Chapter 2

ON THE GROWTH OF DIFFERENTIAL MONOMIALS

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2.1 Introduction, Definitions and Notations.

Let f and g be two transcendental entire functions defined in the open complex plane $\mathbb{C}$. It is well known that

$$\lim_{r\to\infty}\frac{M(r,fog)}{M(r,f)}=\lim_{r\to\infty}\frac{M(r,fog)}{M(r,g)}=\infty.$$

Clunie [7] discussed on the behaviour of

$$\frac{\log M(r,fog)}{\log M(r,f)} \quad \text{and} \quad \frac{\log M(r,fog)}{\log M(r,g)}$$

as $r\to\infty$. Song and Yang [42] worked on

$$\frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r,f)} \quad \text{and} \quad \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r,g)}$$

as $r\to\infty$ where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k=1,2,3,...$ and $\log^{[0]} x = x$.

Replacing maximum modulus functions by Nevanlinna's characteristic functions Clunie [7] proved for any two transcendental entire functions defined in the open complex plane $\mathbb{C}$,

$$\lim_{r\to\infty}\frac{T(r,fog)}{T(r,f)}=\infty \quad \text{and} \quad \lim_{r\to\infty}\frac{T(r,fog)}{T(r,g)}=\infty.$$

The results of this chapter have been published in **International Journal of Pure and Applied Mathematics,** see **[9]**.

Singh [43] proved some comparative growth properties of $\log T(r, fog)$ and $T(r, f)$. Singh [43] also raised the problem of investigating the comparative growth of $\log T(r, fog)$ and $T(r, g)$ and some results on the comparative growth of $\log T(r, fog)$ and $T(r, g)$ are proved in Lahiri [27].

Since $M(r, f)$ and $M(r, g)$ are increasing functions of r, Singh and Baloria [45] asked whether for any two entire functions f, g and for sufficiently large $R = R(r)$,

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R, f)} < \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R, g)} < \infty.$$

Singh and Baloria [45], Lahiri and Sharma [28], Liao and Yang [31] worked on this question.

Let f be a transcendental meromorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_0, n_1, ..., n_k$ be non-negative integers such that $\sum_{i=0}^{k} n_i \geq 1$. We call $P[f] = bf^{n_0}(f^{(1)})^{n_1}...(f^{(k)})^{n_k}$ where $T(r, b) = S(r, f)$, to be a differential monomial generated by f. The numbers $\gamma_P = \sum_{i=0}^{k} n_i$ and $\Gamma_P = \sum_{i=0}^{k}(i+1)n_i$ are called the degree and weight of $P[f]$ respectively {cf. [8]}.

In the chapter we develop some new results on the comparative growth properties of composite entire or meromorphic functions and differential monomials generated by one of the factors.

The following definitions are well known.

Definition 2.1.1. *The order ρ_f and lower order λ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If f is entire, one can easily verify that,

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

Definition 2.1.2. *The type σ_f of a meromorphic function f is defined as follows*

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

When f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0<\rho_f<\infty.$$

Definition 2.1.3. *[50] For $a \in \mathbb{C}\cup\{\infty\}$, we denote by $n(r,a;f \mid= 1)$,the number of simple zeros of $f-a$ in $|z|\le r$. $N(r,a;f\mid=1)$ is defined in terms of $n(r,a;f\mid=1)$ in the usual way. We put*

$$\delta_1(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f\mid=1)}{T(r,f)},$$

the deficiency of 'a' corresponding to the simple a- points of f i,e.simple zeros of $f-a$.

