



**Chapter 1**

**INTRODUCTION**

# Chapter 1

## INTRODUCTION

Let  $f$  be an entire function i.e., a function of a complex variable regular in the open complex plane. Then it has a Taylor's series expansion about  $z = 0$  as

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n + \cdots \quad (1.1)$$

which can be considered as an extension of a polynomial. The degree of a polynomial, which is equal to the number of zeros, estimates the rate of growth of the polynomial as the independent variable moves without bound. So more zeros, greater is the growth.

To characterise growth of an entire function and the distribution of its zeros a special growth scale called maximum modulus function of  $f$  on  $|z| = r$  is introduced as  $M(r, f) = \max_{|z|=r} |f(z)|$ . Obviously  $M(r, f)$  is unbounded for any non-constant entire function  $f$ . Also by the maximum modulus theorem  $M(r, f)$  increases monotonically as  $r$  increases {p.17, [47]}.

Again we know that  $\log M(r, f)$  is a continuous, convex and increasing function of  $\log r$ . In case of transcendental entire function i.e., the function which has an essential singularity at the point at infinity,  $M(r, f)$  grows faster than any positive power of  $r$ . Thus in order to estimate the growth of transcendental entire functions we choose a comparison function  $\exp(r^k)$ ,  $k > 0$ . Set

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r},$$

which is called the order of  $f$ . If  $0 \leq \rho < \infty$  then  $f$  is said to be of

finite order. The number

$$\lambda = \liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}$$

is called the lower order of  $f$ . Obviously  $\lambda \leq \rho$ . If  $\lambda = \rho$  then  $f$  is said to be of regular growth.

With  $0 < \rho < \infty$  the growth of an entire function is characterised more precisely by the type of the function. The number  $\tau$  given by

$$\tau = \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^\rho}$$

is called the type of  $f$ . Between two functions of same order one can be characterised to be of faster growth if its type is greater.

A function in the open complex plane  $\mathbb{C}$  is called a meromorphic function if the only possible singularities in  $\mathbb{C}$  are poles. In the last century the modern theory of value distribution of meromorphic functions, developed by Rolf Nevanlinna was one of the most significant achievements in the function theory. He elevated the theory of meromorphic functions to a new height by introducing his first and second fundamental theorems.

The first fundamental theorem provides an upper bound to the number of  $a$ -points of  $f$  whereas the second fundamental theorem deals with the more difficult question of lower bounds for the number of  $a$ -points of  $f$ . The second fundamental theorem is nothing but the extension of Picard's famous theorem that  $f$  must assume all values in the complex plane with at most two exceptions.

To state the above two fundamental theorems we need the following notations:

Let  $f$  be a meromorphic function in the finite complex plane  $\mathbb{C}$  and  $n(r, a; f) = n(r, \frac{1}{f-a})$  which is a non-negative integer for each  $r$ , denotes the number of  $a$ -points of  $f$  in  $|z| \leq r$ , counted with proper multiplicities, for a complex number ' $a$ ', finite or infinite. Obviously  $n(r, \infty; f) = n(r, f)$  represents the number of poles of  $f$  in  $|z| \leq r$  counted with proper multiplicities. Similarly we denote by  $\bar{n}(r, a; f)$  the number of distinct  $a$ -points of  $f$  in  $|z| \leq r$ , where  $a \in \mathbb{C} \cup \{\infty\}$ .

The functions  $N(r, a; f)$  and  $\bar{N}(r, a; f)$  are defined as follows:

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r$$

and  $\bar{N}(r, a; f) = \int_0^r \frac{\bar{n}(t, a; f) - \bar{n}(0, a; f)}{t} dt + \bar{n}(0, a; f) \log r.$

Also we write

$$N(r, \infty; f) = N(r, f) \quad \text{and} \quad \bar{N}(r, \infty; f) = \bar{N}(r, f).$$

The function  $N(r, f)$  is called the enumerative function.

Let us define

$$\begin{aligned} \log^+ x &= \log x \quad \text{for } x \geq 1 \\ &= 0 \quad \text{for } 0 \leq x < 1. \end{aligned}$$

Then immediately we get the following :

- (i)  $\log^+ x \geq 0$  if  $x \geq 0$ ,
- (ii)  $\log^+ x \geq \log x$  if  $x > 0$ ,
- (iii)  $\log^+ x \geq \log^+ y$  if  $x > y$ ,
- (iv)  $\log x = \log^+ x - \log^+ \frac{1}{x}$  if  $x > 0$ .

