp(r^{\mu}), L(f)) = \log T(\exp r, \exp z) = \frac{\exp r}{\pi}.$$

Therefore,

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), fog)}{\log T(\exp(r^{\mu}), L(f))} = \limsup_{r \to \infty} \frac{O(1)}{(\frac{\exp r}{\pi})} = 0.$$

Theorem 3.3.8. *Let f be rational and g be transcendental meromorphic satisfying (i) $0 < \bar{\lambda}_{fog} \le \bar{\rho}_{fog} < \infty$, (ii) $0 < \bar{\lambda}_g \le \bar{\rho}_g < \infty$ and (iii) $\sum_{a \neq \infty} \delta(a;g) + \delta(\infty;g) = 2$. Then for any positive number A,*

$$\begin{aligned}\frac{\bar{\lambda}_{fog}}{A\bar{\rho}_g} &\le \liminf_{r\to\infty} \frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r^A, L(g))} \le \frac{\bar{\lambda}_{fog}}{A\bar{\lambda}_g} \\ &\le \limsup_{r\to\infty} \frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r^A, L(g))} \le \frac{\bar{\rho}_{fog}}{A\bar{\lambda}_g}.\end{aligned}$$

Proof. From the definition of hyper order and hyper lower order and by Lemma 3.2.6, we get for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[2]} T(r,fog) \ge (\bar{\lambda}_{fog} - \varepsilon)\log r \tag{3.22}$$

and

$$\begin{aligned}\log^{[2]} T(r^A, L(g)) &\le (\bar{\rho}_{L(g)} + \varepsilon)\log r^A \\ \text{i.e.,} \log^{[2]} T(r^A, L(g)) &\le A(\bar{\rho}_g + \varepsilon)\log r. \end{aligned} \tag{3.23}$$

Combining (3.22) and (3.23), we obtain for all sufficiently large values of r,

$$\frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r^A, L(g))} \ge \frac{(\bar{\lambda}_{fog} - \varepsilon)\log r}{A(\bar{\rho}_g + \varepsilon)\log r}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r^A, L(g))} \ge \frac{\bar{\lambda}_{fog}}{A\bar{\rho}_g}. \tag{3.24}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r,fog) \le (\bar{\lambda}_{fog} + \varepsilon)\log r. \tag{3.25}$$

Also in view of Lemma 3.2.6, we have for all sufficiently large values of r,

$$\begin{aligned}\log^{[2]} T(r^A, L(g)) &\ge (\bar{\lambda}_{L(g)} - \varepsilon)\log r^A \\ \text{i.e.,} \log^{[2]} T(r^A, L(g)) &\ge A(\bar{\lambda}_g - \varepsilon)\log r. \end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26), we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, L(g))} \leq \frac{(\bar{\lambda}_{fog} + \varepsilon)\log r}{A(\bar{\lambda}_g - \varepsilon)\log r}.$$

As $\varepsilon(> 0)$ is arbitrary it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, L(g))} \leq \frac{\bar{\lambda}_{fog}}{A\bar{\lambda}_g}. \tag{3.27}$$

Also for a sequence of values of r tending to infinity and by Lemma 3.2.6,

$$\begin{aligned} \log^{[2]} T(r^A, L(g)) &\leq A(\bar{\lambda}_{L(g)} + \varepsilon)\log r \\ \text{i.e.,} \log^{[2]} T(r^A, L(g)) &\leq A(\bar{\lambda}_g + \varepsilon)\log r. \end{aligned} \tag{3.28}$$

Combining (3.22) and (3.28) we have for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, L(g))} \geq \frac{(\bar{\lambda}_{fog} - \varepsilon)\log r}{A(\bar{\lambda}_g + \varepsilon)\log r}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, L(g))} \geq \frac{\bar{\lambda}_{fog}}{A\bar{\lambda}_g}. \tag{3.29}$$