Yang [49] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C}\cup\{\infty\}$ for which

$$\delta_1(a;f) > 0 \text{ and } \sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) \le 4.$$

2.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.2.1. *[1] If f is meromorphic and g is entire then for all sufficiently large values of r ,*

$$T(r,fog) \le \{1+o(1)\}\frac{T(r,g)}{\log M(r,g)} T(M(r,g),f).$$

Lemma 2.2.2. *[4] Let f be meromorphic and g be entire and suppose that $0<\mu<\rho_g\le\infty$. Then for a sequence of values of r tending to infinity,*

$$T(r,fog) \ge T(\exp(r^{\mu}),f)).$$

Lemma 2.2.3. *[32] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$, then*

$$\lim_{r\to\infty} \frac{T(r,P[f])}{T(r,f)} = \Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty;f)$$

where

$$\Theta(\infty; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)}.$$

Lemma 2.2.4. *If f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$, then the order and lower order of $P[f]$ are same as those of f. Also the type of $P[f]$ is $\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}$ times that of f when f is of finite positive order.*

Proof. By Lemma 2.2.3,

$$\lim_{r\to\infty} \frac{\log T(r, P[f])}{\log T(r,f)}$$

exists and is equal to 1. So

$$\rho_{P[f]} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} . \lim_{r\to\infty} \frac{\log T(r, P[f])}{\log T(r,f)} = \rho_f . 1 = \rho_f.$$

In a similar manner, $\lambda_{P[f]} = \lambda_f$. Again

$$\begin{aligned}\sigma_{P[f]} &= \limsup_{r\to\infty} \frac{T(r, P[f])}{r^{\rho_{P[f]}}} = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} . \lim_{r\to\infty} \frac{T(r, P[f])}{T(r,f)} \\ &= \sigma_f . \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}.\end{aligned}$$

This proves the lemma. ■

2.3 Theorems.

In this section we present the main results of the chapter.

Theorem 2.3.1. *Let f be transcendental meromorphic and g be entire satisfying the following conditions: (i) ρ_f and ρ_g are both finite (ii) λ_f is positive and (iii) $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$.*

Then for $p' > 0$ and each $\alpha \in (-\infty, \infty)$,

$$\lim_{r\to\infty} \frac{\{\log T(r, fog)\}^{1+\alpha}}{\log T(\exp(r^{p'}), P[f])} = 0 \quad \text{if } p' > (1+\alpha)\rho_g.$$

Proof. If $1+\alpha \le 0$, the theorem is trivial. So we take $1+\alpha > 0$. Since $T(r,g) \le \log^+ M(r,g)$, by Lemma 2.2.1, we get for all sufficiently large values of r,

$$\begin{aligned} T(r,fog) &\le \{1+o(1)\}T(M(r,g),f) \\ \text{i.e.,} \log T(r,fog) &\le \log\{1+o(1)\} + \log T(M(r,g),f) \\ \text{i.e.,} \log T(r,fog) &\le o(1) + (\rho_f+\varepsilon) r^{(\rho_g+\varepsilon)} \\ \text{i.e.,} \log T(r,fog) &\le r^{(\rho_g+\varepsilon)}\{(\rho_f+\varepsilon)+o(1)\}. \end{aligned} \quad (2.1)$$

Again for all sufficiently large values of r, we obtain by Lemma 2.2.4,

$$\begin{aligned} \log T(\exp(r^{p'}), P[f]) &\ge (\lambda_{P[f]}-\varepsilon)\log(\exp(r^{p'})) \\ \text{i.e.,} \log T(\exp(r^{p'}), P[f]) &\ge (\lambda_f-\varepsilon) r^{p'}. \end{aligned} \quad (2.2)$$

Now from (2.1) and (2.2) we get for all sufficiently large values of r,

$$\frac{\{\log T(r,fog)\}^{1+\alpha}}{\log T(\exp(r^{p'}), P[f])} \le \frac{r^{(\rho_g+\varepsilon)(1+\alpha)}\{(\rho_f+\varepsilon)+o(1)\}^{1+\alpha}}{(\lambda_f-\varepsilon)r^{p'}} \quad (2.3)$$

from which the theorem follows because we can choose ε such that $0<\varepsilon<\min\{\lambda_f, \frac{p'}{1+\alpha}-\rho_g\}$.

