Chapter 8

ON RELATIVE PROXIMATE DEFICIENCY OF THE COMMON ROOTS OF TWO MEROMORPHIC FUNCTIONS

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8.1 Introduction, Definitions and Notations.

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. The order ρ_f of f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If $\rho_f < \infty$ then f is of finite order. A function $\rho_f(r)$ is called a proximate order of f if the following conditions hold:

(i)$\rho_f(r)$ is non-negative and continuous for $r > r_0$, say,

(ii)$\rho_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\rho'_f(r+0)$ and $\rho'_f(r-0)$ exist,

(iii) $\lim\limits_{r\to\infty} \rho_f(r) = \rho_f$,

(iv) $\lim\limits_{r\to\infty} r\rho'_f(r) \log r = 0$ and

(v)$\limsup\limits_{r\to\infty} \dfrac{T(r,f)}{r^{\rho_f(r)}} = 1$ where $T(r,f)$ is the Nevanlinna's characteristic function of f.

The existence of such a proximate order is proved by Lahiri [26].

Let f_1 and f_2 be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let $n_0(r,a)$ denotes the number of common

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roots in the disk $|z| \leq r$ of the two equations $f_1 = a$ and $f_2 = a$ where a is any complex number and $\bar{n}_0(r, a)$ denotes the number of distinct common roots in the disk $|z| \leq r$ of the two equations $f_1 = a$ and $f_2 = a$. Singh [40] found some relations on the relative defects corresponding to the common roots of two meromorphic functions. In the chapter, we further investigate the results of Singh [40] and prove some new results on relative defects of the common roots of $f_1 = a$ and $f_2 = a$ in terms of their proximate order.

In fact we estimate the ratio $\chi(r) = \dfrac{T(r, f_1) + T(r, f_2)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}}$ as r tending to infinity in this chapter.

To start this chapter, we require the following:

Let

$$\bar{N}_0(r, a) = \int_0^r \frac{\bar{n}_0(t, a) - \bar{n}_0(0, a)}{t} dt + \bar{n}_0(0, a) \log r$$

and

$$\bar{N}_{1,2}(r, a) = \bar{N}(r, \frac{1}{f_1 - a}) + \bar{N}(r, \frac{1}{f_2 - a}) - 2\bar{N}_0(r, a).$$

Also let $\bar{N}_0^{(k)}(r, a)$, $\bar{N}_{1,2}^{(k)}(r, a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$ and $f_2^{(k)}$, where k is any non negative integer.

Now we set the following quantities:

$$\delta_{\rho_f}^{(1,2)}(a) = 1 - \limsup_{r \to \infty} \frac{N_{1,2}(r, a)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}},$$

$$^{(k)}\delta_{\rho_f}^{(1,2)}(a) = 1 - \limsup_{r \to \infty} \frac{N_{1,2}^{(k)}(r, a)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}},$$

$$\delta_{\rho_f}^{(0)}(a) = 1 - \limsup_{r \to \infty} \frac{N_0(r, a)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}},$$

$$\Theta_{\rho_f}^{(1,2)}(a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_{1,2}(r, a)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}},$$

$$^{(k)}\Theta_{\rho_f}^{(1,2)}(a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_{1,2}^{(k)}(r, a)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}}$$

and

$$\Theta_{\rho_f}^{(0)}(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_0(r,a)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}}.$$

The term $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise.

8.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 8.2.1. *[41] Let f be a meromorphic function of finite order such that $\sum\limits_{a\neq\infty} \delta(a;f) = 1$ and $\delta(\infty;f) = 1$. Then for any non-negative integer k, $\lim\limits_{r\to\infty} \dfrac{T(r,f^{(k)})}{T(r,f)} = 1$.*

The following lemma is due to Milloux {p.55, [22]}.

Lemma 8.2.2. *[22] Let k be a positive integer and $\psi = \sum\limits_{i=0}^{k} a_i f^{(i)}$ where a_i are meromorphic functions such that $T(r,a_i) = S(r,f)$, for $i = 0,1,2,...,k$. Then $m(r,\dfrac{\psi}{f}) = S(r,f)$.*

8.3 Theorems.

In this section we present the main results of the chapter.

Theorem 8.3.1. *Let f_1 and f_2 be any two meromorphic functions of finite order such that $\bar{N}(r,f_1) = S(r,f_1)$ and $\bar{N}(r,f_2) = S(r,f_2)$. Also let a be a finite non-zero complex number. Then*

$$\delta_{\rho_f}^{(1,2)}(0) + 2\delta_{\rho_f}^{(0)}(0) + \delta_{\rho_f}^{(1,2)}(a) + 2\delta_{\rho_f}^{(0)}(a) + \limsup_{r\to\infty} \chi(r) \leq 6.$$

Proof. For any positive integer k, let us consider the following identity

$$\frac{a}{f} = 1 - \frac{f-a}{f^{(k)}}.\frac{f^{(k)}}{f}.$$

Since $m(r, \frac{1}{f}) \leq m(r, \frac{a}{f}) + O(1)$ we get from the above identity that

$$m(r, \frac{1}{f}) \leq m\left(r, \frac{f-a}{f^{(k)}}\right) + S(r, f). \tag{8.1}$$