The quantity  $m(r, f)$  is defined by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

and is called the proximity function of  $f$ . The term  $m(r, f)$  is a sort of averaged magnitude of  $\log |f(z)|$  on arcs of  $|z| = r$  where  $|f(z)|$  is large. We write  $T(r, f) = m(r, f) + N(r, f)$ . The function  $T(r, f)$  is called the Nevanlinna's characteristic function of the meromorphic function  $f$  which plays an important role in the theory of meromorphic functions.

If  $f$  is an entire function then  $N(r, f) = 0$  and so  $T(r, f) = m(r, f)$ . For an entire function  $f$  the study of the comparative growth properties of  $T(r, f)$  and  $\log M(r, f)$  is a popular problem for the researchers.

Here we note that for a meromorphic function  $f$ ,  $T(r, f)$  is an increasing convex function of  $\log r$  {p.9, [22]}. If  $f$  is regular in  $|z| \leq R$  the following result establishes a relation between  $T(r, f)$  and  $M(r, f)$ , which is of frequent use.

**Theorem 1.1.** {p.18, [22]} *If  $f$  is regular for  $|z| \leq R$  then*

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f), \quad 0 \leq r < R.$$

Since for any positive integer  $p$  and complex numbers  $a_\nu (\nu = 1, 2, \dots, p)$ ,

$$\begin{aligned} \log^+ \left| \prod_{\nu=1}^p a_\nu \right| &\leq \sum_{\nu=1}^p \log^+ |a_\nu| \\ \text{and } \log^+ \left| \sum_{\nu=1}^p a_\nu \right| &\leq \sum_{\nu=1}^p \log^+ |a_\nu| + \log p, \end{aligned}$$

it is easy to show that for any arbitrary  $p$  meromorphic functions  $f_1, f_2, \dots, f_p$

$$\begin{aligned} m \left( r, \prod_{\nu=1}^p f_\nu(z) \right) &\leq \sum_{\nu=1}^p m(r, f_\nu(z)) \\ \text{and } m \left( r, \sum_{\nu=1}^p f_\nu(z) \right) &\leq \sum_{\nu=1}^p m(r, f_\nu(z)) + \log p. \end{aligned}$$

Again

$$\begin{aligned} N \left( r, \prod_{\nu=1}^p f_\nu(z) \right) &\leq \sum_{\nu=1}^p N(r, f_\nu(z)) \\ \text{and } N \left( r, \sum_{\nu=1}^p f_\nu(z) \right) &\leq \sum_{\nu=1}^p N(r, f_\nu(z)). \end{aligned}$$

From these we get

$$\begin{aligned} T \left( r, \prod_{\nu=1}^p f_\nu(z) \right) &\leq \sum_{\nu=1}^p T(r, f_\nu(z)) \\ \text{and } T \left( r, \sum_{\nu=1}^p f_\nu(z) \right) &\leq \sum_{\nu=1}^p T(r, f_\nu(z)) + \log p. \end{aligned}$$

We now state Nevanlinna's first fundamental theorem in the following form:

**Theorem 1.2.** {p.6, [22]} *If  $f$  is a meromorphic function in  $|z| < \infty$  and  $a$  be any complex number, finite or infinite, then*

$$m(r, a; f) + N(r, a; f) = T(r, f) + O(1).$$

The result shows remarkable symmetry exhibited by a meromorphic function in its behaviour relative to different complex numbers, finite or infinite. For different values of ' $a$ ' the sum  $m(r, a; f) + N(r, a; f)$  maintains a total, given by  $T(r, f)$  which is invariant apart from a bounded additive term.

The first term  $m(r, a; f)$  of this invariant sum is termed as the mean value of  $\log^+ \left| \frac{1}{f-a} \right|$  (or,  $\log^+ |f|$  if  $a = \infty$ ) on the circle  $|z| = r$ , contributes remarkably for those arcs on the circle where the functional values differ very little from the given value ' $a$ '. The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle  $|z| = r$  of the functional value  $f$  from the value ' $a$ '.