This proves the theorem. ∎

Remark 2.3.1. *Theorem 2.3.1 improves Theorem 2 of Lahiri and Sharma [28].*

Remark 2.3.2. *If we choose f to be meromorphic and g to be transcendental entire satisfying $0<\lambda_g \le \rho_g < \infty$, $\rho_f < \infty$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$, the theorem remains true with $P[f]$ replaced by $P[g]$ in the denominator.*

Remark 2.3.3. *If we take $\rho_f > 0$ instead of $\lambda_f > 0$ and the other conditions remain same, the conclusion of Theorem 2.3.1 remains valid with 'limit' replaced by 'limit inferior'.*

Lahiri [27] proved the following theorem on the comparative growth of $\log T(r,fog)$ and $T(r,f)$.

Theorem 2.3.A. Let f and g be two non-constant entire functions such that $\lambda_g < \lambda_f \le \rho_f < \infty$. Then

$$\liminf_{r\to\infty} \frac{\log T(r,fog)}{T(r,f)} = 0.$$

In the following theorem we improve Theorem 2.3.A by considering f to be transcendental meromorphic. In fact we prove something more.

Theorem 2.3.2. *Let f be transcendental meromorphic and g be entire with $\lambda_g < \lambda_f \le \rho_f < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Then*

$$\liminf_{r\to\infty} \frac{\log\{T(r, fog)\log M(r,g)\}}{T(r, P[f])} = 0.$$

Proof. In view of Lemma 2.2.1, for all sufficiently large values of r, we have

$$T(r, fog)\log M(r,g) \le \{1 + o(1)\}T(r,g)T(M(r,g), f)$$

$$\begin{aligned} &\text{i.e.,} \log\{T(r, fog)\log M(r,g)\} \\ \le\ & \log\{1 + o(1)\} + \log T(r,g) + \log T(M(r,g), f). \end{aligned}$$

From above we get for a sequence of values of r tending to infinity,

$$\begin{aligned} &\log\{T(r, fog)\log M(r,g)\} \\ \le\ & o(1) + (\lambda_g + \varepsilon)\log r + (\rho_f + \varepsilon)\log M(r,g) \\ \le\ & o(1) + (\lambda_g + \varepsilon)\log r + (\rho_f + \varepsilon)r^{(\lambda_g+\varepsilon)}. \end{aligned} \tag{2.4}$$

Again for all sufficiently large values of r, we obtain by Lemma 2.2.4,

$$T(r, P[f]) \ge r^{(\lambda_{P[f]}-\varepsilon)} = r^{(\lambda_f - \varepsilon)}. \tag{2.5}$$

In view of (2.4) and (2.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log\{T(r, fog)\log M(r,g)\}}{T(r, P[f])} \le \frac{o(1) + (\lambda_g + \varepsilon)\log r + (\rho_f + \varepsilon)r^{(\lambda_g+\varepsilon)}}{r^{(\lambda_f-\varepsilon)}}. \tag{2.6}$$

Now as $\lambda_g < \lambda_f$, we can choose $\varepsilon(> 0)$ in such a way that $\lambda_g + \varepsilon < \lambda_f - \varepsilon$ and the theorem follows from (2.6). ∎

Remark 2.3.4. *If we take $\rho_g < \lambda_f \le \rho_f < \infty$ instead of $\lambda_g < \lambda_f \le \rho_f < \infty$ and the other conditions remain same, the conclusion of Theorem 2.3.2 remains valid with 'limit inferior' replaced by 'limit' as we see in the following theorem.*