Now by the relation $T(r, f) = T(r, \frac{1}{f}) + O(1)$ and Lemma 8.2.2, it follows from (8.1) that

$$\begin{aligned} m(r, \frac{1}{f}) &\leq T(r, \frac{f-a}{f^{(k)}}) - N\left(r, \frac{f-a}{f^{(k)}}\right) + S(r, f) \\ i.e., \ m(r, \frac{1}{f}) &\leq T(r, \frac{f^{(k)}}{f-a}) - N\left(r, \frac{f-a}{f^{(k)}}\right) + S(r, f) + O(1) \\ i.e., \ m(r, \frac{1}{f}) &\leq N(r, \frac{f^{(k)}}{f-a}) - N\left(r, \frac{f-a}{f^{(k)}}\right) \\ &\quad + S(r, f) + O(1). \end{aligned} \tag{8.2}$$

In view of {p.34, [22]}, the relation $N(r, f^{(k)}) = N(r, f) + k\bar{N}(r, f)$ and as $N\left(r, \frac{1}{f^{(k)}}\right) \geq 0$, it follows from (8.2) that

$$\begin{aligned} m(r, \frac{1}{f}) &\leq N(r, f^{(k)}) + N(r, \frac{1}{f-a}) - N(r, f-a) \\ &\quad - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) + O(1) \\ i.e., m(r, \frac{1}{f}) &\leq N(r, f) + k\bar{N}(r, f) + N(r, \frac{1}{f-a}) \\ &\quad - N(r, f) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) + O(1) \\ i.e., T(r, f) &\leq k\bar{N}(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{f-a}) \\ &\quad + S(r, f) + O(1). \end{aligned} \tag{8.3}$$

Applying this inequality for f_1 and f_2 we get that

$$\begin{aligned} T(r, f_1) + T(r, f_2) &\leq k[\bar{N}(r, f_1) + \bar{N}(r, f_2)] + N(r, \frac{1}{f_1}) + N(r, \frac{1}{f_2}) \\ &\quad + N(r, \frac{1}{f_1 - a}) + N(r, \frac{1}{f_2 - a}) \\ &\quad + S(r, f_1) + S(r, f_2) + O(1). \end{aligned} \tag{8.4}$$

In view of $\bar{N}(r,f_1) = S(r,f_1)$ and $\bar{N}(r,f_2) = S(r,f_2)$ and dividing both sides of (8.4) by $\{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}\}$ and taking limit superior we obtain that

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{T(r,f_1)+T(r,f_2)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}} \\
\leq\ & \{1-\delta_{\rho_f}^{(1,2)}(0)\} + 2\{1-\delta_{\rho_f}^{(0)}(0)\} + \{1-\delta_{\rho_f}^{(1,2)}(a)\} + 2\{1-\delta_{\rho_f}^{(0)}(a)\} \\
\leq\ & 6 - \delta_{\rho_f}^{(1,2)}(0) - 2\delta_{\rho_f}^{(0)}(0) - \delta_{\rho_f}^{(1,2)}(a) - 2\delta_{\rho_f}^{(0)}(a)
\end{aligned}$$

$$i.e., \delta_{\rho_f}^{(1,2)}(0) + 2\delta_{\rho_f}^{(0)}(0) + \delta_{\rho_f}^{(1,2)}(a) + 2\delta_{\rho_f}^{(0)}(a) + \limsup_{r\to\infty}\chi(r) \leq 6.$$

This proves the theorem. ■

Remark 8.3.1. *The condition that 'a be a finite non-zero complex number' in Theorem 8.3.1 is essential which is evident from the following two examples.*

Example 8.3.1. *Let* $f_1 = e^z$, $f_2 = e^{-z}$ *and* $a = 0$.
Then

$$T(r,f_1) = T(r,f_2) = \frac{r}{\pi}$$
$$\text{and } \delta_{\rho_f}^{(1,2)}(0) = \delta_{\rho_f}^{(0)}(0) = 1.$$

Also

$$\bar{N}(r,f_1) = S(r,f_1) \quad \text{and} \quad \bar{N}(r,f_2) = S(r,f_2).$$

Now in view of condition (v) in the definition of proximate order of a meromorphic function we get for all sufficiently large values of r and for arbitrary $\varepsilon_1(>0)$ *and* $\varepsilon_2(>0)$

$$\begin{aligned}
\chi(r) &= \frac{T(r,f_1)+T(r,f_2)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}} \\
&\leq \frac{T(r,f_1)+T(r,f_2)}{\frac{T(r,f_1)}{(1+\varepsilon_1)} + \frac{T(r,f_2)}{(1+\varepsilon_2)}} \\
&= \frac{\frac{r}{\pi}+\frac{r}{\pi}}{\frac{\frac{r}{\pi}}{(1+\varepsilon_1)} + \frac{\frac{r}{\pi}}{(1+\varepsilon_2)}} \\
&= \frac{2.\frac{r}{\pi}}{\frac{r}{\pi}\left(\frac{1}{(1+\varepsilon_1)} + \frac{1}{(1+\varepsilon_2)}\right)}
\end{aligned}$$