Also for all sufficiently large values of r,

$$\log^{[2]} T(r, fog) \leq (\bar{\rho}_{fog} + \varepsilon)\log r. \tag{3.30}$$

From (3.26) and (3.30) we obtain for all sufficiently large values of r,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, L(g))} \leq \frac{(\bar{\rho}_{fog} + \varepsilon)\log r}{A(\bar{\lambda}_g - \varepsilon)\log r}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, L(g))} \leq \frac{\bar{\rho}_{fog}}{A\bar{\lambda}_g}. \tag{3.31}$$

Thus the theorem follows from (3.24), (3.27), (3.29) and (3.31). ■

Theorem 3.3.9. *Let f be meromorphic and g be transcendental entire such that (i) $0<\rho_g<\infty$, (ii) $\sigma_g>0$, (iii) $0<\rho_{fog}<\infty$, (iv) $\sigma_{fog}<\infty$, (v) $\rho_f^*<1$ and (vi) $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\liminf_{r\to\infty}\frac{T(r,fog)}{T(r,L(g))}=0.$$

Proof. From the definition of type, we have for arbitrary positive ε and for all sufficiently large values of r,

$$T(r,fog)\leq(\sigma_{fog}+\varepsilon)r^{\rho_{fog}}. \tag{3.32}$$

Again in view of Lemma 3.2.5, we get for a sequence of values of r tending to infinity that

$$\begin{aligned} T(r,L(g)) &\geq (\sigma_{L(g)}-\varepsilon)r^{\rho_{L(g)}} \\ \text{i.e.,} T(r,L(g)) &\geq [\{1+k-k\delta(\infty;g)\}\sigma_g-\varepsilon]r^{\rho_g}. \end{aligned} \tag{3.33}$$

Since $\rho_{fog}<\infty$, it follows that $\rho_f=0$ [cf.[20]].

So in view of Lemma 3.2.7, from (3.32) and (3.33) we obtain for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{T(r,fog)}{T(r,L(g))} &\leq \frac{(\sigma_{fog}+\varepsilon)r^{\rho_f^*\rho_g}}{[\{1+k-k\delta(\infty;g)\}\sigma_g-\varepsilon]r^{\rho_g}} \\ \text{i.e.,}\frac{T(r,fog)}{T(r,L(g))} &\leq \frac{(\sigma_{fog}+\varepsilon)r^{(\rho_f^*-1)\rho_g}}{[\{1+k-k\delta(\infty;g)\}\sigma_g-\varepsilon]}. \end{aligned}$$

Since $\varepsilon(>0)$ is arbitrary, in view of condition (v), it follows that

$$\liminf_{r\to\infty}\frac{T(r,fog)}{T(r,L(g))}=0.$$

This proves the theorem. ■

Remark 3.3.7. *The condition $\rho_f^*<1$ in Theorem 3.3.9 is essential which is evident from the following example.*

Example 3.3.7. *Let $f=z$ and $g=\exp z$.*
Then

$$\rho_g=1=\sigma_g,\ \rho_{fog}=1=\sigma_{fog},\ \rho_f=0$$

and

$$\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2.$$

Also let

$$L(g)=\begin{vmatrix} a_1 & g \\ a_1' & g' \end{vmatrix}.$$

Then taking $a_1 = 1$, *we obtain*

$$L(g)=e^z.$$

Also we have

$$\rho_f^* = \limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log^{[2]} r} = \limsup_{r\to\infty}\frac{\log^{[2]} r}{\log^{[2]} r} = 1.$$

Again

$$T(r,fog)=\frac{r}{\pi} \text{ and } T(r,L(g))=\frac{r}{\pi}.$$

Therefore

$$\liminf_{r\to\infty}\frac{T(r,fog)}{T(r,L(g))}=\liminf_{r\to\infty}\frac{\left(\frac{r}{\pi}\right)}{\left(\frac{r}{\pi}\right)}=1.$$

$-----X-----$