Theorem 2.3.3. *Let f be transcendental meromorphic and g be entire with $\rho_g < \lambda_f \leq \rho_f < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Then*

$$\lim_{r \to \infty} \frac{\log\{T(r, fog) \log M(r, g)\}}{T(r, P[f])} = 0.$$

Proof. In view of Lemma 2.2.1, we get for all sufficiently large values of r,

$$\begin{aligned} & \log\{T(r, fog) \log M(r, g)\} \\ \leq\ & \log\{1 + o(1)\} + \log T(r, g) + \log T(M(r, g), f) \\ \leq\ & o(1) + (\rho_g + \varepsilon) \log r + (\rho_f + \varepsilon) \log M(r, g) \end{aligned}$$

$$\text{i.e.,} \log\{T(r, fog) \log M(r, g)\} \leq o(1) + (\rho_g + \varepsilon) \log r + (\rho_f + \varepsilon) r^{(\rho_g + \varepsilon)}. \tag{2.7}$$

Now combining (2.5) and (2.7) it follows for all sufficiently large values of r,

$$\frac{\log\{T(r, fog) \log M(r, g)\}}{T(r, P[f])} \leq \frac{o(1) + (\rho_g + \varepsilon) \log r + (\rho_f + \varepsilon) r^{(\rho_g + \varepsilon)}}{r^{(\lambda_f - \varepsilon)}}. \tag{2.8}$$

As $\rho_g < \lambda_f$ we can choose $\varepsilon (> 0)$ in such a way that $\rho_g + \varepsilon < \lambda_f - \varepsilon$ and thus the theorem follows from (2.8). ■

Theorem 2.3.4. *Let f be transcendental meromorphic and g be transcendental entire such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. If $\sigma_g < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$. Then*

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(\exp(r^{\rho_g}), P[f])} < \infty$$

$$\textit{and} \quad \limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(\exp(r^{\rho_g}), P[g])} < \infty.$$

Proof. Since $T(r, g) \leq \log^+ M(r, g)$, by Lemma 2.2.1 we get for all sufficiently large values of r,

$$\log T(r, fog) \leq o(1) + \log T(M(r, g), f).$$

So in view of Lemma 2.2.4, we obtain

$$\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(\exp(r^{\rho_g}),P[f])}\leq\limsup_{r\to\infty}\frac{o(1)+\log T(M(r,g),f)}{\log T(\exp(r^{\rho_g}),P[f])}$$

$$\begin{aligned}&\text{i.e., }\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(\exp(r^{\rho_g}),P[f])}\\ \leq\ &\limsup_{r\to\infty}\frac{\log T(M(r,g),f)}{\log M(r,g)}\cdot\limsup_{r\to\infty}\frac{\log M(r,g)}{r^{\rho_g}}\\ &\cdot\limsup_{r\to\infty}\frac{\log(\exp(r^{\rho_g}))}{\log T(\exp(r^{\rho_g}),P[f])}\end{aligned}$$

$$i.e., \limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(\exp(r^{\rho_g}),P[f])}\leq\rho_f\cdot\sigma_g\frac{1}{\lambda_{P[f]}}=\rho_f\cdot\sigma_g\frac{1}{\lambda_f}<\infty.$$

Similarly we obtain by Lemma 2.2.4,

$$\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(\exp(r^{\rho_g}),P[g])}\leq\rho_f\ \sigma_g\frac{1}{\lambda_{P[g]}}=\rho_f\ \sigma_g\frac{1}{\lambda_g}<\infty.$$

This proves the theorem. ■

Remark 2.3.5. *Theorem 2.3.4 is an analogue to Theorem 5 [31] for composite meromorphic functions.*

Theorem 2.3.5. *Let h be meromorphic and k,g be entire such that $\lambda_h>0$ and $\rho_g<\rho_k$. Then for every transcendental meromorphic function f of finite order and satisfying $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$,*

$$\limsup_{r\to\infty}\frac{T(r,hok)}{T(r,fog)T(r,P[f])}=\infty.$$

Proof. In view of Lemma 2.2.2, we get for a sequence of values of r tending to infinity,

$$T(r,hok)\geq T(\exp(r^{\mu}),h),\quad 0<\mu<\rho_k. \tag{2.9}$$

Again for all sufficiently large values of r,

$$\begin{aligned}\log T(\exp(r^{\mu}),h)&\geq(\lambda_h-\varepsilon)\log\{\exp(r^{\mu})\}\\ \text{i.e.,}T(\exp(r^{\mu}),h)&\geq\exp\{(\lambda_h-\varepsilon)r^{\mu}\}.\end{aligned} \tag{2.10}$$