$$i.e., \limsup_{r\to\infty} \chi(r) \le 1.$$

Again by condition (v) in the definition of proximate order of a meromorphic function we get for a sequence of values of r tending to infinity and for arbitrary $\varepsilon_1(>0)$ *and* $\varepsilon_2(>0)$

$$\begin{aligned}
\chi(r) &= \frac{T(r,f_1)+T(r,f_2)}{r^{\rho_{f_1}(r)}+r^{\rho_{f_2}(r)}} \\
&\ge \frac{T(r,f_1)+T(r,f_2)}{\frac{T(r,f_1)}{(1-\varepsilon_1)}+\frac{T(r,f_2)}{(1-\varepsilon_2)}} \\
&= \frac{\frac{r}{\pi}+\frac{r}{\pi}}{\frac{\frac{r}{\pi}}{(1-\varepsilon_1)}+\frac{\frac{r}{\pi}}{(1-\varepsilon_2)}} \\
&= \frac{2.\frac{r}{\pi}}{\frac{r}{\pi}\left(\frac{1}{(1-\varepsilon_1)}+\frac{1}{(1-\varepsilon_2)}\right)}
\end{aligned}$$

$$i.e., \limsup_{r\to\infty} \chi(r) \ge 1.$$

So from the above two inequalities we have

$$\limsup_{r\to\infty} \chi(r) = 1.$$

Hence

$$\begin{aligned}
&\delta_{\rho_f}^{(1,2)}(0)+2\delta_{\rho_f}^{(0)}(0)+\delta_{\rho_f}^{(1,2)}(a)+2\delta_{\rho_f}^{(0)}(a)+\limsup_{r\to\infty}\chi(r) \\
=\; &\delta_{\rho_f}^{(1,2)}(0)+2\delta_{\rho_f}^{(0)}(0)+\delta_{\rho_f}^{(1,2)}(0)+2\delta_{\rho_f}^{(0)}(0)+\limsup_{r\to\infty}\chi(r) \\
=\; &1+2+1+2+1=7.
\end{aligned}$$

Example 8.3.2. *Let* $f_1 = e^z$, $f_2 = e^{-z}$ *and* $a = \infty$. *Then*

$$T(r,f_1) = T(r,f_2) = \frac{r}{\pi}$$

$$\text{and}\;\; \delta_{\rho_f}^{(1,2)}(0) = \delta_{\rho_f}^{(0)}(0) = \delta_{\rho_f}^{(1,2)}(\infty) = \delta_{\rho_f}^{(0)}(\infty) = 1.$$

Also

$$\bar{N}(r,f_1) = S(r,f_1), \quad \bar{N}(r,f_2) = S(r,f_2)$$

$$\text{and}\;\; \limsup_{r\to\infty}\chi(r) = 1.$$

Hence

$$\begin{aligned}
& \delta^{(1,2)}_{\rho_f}(0) + 2\delta^{(0)}_{\rho_f}(0) + \delta^{(1,2)}_{\rho_f}(a) + 2\delta^{(0)}_{\rho_f}(a) + \limsup_{r\to\infty} \chi(r) \\
= \; & \delta^{(1,2)}_{\rho_f}(0) + 2\delta^{(0)}_{\rho_f}(0) + \delta^{(1,2)}_{\rho_f}(\infty) + 2\delta^{(0)}_{\rho_f}(\infty) + \limsup_{r\to\infty} \chi(r) \\
= \; & 1+2+1+2+1 = 7.
\end{aligned}$$

Theorem 8.3.2. *Let f_1 and f_2 be two meromorphic functions of finite order such that $T(r, f_1^{(k)}) \sim lT(r, f_1)$ and $T(r, f_2^{(k)}) \sim lT(r, f_2)$, where k is any positive integer and $l \geq 1$. Then*

$$\left(\frac{l-1}{k}\right) \limsup_{r\to\infty} \chi(r) + \delta^{(1,2)}_{\rho_f}(\infty) + 2\delta^{(0)}_{\rho_f}(\infty) \leq 3.$$

Proof. In view of Lemma 8.2.2 we obtain that

$$\begin{aligned}
T(r, f^{(k)}) &= N(r, f^{(k)}) + m(r, f^{(k)}) \\
&= N(r, f) + k\bar{N}(r, f) + m(r, f) + S(r, f) \\
&= T(r, f) + k\bar{N}(r, f) + S(r, f) \\
&\leq T(r, f) + kN(r, f) + S(r, f). \qquad (8.5)
\end{aligned}$$

Applying (8.5) on f_1 and f_2 it follows that

$$\begin{aligned}
T(r, f_1^{(k)}) + T(r, f_2^{(k)}) \leq \; & [T(r, f_1) + T(r, f_2)] + k[N(r, f_1) + N(r, f_2)] \\
& + S(r, f_1) + S(r, f_2). \qquad (8.6)
\end{aligned}$$

Since $T(r, f_i^{(k)}) \sim l.T(r, f_i)$ for $i = 1, 2$ we get from (8.6) that

$$(l-1)[T(r, f_1) + T(r, f_2)] \leq k[N(r, f_1) + N(r, f_2)] + S(r, f_1) + S(r, f_2). \qquad (8.7)$$