The term  $N(r, a; f)$  indicates the density of the average distribution of the roots of the equation  $f(z) = a$  in the disc  $|z| < r$ . The larger the number of  $a$ -points the faster this counting function for  $a$ -points grows with  $r$ .

At this junction we introduce some definitions.

**Definition 1.1.** {p.16, [22]} *Let  $S(r)$  be a real and non-negative increasing function for  $r_0 < r < \infty$ ,  $r_0 > 0$ . The order  $k$  and lower order  $\lambda$  of the function  $S(r)$  are defined as*

$$k = \limsup_{r \rightarrow \infty} \frac{\log S(r)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \rightarrow \infty} \frac{\log S(r)}{\log r}$$

respectively.

Moreover if  $0 < k < \infty$ , we set  $C = \limsup_{r \rightarrow \infty} \frac{S(r)}{r^k}$  and distinguish the following possibilities:

- (a)  $S(r)$  has maximal type if  $C = +\infty$ ,
- (b)  $S(r)$  has mean type if  $0 < C < +\infty$ ,
- (c)  $S(r)$  has minimal type if  $C = 0$  and

(d)  $S(r)$  has convergence class if  $\int_r^\infty \frac{S(t)}{t^{k+1}} dt$  converges.

The following theorem can be proved easily.

**Theorem 1.3.** {p.18, [22]} If  $f$  is an entire function then the order  $k$  of the functions  $S_1(r) = \log^+ M(r, f)$  and  $S_2(r) = T(r, f)$  is the same. Further if  $0 < k < \infty$ ,  $S_1(r)$  and  $S_2(r)$  belong to the same classes (a), (b), (c) or (d). Also lower order of  $S_1(r)$  and  $S_2(r)$  are same.

A meromorphic function is said to have order  $\rho$ , lower order  $\lambda$  and maximal, minimal, mean type or convergence class if the function  $T(r, f)$  has this property. For an entire function these coincides by the above theorem with corresponding definition in terms of  $\log^+ M(r, f)$  which is classical. The quantity  $\tau$  defined by

$$\tau = \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^\rho}$$

where  $\rho$  is the positive finite order of  $f$ , is called the type of  $f$ .

Nevanlinna's second fundamental theorem is the following:

**Theorem 1.4.** {p.31, [22]} Suppose that  $f$  is a non-constant meromorphic function in  $|z| \leq r$  and  $f(0) \neq \infty$ ,  $f'(0) \neq 0$ . Let  $a_1, a_2, \dots, a_q$  where  $q > 2$ , be distinct finite complex numbers,  $\delta > 0$  and suppose that  $|a_\mu - a_\nu| \geq \delta$  ( $0 < \delta < 1$ ) for  $1 \leq \mu < \nu \leq q$ . Then

$$m(r, \infty; f) + \sum_{\nu=1}^q m(r, a_\nu; f) \leq 2T(r, f) - N_1(r) + S(r, f),$$

where

$$N_1(r) = N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f')$$

and

$$S(r, f) = m\left(r, \frac{f'}{f}\right) + m\left(r, \sum_{\nu=1}^q \frac{f'}{f - a_\nu}\right) - q \log^+ \frac{3q}{\delta} + \log 2 + \log \frac{1}{|f'(0)|}.$$

The cases  $f(0) = \infty$  or  $f'(0) = 0$  can be dealt with suitable modification.

The quantity  $S(r, f)$  will in general play the role of an unimportant error term. The combination of this fact with the above theorem yields the second fundamental theorem.

We can estimate  $S(r, f)$  from the following theorem:

**Theorem 1.5.** {p.34, [22]} *Let  $f$  be a non-constant meromorphic function defined in  $|z| < R_0 \leq \infty$  and that  $S(r, f)$  is as in the above theorem. Then we have the following:*

(i) *If  $R_0 = +\infty$ ,  $S(r, f) = O\{\log T(r, f)\} + O(\log r)$ , as  $r \rightarrow \infty$  through all values if  $f$  has finite order and as  $r \rightarrow \infty$  outside a set  $E$  of finite linear measure otherwise.*

(ii) *If  $0 < R_0 < +\infty$ ,  $S(r, f) = O\{\log^+ T(r, f) + \log \frac{1}{R_0 - r}\}$  as  $r \rightarrow R_0$  outside a set  $E$  such that  $\int_E \frac{dr}{R_0 - r} < +\infty$ .*

*Further there is a point  $r$  outside  $E$  for which  $\rho < r < \rho'$  provided that*

$$0 < R - \rho' < e^{-2}(R - \rho).$$

As a consequence we get the following theorem immediately.