Now combining (2.9) and (2.10), we get for a sequence of values of r tending to infinity,

$$T(r, hok) \geq \exp\{(\lambda_h - \varepsilon)r^{\mu}\}. \tag{2.11}$$

Since $T(r, g) \leq \log^+ M(r, g)$, for all sufficiently large values of r we obtain from Lemma 2.2.1

$$\begin{aligned}
\log T(r, fog) &\leq \log\{1 + o(1)\} + \log T(M(r, g), f) \\
\text{i.e.}, \log T(r, fog) &\leq \log\{1 + o(1)\} + (\rho_f + \varepsilon) \log M(r, g) \\
\text{i.e.}, \log T(r, fog) &\leq \log\{1 + o(1)\} + (\rho_f + \varepsilon) r^{(\rho_g + \varepsilon)} \\
\text{i.e.}, T(r, fog) &\leq \{1 + o(1)\} \exp\{(\rho_f + \varepsilon) r^{(\rho_g + \varepsilon)}\}. \qquad (2.12)
\end{aligned}$$

In view of Lemma 2.2.4, we have for all sufficiently large values of r,

$$\begin{aligned}
T(r, P[f]) &\leq r^{(\rho_{P[f]} + \varepsilon)} \\
\text{i.e.}, T(r, P[f]) &\leq r^{(\rho_f + \varepsilon)}. \qquad (2.13)
\end{aligned}$$

From (2.12) and (2.13) it follows for all large values of r,

$$T(r, fog) T(r, P[f]) \leq \{1 + o(1)\} r^{(\rho_f + \varepsilon)} \exp\{(\rho_f + \varepsilon) r^{(\rho_g + \varepsilon)}\}. \tag{2.14}$$

Now combining (2.11) and (2.14), we get for a sequence of values of r tending to infinity,

$$\frac{T(r, hok)}{T(r, fog) T(r, P[f])} \geq \frac{\exp\{(\lambda_h - \varepsilon) r^{\mu}\}}{\{1 + o(1)\} r^{(\rho_f + \varepsilon)} \exp\{(\rho_f + \varepsilon) r^{(\rho_g + \varepsilon)}\}}. \tag{2.15}$$

Since $\rho_g < \rho_k$, we can choose $\varepsilon(> 0)$ in such a manner that

$$\rho_g + \varepsilon < \mu < \rho_k. \tag{2.16}$$

Thus the theorem follows from (2.15) and (2.16). ■

Theorem 2.3.6. *Let f be meromorphic and g be transcendental entire such that $\rho_g < \infty$, $\lambda_{fog} = \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$, then for every $A(> 0)$,*

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, P[g])} = \infty.$$

Proof. If possible let there exists a constant β such that for a sequence of values of r tending to infinity

$$\log T(r, fog) \leq \beta . \log T(r^A, P[g]). \tag{2.17}$$

Again in view of Lemma 2.2.4, we obtain for all sufficiently large values of r,

$$\begin{aligned} \log T(r^A, P[g]) &\leq (\rho_{P[g]} + \varepsilon) \log r^A \\ \text{i.e.,} \log T(r^A, P[g]) &\leq (\rho_g + \varepsilon) A \log r. \end{aligned} \tag{2.18}$$

Now combining (2.17) and (2.18), we have for a sequence of values of r tending to infinity,

$$\begin{aligned} \log T(r, fog) &\leq \beta .(\rho_g + \varepsilon) A \log r \\ \text{i.e.,} \lambda_{fog} &\leq \beta . A(\rho_g + \varepsilon) \end{aligned}$$

which contradicts the condition $\lambda_{fog} = \infty$.