Now dividing both sides of (8.7) by $\{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}\}$ and taking limit superior we obtain

$$\begin{aligned}
& (l-1) \limsup_{r\to\infty} \frac{T(r, f_1) + T(r, f_2)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}} \\
\leq \; & k[\{1 - \delta^{(1,2)}_{\rho_f}(\infty)\} + 2\{1 - \delta^{(0)}_{\rho_f}(\infty)\}]
\end{aligned}$$

$$\text{i.e.,} \left(\frac{l-1}{k}\right) \limsup_{r\to\infty} \chi(r) \leq 3 - \delta^{(1,2)}_{\rho_f}(\infty) - 2\delta^{(0)}_{\rho_f}(\infty). \qquad (8.8)$$

Thus the theorem follows from (8.8). ■

Remark 8.3.2. *The sign '$\leq$' cannot be replaced by '$<$' only in Theorem 8.3.2 as we see in the following example.*

Example 8.3.3. *Let $f_1 = e^z$, $f_2 = e^{-z}$, $k = 2$ and $l = 1$. Then*

$$T(r, f_1) = T(r, f_2) = \frac{r}{\pi}.$$

So,

$$T(r, f_1^{(2)}) \sim T(r, f_1) \quad \text{and} \quad T(r, f_2^{(2)}) \sim T(r, f_2).$$

Also

$$\delta_{\rho_f}^{(1,2)}(\infty) = \delta_{\rho_f}^{(0)}(\infty) = 1.$$

and

$$\limsup_{r \to \infty} \chi(r) = 1.$$

Hence

$$\left(\frac{l-1}{k}\right) \limsup_{r \to \infty} \chi(r) + \delta_{\rho_f}^{(1,2)}(\infty) + 2\delta_{\rho_f}^{(0)}(\infty) = 3.$$

Theorem 8.3.3. *Let f_1 and f_2 be any two meromorphic functions of finite order such that $\bar{N}(r, f_1) = S(r, f_1)$ and $\bar{N}(r, f_2) = S(r, f_2)$. Then for any positive integer k and for any two distinct finite complex numbers a and b*

$$2\delta_{\rho_f}^{(1,2)}(a) + {}^{(k)}\delta_{\rho_f}^{(1,2)}(b) + 4\delta_{\rho_f}^{(0)}(a) + 2\delta_{\rho_f}^{(0)}(b) + \limsup_{r \to \infty} \chi(r) \leq 8.$$

Proof. Considering the identity

$$\frac{b-a}{f-a} = \frac{f^{(k)}}{f-a} \cdot \left\{ \frac{f-a}{f^{(k)}} - \frac{f-b}{f^{(k)}} \right\},$$

we obtain in view of Lemma 8.2.2

$$\begin{aligned} m(r, \frac{b-a}{f-a}) &\leq m(r, \frac{f-a}{f^{(k)}}) + m(r, \frac{f-b}{f^{(k)}}) + S(r, f) \\ \text{i.e., } m(r, \frac{b-a}{f-a}) &\leq T(r, \frac{f-a}{f^{(k)}}) - N(r, \frac{f-a}{f^{(k)}}) + T(r, \frac{f-b}{f^{(k)}}) \\ &\quad - N(r, \frac{f-b}{f^{(k)}}) + S(r, f). \end{aligned} \tag{8.9}$$

Since $m(r, \frac{1}{f-a}) \le m(r, \frac{b-a}{f-a}) + O(1)$ and $T(r,f) = T(r, \frac{1}{f}) + O(1)$, it follows from (8.9) that

$$\begin{aligned} m(r, \frac{1}{f-a}) &\le N(r, \frac{f^{(k)}}{f-a}) - N(r, \frac{f-a}{f^{(k)}}) + N(r, \frac{f^{(k)}}{f-b}) \\ &\quad -N(r, \frac{f-b}{f^{(k)}}) + S(r,f) + O(1). \qquad (8.10) \end{aligned}$$

In view of {p.34, [22]} and as $N(r, \frac{1}{f^{(k)}}) \ge 0$ we get from (8.10) that

$$\begin{aligned} m(r, \frac{1}{f-a}) &\le N(r, f^{(k)}) + N(r, \frac{1}{f-a}) - N(r, f-a) \\ &\quad -N(r, \frac{1}{f^{(k)}}) + N(r, f^{(k)}) + N(r, \frac{1}{f-b}) \\ &\quad -N(r, f-b) - N(r, \frac{1}{f^{(k)}}) + S(r,f) + O(1) \\ &= 2N(r, f^{(k)}) - 2N(r,f) - 2N(r, \frac{1}{f^{(k)}}) + N(r, \frac{1}{f-a}) \\ &\quad +N(r, \frac{1}{f-b}) + S(r,f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e., T(r,f) &\le 2k.\bar{N}(r,f) + 2N(r, \frac{1}{f-a}) + N(r, \frac{1}{f-b}) \\ &\quad +S(r,f) + O(1). \qquad (8.11) \end{aligned}$$