**Theorem 1.6.** {p.41, [22]} *Let  $f$  be a non-constant and meromorphic function in  $|z| < R_0$ . Then*

$$\frac{S(r, f)}{T(r, f)} \rightarrow 0 \tag{1.2}$$

*as  $r \rightarrow R_0$  with the following provisions:*

(a) *(1.2) holds without restrictions if  $R_0 = +\infty$  and  $f$  is of finite order in the plane.*

(b) *If  $f$  has infinite order in the plane and  $R_0 = +\infty$ , (1.2) still holds as  $r \rightarrow \infty$  outside a certain exceptional set  $E$  of finite length. Here  $E$  depends only on  $f$ .*

(c) *If  $R_0 < +\infty$  and  $\limsup_{r \rightarrow R_0} \frac{T(r, f)}{\log\{\frac{1}{R_0 - r}\}} = +\infty$ , then (1.2) holds as  $r \rightarrow R_0$  through a suitable sequence  $r_n$  which depends on  $f$  only.*

This theorem indicates why  $S(r, f)$  plays the role of an unimportant error term.

Let  $f$  be a non-constant meromorphic function in the plane. By  $S(r, f)$  we shall denote any quantity satisfying  $S(r, f) = o\{T(r, f)\}$  as  $r \rightarrow \infty$  possibly outside a set  $E$  of finite linear measure.

We shall now present an important consequence of Nevanlinna's second fundamental theorem, which is called Nevanlinna's theorem on deficient values. To rewrite this the following definitions are useful.

**Definition 1.2.** For a complex number 'a' we put

$$\delta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)}$$

and

$$\Theta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{T(r, f)},$$

where  $\bar{N}(r, a; f)$  is the counting function for distinct  $a$ -points of  $f$ .

Also

$$\theta(a; f) = \liminf_{r \rightarrow \infty} \frac{N(r, a; f) - \bar{N}(r, a; f)}{T(r, f)}.$$

The quantities  $\delta(a; f)$ ,  $\Theta(a; f)$  and  $\theta(a; f)$  are called respectively the deficiency, the ramification index and the index of multiplicity of the value 'a'. Evidently  $\delta(a; f)$  is positive only if there are relatively few roots of the equation  $f(z) = a$ . Maximum value of deficiency is 1 when the roots of the equation  $f(z) = a$  is very sparsely distributed, in particular when the value 'a' is a Picard's exceptional value. In general  $0 \leq m(r, a; f) \leq T(r, f)$  so we have  $0 \leq \delta(a; f) \leq 1$ . Thus the quantity gives us a very accurate measure for relative density of the points where the function  $f$  assumes the value 'a'. We shall call a value normal if  $\delta(a; f)$  vanishes. Again  $\theta(a; f)$  is positive only when  $f(z) = a$  has relatively many multiple roots.

Let  $\varepsilon (> 0)$  be an arbitrary small number. Then from the above definitions for sufficiently large values of  $r$  we have

$$N(r, a; f) - \bar{N}(r, a; f) > \{\theta(a; f) - \varepsilon\}T(r, f),$$

$$N(r, a; f) < \{1 - \delta(a; f) + \varepsilon\}T(r, f)$$

and so

$$\bar{N}(r, a; f) < \{1 - \delta(a; f) - \theta(a; f) + 2\varepsilon\}T(r, f)$$

so that

$$\Theta(a; f) \geq \delta(a; f) + \theta(a; f).$$

Now we state Nevanlinna's deficient value theorem.

**Theorem 1.7.** {p.43, [22]} *Let  $f$  be a non-constant meromorphic function defined on the plane. Then the set of values 'a' for which  $\Theta(a; f) > 0$  is countable and summing over all such values 'a'*

$$\sum_a \{\delta(a; f) + \theta(a; f)\} \leq \sum_a \Theta(a; f) \leq 2.$$

Now Picard's theorem, that a non-constant meromorphic function can have at most two Picard's exceptional values, follows easily from Nevanlinna's deficient value theorem because for a Picard's exceptional value  $a$ ,  $\delta(a; f) = 1$ .