So for all sufficiently large values of r, we get

$$\log T(r, fog) > \beta . \log T(r^A, P[g]),$$

from which the theorem follows. ■

Corollary 2.3.1. *Under the assumptions of Theorem 2.3.6,*

$$\lim_{r \to \infty} \frac{T(r, fog)}{T(r^A, P[g])} = \infty.$$

Proof. By Theorem 2.3.6, we obtain for all sufficiently large values of r and $K > 1$,

$$\begin{aligned} \log T(r, fog) &> K . \log T(r^A, P[g]) \\ \text{i.e.,} T(r, fog) &> \{T(r^A, P[g])\}^K \end{aligned}$$

from which the corollary follows. ■

Remark 2.3.6. *If we take $\rho_f < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ instead of $\rho_g < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ respectively, then Theorem 2.3.6 and Corollary 2.3.1 remain valid with $P[g]$ replaced by $P[f]$ in the denominator.*

Theorem 2.3.7. *If f be meromorphic and g be transcendental entire satisfying (i) $\rho_f < \infty$ (ii) $0 < \lambda_g \leq \rho_g < \infty$ and (iii) $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$, then*

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, P[g])} \leq \frac{\rho_g}{\lambda_g}.$$

Proof. In view of Lemma 2.2.1 and Lemma 2.2.4 and since $T(r, g) \leq \log^+ M(r, g)$, we get for all sufficiently large values of r,

$$\begin{aligned}
\log T(r, fog) &\leq \log\{1 + o(1)\} + \log T(M(r, g), f) \\
\text{i.e.,} \log T(r, fog) &\leq o(1) + (\rho_f + \varepsilon) \log M(r, g) \\
\text{i.e.,} \log T(r, fog) &\leq \{(\rho_f + \varepsilon) + o(1)\} \log M(r, g) \\
\text{i.e.,} \log^{[2]} T(r, fog) &\leq \log^{[2]} M(r, g) + O(1) \\
\text{i.e.,} \frac{\log^{[2]} T(r, fog)}{\log T(r, P[g])} &\leq \frac{\log^{[2]} M(r, g) + O(1)}{\log r} \cdot \frac{\log r}{\log T(r, P[g])}
\end{aligned}$$

$$\begin{aligned}
&\text{i.e.,} \limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, P[g])} \\
\leq\ & \limsup_{r \to \infty} \frac{\{\log^{[2]} M(r, g) + O(1)\}}{\log r} \cdot \limsup_{r \to \infty} \frac{\log r}{\log T(r, P[g])} \\
=\ & \rho_g \frac{1}{\lambda_{P[g]}} = \frac{\rho_g}{\lambda_g}.
\end{aligned}$$

This proves the theorem. ■

Theorem 2.3.8. *Let f be meromorphic and g be transcendental entire satisfying $\lambda_f > 0$, $0 < \rho_g < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$. Then*

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(\exp(r^{\mu}), P[g])} = \infty, \quad \textit{where } 0 < \mu < \rho_g.$$

Proof. Let $0 < \mu < \mu' < \rho_g$. Then in view of Lemma 2.2.2 for a sequence of values of r tending to infinity,we get

$$\begin{aligned}
T(r, fog) &\geq T(\exp(r^{\mu'}), f) \\
\text{i.e.,} \log T(r, fog) &\geq \log T(\exp(r^{\mu'}), f) \\
\text{i.e.,} \log T(r, fog) &\geq (\lambda_f - \varepsilon) \log(\exp(r^{\mu'})) \\
\text{i.e.,} \log T(r, fog) &\geq (\lambda_f - \varepsilon) r^{\mu'}. \qquad (2.19)
\end{aligned}$$

Again in view of Lemma 2.2.4, since $\rho_f < \infty$ then for $\varepsilon(>0)$ and for all sufficiently large values of r,

$$\begin{aligned} \log T(\exp(r^{\mu}), P[g]) &\leq (\rho_{P[g]} + \varepsilon)\log\{\exp(r^{\mu})\} \\ \text{i.e.,} \log T(\exp(r^{\mu}), P[g]) &\leq (\rho_g + \varepsilon) r^{\mu}. \end{aligned} \tag{2.20}$$