Now applying (8.11) for f_1 and f_2 we obtain that

$$\begin{aligned} [T(r,f_1) + T(r,f_2)] &\le 2k[\bar{N}(r,f_1) + \bar{N}(r,f_2)] + 2[N(r, \frac{1}{f_1-a}) \\ &\quad +N(r, \frac{1}{f_2-a})] + [N(r, \frac{1}{f_1-b}) \\ &\quad +N(r, \frac{1}{f_2-b})] + S(r,f_1) \\ &\quad +S(r,f_2) + O(1). \qquad (8.12) \end{aligned}$$

As $\bar{N}(r,f_i) = S(r,f_i)$ for $i = 1,2$; dividing both sides of (8.12) by $\{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}\}$ and taking limit superior we get that

$$\begin{aligned} &\limsup_{r\to\infty} \frac{T(r,f_1) + T(r,f_2)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}} \\ &\le 2\{1 - \delta_{\rho_f}^{(1,2)}(a)\} + 4\{1 - \delta_{\rho_f}^{(0)}(a)\} + \{1 - {}^{(k)}\delta_{\rho_f}^{(1,2)}(b)\} + 2\{1 - \delta_{\rho_f}^{(0)}(b)\} \end{aligned}$$

$$i.e., 2\delta_{\rho_f}^{(1,2)}(a) + {}^{(k)}\delta_{\rho_f}^{(1,2)}(b) + 4\delta_{\rho_f}^{(0)}(a) + 2\delta_{\rho_f}^{(0)}(b) + \limsup_{r\to\infty} \chi(r) \le 8.$$

This proves the theorem. ■

Remark 8.3.3. *The condition that 'a and b are two distinct finite complex number' in Theorem 8.3.3 is essential which is evident from the following example.*

Example 8.3.4. *Let* $f_1 = e^z$ *and* $f_2 = e^{-z}$.
Also let $a = 0$, $b = \infty$ *and* $k = 2$.
Then

$$\bar{N}(r, f_1) = S(r, f_1), \quad \bar{N}(r, f_2) = S(r, f_2)$$
$$\text{and } \delta_{\rho_f}^{(1,2)}(0) = \delta_{\rho_f}^{(0)}(0) = {}^{(2)}\delta_{\rho_f}^{(1,2)}(\infty) = \delta_{\rho_f}^{(0)}(\infty) = 1.$$

Also

$$\limsup_{r\to\infty} \chi(r) = 1.$$

Hence

$$\begin{aligned} & 2\delta_{\rho_f}^{(1,2)}(a) +^{(k)} \delta_{\rho_f}^{(1,2)}(b) + 4\delta_{\rho_f}^{(0)}(a) + 2\delta_{\rho_f}^{(0)}(b) + \limsup_{r\to\infty} \chi(r) \\ = & \ 2\delta_{\rho_f}^{(1,2)}(0) +^{(2)} \delta_{\rho_f}^{(1,2)}(\infty) + 4\delta_{\rho_f}^{(0)}(0) + 2\delta_{\rho_f}^{(0)}(\infty) + \limsup_{r\to\infty} \chi(r) \\ = & \ 2 + 1 + 4 + 2 + 1 = 10. \end{aligned}$$

Theorem 8.3.4. *Let* f_1 *and* f_2 *be any two meromorphic functions both of finite orders such that* $\bar{N}(r, f_1) = S(r, f_1)$ *and* $\bar{N}(r, f_2) = S(r, f_2)$. *Then for any positive integer* k *and for any non-zero finite complex number* a,

$${}^{(k)}\delta_{\rho_f}^{(1,2)}(a) + \delta_{\rho_f}^{(1,2)}(a) + 2^{(k)}\delta_{\rho_f}^{(0)}(a) + 2\delta_{\rho_f}^{(0)}(a) + \limsup_{r\to\infty} \chi(r) \le 6.$$

Proof. From the identity

$$\frac{1}{f-a} = \frac{1}{a}.\left\{\frac{f^{(k)}}{f-a} - \frac{f^{(k)}-a}{f^{(n)}}.\frac{f^{(n)}}{f-a}\right\}$$

where $0 \le k < n$ and by Lemma 8.2.2 we get that

$$m(r, \frac{1}{f-a}) \le m(r, \frac{f^{(k)}-a}{f^{(n)}}) + S(r, f). \tag{8.13}$$

Now by the relation $T(r,f) = T(r,\frac{1}{f}) + O(1)$ and Lemma 8.2.2 it follows from (8.13) that

$$\begin{aligned} m(r,\frac{1}{f-a}) &\le T(r,\frac{f^{(k)}-a}{f^{(n)}}) - N(r,\frac{f^{(k)}-a}{f^{(n)}}) + S(r,f) \\ i.e.,\ m(r,\frac{1}{f-a}) &\le T(r,\frac{f^{(n)}}{f^{(k)}-a}) - N(r,\frac{f^{(k)}-a}{f^{(n)}}) \\ &\quad +S(r,f) + O(1) \\ i.e.,\ m(r,\frac{1}{f-a}) &\le N(r,\frac{f^{(n)}}{f^{(k)}-a}) - N(r,\frac{f^{(k)}-a}{f^{(n)}}) \\ &\quad +S(r,f) + O(1). \end{aligned} \tag{8.14}$$