Now we state Milloux's theorem which plays an important role in studying the properties of meromorphic derivatives.

**Theorem 1.8.** {p.55, [22]} *Let  $l$  be a positive integer and  $\psi = \sum_{\nu=0}^l a_\nu f^{(\nu)}$ , where  $T(r, a_\nu) = S(r, f)$ ,  $\nu = 0, 1, \dots, l$ . Then*

$$m\left(r, \frac{\psi}{f}\right) = S(r, f)$$

and  $T(r, \psi) \leq (l+1)T(r, f) + S(r, f)$ .

Milloux showed that in the second fundamental theorem we can replace the counting functions for certain roots of  $f(z) = a$  by roots of the equation  $\psi(z) = b$  where  $\psi$  is as above. The following theorem explains this fact.

**Theorem 1.9.** {p.57, [22]} *Let  $f$  be meromorphic and non-constant and  $\psi = \sum_{\nu=0}^l a_\nu f^{(\nu)}$ ,  $l$  is a positive integer, be non-constant in the complex plane. Then*

$$T(r, f) \leq \bar{N}(r, f) + N(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{\psi - 1}) - N_0(r, \frac{1}{\psi'}) + S(r, f),$$

where in  $N_0(r, \frac{1}{\psi'})$  only zeros of  $\psi'$  not corresponding to the repeated roots of  $\psi(z) = 1$  are to be considered.

If we take  $\psi = f$  then the above result reduces to the second fundamental theorem for  $q = 3$ .

Apart from **Chapter 1** the thesis contains seven chapters which are structured as follows:

- In **Chapter 2** we study the comparative growth of composite entire or meromorphic functions and differential monomials generated by one of the factors. The results of this chapter have been published in **International Journal of Pure and Applied Mathematics**, see [9].
- In **Chapter 3** we discuss about the comparative growth properties of composite entire or meromorphic functions and wronskians generated by one of the factors. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [10].
- In **Chapter 4** we study the comparative growth properties of composite entire or meromorphic functions and differential polynomials generated by one of the factors. The results of this chapter have been published in **International Journal of Mathematical Sciences and Engineering Applications**, see [14].
- In **Chapter 5** we study the comparative growth properties of composite entire functions whose left or right factors are of  $L$ -bounded index or of non-uniform  $L$ -bounded index. Further we introduce the notion of relative  $L$ -order and relative  $L$ -lower order of entire functions and discuss some of their properties. In this chapter we also discuss about the relationship between  $L$ -orders ( $L$ -lower orders,  $L$ -types) of two entire and meromorphic functions based on sharing of values of them. The results of this chapter have been published in **Wesleyan Journal of Research**, see [11] and **Journal of Mathematics**, see {[12], [13]}.
- In **Chapter 6** we discuss about the comparative growth properties of composite entire functions using generalised  $L^*$ -order and generalised  $L^*$ -type. Further we introduce the notion of generalised relative  $L^*$ -order (generalised relative  $L^*$ -lower order) of entire functions and discuss some of their properties. Some portion of the results of this chapter have been published in **International Journal of Pure and Applied Mathematics**, see [15]

and the remaining portion have been accepted for publication and to appear in **International Journal of Contemporary Mathematical Sciences**, see [18].

- In **Chapter 7** we compare the relative Valiron defect with the relative Nevanlinna defect of differential monomials generated by a transcendental meromorphic function. The results of this chapter have been accepted for publication and to appear in **Wesleyan Journal of Research**, see [17].
- In **Chapter 8** we consider two different meromorphic functions having common roots and find some relations involving their relative proximate deficiency. The results of this chapter have been accepted for publication and to appear in **International Mathematical Forum**, see [16].

From **Chapter 2** onwards when we write **Theorem  $a.b.c$**  (or **Corollary  $a.b.c$**  etc.) where  $a$ ,  $b$  and  $c$  are positive integers, we mean the  $c$ -th theorem (or  $c$ -th corollary etc.) of the  $b$ -th section in the  $a$ -th chapter. Also by **equation number  $(a.b)$**  we mean the  $b$ -th equation in the  $a$ -th chapter for positive integers  $a$  and  $b$ . Individual chapters have been presented in such a manner that they are almost independent of the other chapters. The references to books and journals have been classified as bibliography and are given at the end of the thesis.

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