Now combining (2.19) and (2.20), we obtain for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(\exp(r^{\mu}), P[g])} \geq \frac{(\lambda_f - \varepsilon) r^{\mu'}}{(\rho_g + \varepsilon) r^{\mu}}. \tag{2.21}$$

Since $\mu < \mu'$, the theorem follows from (2.21). ∎

Corollary 2.3.2. *Under the assumptions of Theorem 2.3.8,*

$$\limsup_{r\to\infty} \frac{T(r, fog)}{T(\exp(r^{\mu}), P[g])} = \infty, \quad \text{where } 0 < \mu < \rho_g.$$

Proof. In view of Theorem 2.3.8, we get for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T(r, fog) &\geq K \log T(\exp(r^{\mu}), P[g]) \quad \text{for } K > 1 \\ \text{i.e.,} T(r, fog) &\geq \{T(\exp(r^{\mu}), P[g])\}^{K}, \end{aligned}$$

from which the corollary follows. ∎

Theorem 2.3.9. *Let f be meromorphic and g be transcendental entire such that (i) $0 < \rho_g < \infty$, (ii) $0 < \sigma_g < \infty$ (iii) $\rho_f < \infty$ and (iv) $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; g) = 4$. Then*

$$\liminf_{r\to\infty} \frac{T(r, fog)}{T(r, P[g])} \leq \frac{\rho_f}{\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)\}}.$$

Proof. In view of the inequality $T(r, g) \leq \log^{+} M(r, g)$ and Lemma 2.2.1 we get for all sufficiently large values of r,

$$\begin{aligned} T(r, fog) &\leq \{1 + o(1)\} T(M(r, g), f) \\ \text{i.e.,} \log T(r, fog) &\leq \log\{1 + o(1)\} + \log T(M(r, g), f) \\ \text{i.e.,} \log T(r, fog) &\leq o(1) + (\rho_f + \varepsilon) \log M(r, g) \\ \text{i.e.,} \liminf_{r\to\infty} \frac{\log T(r, fog)}{T(r, P[g])} &\leq (\rho_f + \varepsilon) \liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, P[g])}. \end{aligned} \tag{2.22}$$

From the definition of type of an entire function it follows for all sufficiently large values of r,

$$\log M(r,g) \le (\sigma_g + \varepsilon) r^{\rho_g}. \tag{2.23}$$

Again in view of Lemma 2.2.4, we obtain for a sequence of values of r tending to infinity,

$$\begin{aligned} T(r, P[g]) &\ge (\sigma_{P[g]} - \varepsilon) r^{\rho_{P[g]}} \\ \text{i.e.,} T(r, P[g]) &\ge [\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)\}\sigma_g - \varepsilon] r^{\rho_g}. \end{aligned} \tag{2.24}$$

Now it follows from (2.23) and (2.24) that for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log M(r,g)}{T(r,P[g])} &\le \frac{(\sigma_g + \varepsilon)}{[\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)\}\sigma_g - \varepsilon]} \\ \text{i.e.,} \liminf_{r\to\infty} \frac{\log M(r,g)}{T(r,P[g])} &\le \frac{(\sigma_g + \varepsilon)}{[\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)\}\sigma_g - \varepsilon]}. \end{aligned}$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log M(r,g)}{T(r,P[g])} \le \frac{1}{\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)\}}. \tag{2.25}$$

Now from (2.22) and (2.25) we obtain that

$$\liminf_{r\to\infty} \frac{T(r, fog)}{T(r,P[g])} \le \frac{(\rho_f + \varepsilon)}{\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)\}}. \tag{2.26}$$

As $\varepsilon(> 0)$ is arbitrary, the theorem follows from (2.26). ■

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