Now in view of {p.34, [22]} and as $N(r,\frac{1}{f^{(n)}}) \ge 0$ we obtain from (8.14) that

$$\begin{aligned} m(r,\frac{1}{f-a}) &\le N(r,f^{(n)}) + N(r,\frac{1}{f^{(k)}-a}) - N(r,f^{(k)}-a) \\ &\quad -N(r,\frac{1}{f^{(n)}}) + S(r,f) + O(1) \\ i.e., T(r,\frac{1}{f-a}) &\le [N(r,f) + n\bar{N}(r,f)] + [N(r,\frac{1}{f^{(k)}-a}) \\ &\quad +N(r,\frac{1}{f-a})] - [N(r,f) + k\bar{N}(r,f)] \\ &\quad +S(r,f) + O(1) \\ i.e., T(r,f) &\le (n-k)\bar{N}(r,f) + N(r,\frac{1}{f^{(k)}-a}) \\ &\quad +N(r,\frac{1}{f-a}) + S(r,f) + O(1). \end{aligned} \tag{8.15}$$

Now applying this inequality for f_1 and f_2 it follows from (8.15) that

$$\begin{aligned} T(r,f_1) + T(r,f_2) &\le (n-k)[\bar{N}(r,f_1) + \bar{N}(r,f_2)] + [N(r,\frac{1}{f_1^{(k)}-a}) \\ &\quad +N(r,\frac{1}{f_2^{(k)}-a})] + [N(r,\frac{1}{f_1-a}) + N(r,\frac{1}{f_2-a})] \\ &\quad +S(r,f_1) + S(r,f_2) + O(1). \end{aligned} \tag{8.16}$$

As $\bar{N}(r,f_1)=S(r,f_1)$ and $\bar{N}(r,f_2)=S(r,f_2)$ dividing both sides of (8.16) by $\{r^{\rho_{f_1}(r)}+r^{\rho_{f_2}(r)}\}$ and taking limit superior we get that

$$\begin{aligned}&\limsup_{r\to\infty}\frac{T(r,f_1)+T(r,f_2)}{r^{\rho_{f_1}(r)}+r^{\rho_{f_2}(r)}}\\ \leq\ &\{1-{}^{(k)}\delta_{\rho_f}^{(1,2)}(a)\}+2\{1-{}^{(k)}\delta_{\rho_f}^{(0)}(a)\}+\{1-\delta_{\rho_f}^{(1,2)}(a)\}\\ &+2\{1-\delta_{\rho_f}^{(0)}(a)\}\end{aligned}$$

$$i.e.,{}^{(k)}\delta_{\rho_f}^{(1,2)}(a)+\delta_{\rho_f}^{(1,2)}(a)+2{}^{(k)}\delta_{\rho_f}^{(0)}(a)+2\delta_{\rho_f}^{(0)}(a)+\limsup_{r\to\infty}\chi(r)\leq 6.$$

Thus the theorem is established. ■

Remark 8.3.4. *The condition that 'a be a finite non-zero complex number' in Theorem 8.3.4 is necessary as we see in the following two examples.*

Example 8.3.5. *Let* $f_1=e^z$ *and* $f_2=e^{-z}$.
Also let $a=0$ *and* $k=2$.
Then

$$\bar{N}(r,f_1)=S(r,f_1),\quad \bar{N}(r,f_2)=S(r,f_2).$$

$$\text{and}\ \ \delta_{\rho_f}^{(1,2)}(0)=\delta_{\rho_f}^{(0)}(0)={}^{(2)}\delta_{\rho_f}^{(1,2)}(0)={}^{(2)}\delta_{\rho_f}^{(0)}(0)=1.$$

Also

$$\limsup_{r\to\infty}\chi(r)=1.$$

Hence

$$\begin{aligned}&{}^{(k)}\delta_{\rho_f}^{(1,2)}(a)+\delta_{\rho_f}^{(1,2)}(a)+2{}^{(k)}\delta_{\rho_f}^{(0)}(a)+2\delta_{\rho_f}^{(0)}(a)+\limsup_{r\to\infty}\chi(r)\\ =\ &{}^{(2)}\delta_{\rho_f}^{(1,2)}(0)+\delta_{\rho_f}^{(1,2)}(0)+2{}^{(2)}\delta_{\rho_f}^{(0)}(0)+2\delta_{\rho_f}^{(0)}(0)+\limsup_{r\to\infty}\chi(r)\\ =\ &1+1+2+2+1=7.\end{aligned}$$

Example 8.3.6. *Let* $f_1=e^z$ *and* $f_2=e^{-z}$.
Also let $a=\infty$ *and* $k=2$.
Then

$$\bar{N}(r,f_1)=S(r,f_1),\quad \bar{N}(r,f_2)=S(r,f_2).$$

$$\text{and}\ \ \delta_{\rho_f}^{(1,2)}(\infty)=\delta_{\rho_f}^{(0)}(\infty)={}^{(2)}\delta_{\rho_f}^{(1,2)}(\infty)={}^{(2)}\delta_{\rho_f}^{(0)}(\infty)=1.$$

Also

$$\limsup_{r\to\infty}\chi(r)=1.$$

Hence

$$
\begin{aligned}
& {}^{(k)}\delta^{(1,2)}_{\rho_f}(a) + \delta^{(1,2)}_{\rho_f}(a) + 2\,{}^{(k)}\delta^{(0)}_{\rho_f}(a) + 2\delta^{(0)}_{\rho_f}(a) + \limsup_{r\to\infty} \chi(r) \\
= \; & {}^{(2)}\delta^{(1,2)}_{\rho_f}(\infty) + \delta^{(1,2)}_{\rho_f}(\infty) + 2\,{}^{(2)}\delta^{(0)}_{\rho_f}(\infty) + 2\delta^{(0)}_{\rho_f}(\infty) + \limsup_{r\to\infty} \chi(r) \\
= \; & 1+1+2+2+1 = 7.
\end{aligned}
$$

Theorem 8.3.5. *Let f_1 and f_2 be any two meromorphic functions both of finite orders such that $\sum\limits_{\alpha\neq\infty} \delta(\alpha; f_1) = \delta(\infty; f_1) = 1$ and $\sum\limits_{\alpha\neq\infty} \delta(\alpha; f_2) = \delta(\infty; f_2) = 1$. Also, let a be a finite complex number and b, c be two distinct non-zero complex numbers then for any positive integer k,*

$$
\begin{aligned}
& \delta^{(1,2)}_{\rho_f}(a) + {}^{(k)}\Theta^{(1,2)}_{\rho_f}(b) + {}^{(k)}\Theta^{(1,2)}_{\rho_f}(c) + 2\delta^{(0)}_{\rho_f}(a) \\
& + 2\,{}^{(k)}\Theta^{(0)}_{\rho_f}(b) + 2\,{}^{(k)}\Theta^{(0)}_{\rho_f}(c) + \limsup_{r\to\infty} \chi(r) \leq 9.
\end{aligned}
$$

Proof. Since $\dfrac{1}{f-a} = \dfrac{f^{(k)}}{f-a}.\dfrac{1}{f^{(k)}}$ by Lemma 8.2.2 we obtain that

$$
m(r, \frac{1}{f-a}) \leq m(r, \frac{1}{f^{(k)}}) + S(r, f). \tag{8.17}
$$

Applying the relation $T(r, f) = T(r, \dfrac{1}{f}) + O(1)$ we get from (8.17) that

$$
m(r, \frac{1}{f-a}) \leq T(r, f^{(k)}) - N(r, \frac{1}{f^{(k)}}) + S(r, f) + O(1). \tag{8.18}
$$

Now by Nevanlinna's second fundamental theorem and Lemma 8.2.1 it follows from (8.18) that

$$
\begin{aligned}
m(r, \frac{1}{f-a}) \; \leq \; & \bar{N}(r, \frac{1}{f^{(k)}}) + \bar{N}(r, \frac{1}{f^{(k)}-b}) + \bar{N}(r, \frac{1}{f^{(k)}-c}) \\
& -N(r, \frac{1}{f^{(k)}}) + S(r, f) + O(1). \qquad (8.19)
\end{aligned}
$$

Since $\bar{N}(r, \frac{1}{f^{(k)}}) - N(r, \frac{1}{f^{(k)}}) \leq 0$, we obtain from (8.19) that

$$\begin{aligned} m(r, \frac{1}{f-a}) &\leq \bar{N}(r, \frac{1}{f^{(k)}-b}) + \bar{N}(r, \frac{1}{f^{(k)}-c}) + S(r,f) + O(1) \\ i.e., T(r,f) &\leq N(r, \frac{1}{f-a}) + \bar{N}(r, \frac{1}{f^{(k)}-b}) \\ &\quad + \bar{N}(r, \frac{1}{f^{(k)}-c}) + S(r,f) + O(1). \end{aligned} \tag{8.20}$$

Now applying (8.20) for f_1 and f_2 it follows that

$$\begin{aligned} T(r,f_1) + T(r,f_2) &\leq [N(r, \frac{1}{f_1-a}) + N(r, \frac{1}{f_2-a})] + [\bar{N}(r, \frac{1}{f_1^{(k)}-b}) \\ &\quad + \bar{N}(r, \frac{1}{f_2^{(k)}-b})] + [\bar{N}(r, \frac{1}{f_1^{(k)}-c}) \\ &\quad + \bar{N}(r, \frac{1}{f_2^{(k)}-c})] + S(r,f_1) \\ &\quad + S(r,f_2) + O(1). \end{aligned} \tag{8.21}$$

Dividing both sides of (8.21) by $\{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}\}$ and taking limit superior we get that

$$\begin{aligned} &\limsup_{r\to\infty} \frac{T(r,f_1)+T(r,f_2)}{r^{\rho_{f_1}(r)} + r^{\rho_{f_2}(r)}} \\ \leq\ & \{1 - \delta_{\rho_f}^{(1,2)}(a)\} + 2\{1 - \delta_{\rho_f}^{(0)}(a)\} + \{1 - {}^{(k)}\Theta_{\rho_f}^{(1,2)}(b)\} \\ & + 2\{1 - {}^{(k)}\Theta_{\rho_f}^{(0)}(b)\} + \{1 - {}^{(k)}\Theta_{\rho_f}^{(1,2)}(c)\} + 2\{1 - {}^{(k)}\Theta_{\rho_f}^{(0)}(c)\} \end{aligned}$$

$$\begin{aligned} i.e., \delta_{\rho_f}^{(1,2)}(a) + {}^{(k)}\Theta_{\rho_f}^{(1,2)}(b) + {}^{(k)}\Theta_{\rho_f}^{(1,2)}(c) + 2\delta_{\rho_f}^{(0)}(a) \\ + 2^{(k)}\Theta_{\rho_f}^{(0)}(b) + 2^{(k)}\Theta_{\rho_f}^{(0)}(c) + \limsup_{r\to\infty} \chi(r) \leq 9. \end{aligned}$$

This proves the theorem. ■

Theorem 8.3.6. *Let f_1 and f_2 be any two meromorphic functions of finite order with $\bar{N}(r,f_1) = S(r,f_1)$ and $\bar{N}(r,f_2) = S(r,f_2)$ and α be a non-zero finite complex number. Then*

$${}^{(k)}\delta_{\rho_f}^{(1,2)}(\alpha) + \delta_{\rho_f}^{(1,2)}(0) + 2^{(k)}\delta_{\rho_f}^{(0)}(\alpha) + 2\delta_{\rho_f}^{(0)}(0) + \limsup_{r\to\infty} \chi(r) \leq 6.$$

Proof. Considering the identity

$$\frac{\alpha}{f} = \frac{f^{(k)}}{f} - \frac{f^{(k)} - \alpha}{f^{(k+1)}} . \frac{f^{(k+1)}}{f}$$

we get in view of Lemma 8.2.2 and the relation $T(r, f) = T(r, \frac{1}{f}) + O(1)$,

$$\begin{aligned}
m(r, \frac{1}{f}) &\le m(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) + S(r, f) \\
i.e., m(r, \frac{1}{f}) &\le T(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) - N(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) + S(r, f) \\
i.e., \ m(r, \frac{1}{f}) &\le T(r, \frac{f^{(k+1)}}{f^{(k)} - \alpha}) - N(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) + S(r, f) + O(1) \\
i.e., \ m(r, \frac{1}{f}) &\le N(r, \frac{f^{(k+1)}}{f^{(k)} - \alpha}) - N(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) \\
&\quad + S(r, f) + O(1). \qquad (8.22)
\end{aligned}$$

Now in view of {p.34, [22]} and as $N(r, \frac{1}{f^{(k+1)}}) \ge 0$ it follows from (8.22) that

$$\begin{aligned}
m(r, \frac{1}{f}) &\le N(r, f^{(k+1)}) + N(r, \frac{1}{f^{(k)} - \alpha}) - N(r, f^{(k)} - \alpha) \\
&\quad - N(r, \frac{1}{f^{(k+1)}}) + S(r, f) + O(1) \\
i.e., m(r, \frac{1}{f}) &\le N(r, \frac{1}{f^{(k)} - \alpha}) + N(r, f^{(k+1)}) - N(r, f^{(k)}) \\
&\quad + S(r, f) + O(1) \\
i.e., m(r, \frac{1}{f}) &\le N(r, \frac{1}{f^{(k)} - \alpha}) + \bar{N}(r, f) + S(r, f) + O(1) \\
i.e., T(r, f) &\le N(r, \frac{1}{f^{(k)} - \alpha}) + N(r, \frac{1}{f}) + \bar{N}(r, f) \\
&\quad + S(r, f) + O(1). \qquad (8.23)
\end{aligned}$$

Now applying (8.23) for f_1 and f_2 we obtain that

$$\begin{aligned} T(r,f_1)+T(r,f_2) \leq\ & N(r,\frac{1}{f_1^{(k)}-\alpha})+N(r,\frac{1}{f_2^{(k)}-\alpha})+N(r,\frac{1}{f_1}) \\ & +N(r,\frac{1}{f_2})+\bar{N}(r,f_1)+\bar{N}(r,f_2) \\ & +S(r,f_1)+S(r,f_2)+O(1). \qquad (8.24) \end{aligned}$$

As $\bar{N}(r,f_i) = S(r,f_i)$ for $i = 1,2$; dividing both sides of (8.24) by $\{r^{\rho_{f_1}(r)}+r^{\rho_{f_2}(r)}\}$ and taking limit superior we get that

$$\begin{aligned} & \limsup_{r\to\infty} \frac{T(r,f_1)+T(r,f_2)}{r^{\rho_{f_1}(r)}+r^{\rho_{f_2}(r)}} \\ \leq\ & \{1-\ ^{(k)}\delta_{\rho_f}^{(1,2)}(\alpha)\}+2\{1-\ ^{(k)}\delta_{\rho_f}^{(0)}(\alpha)\}+\{1-\delta_{\rho_f}^{(1,2)}(0)\} \\ & +2\{1-\delta_{\rho_f}^{(0)}(0)\} \end{aligned}$$

$$i.e., ^{(k)}\delta_{\rho_f}^{(1,2)}(\alpha)+\delta_{\rho_f}^{(1,2)}(0)+2^{(k)}\delta_{\rho_f}^{(0)}(\alpha)+2\delta_{\rho_f}^{(0)}(0)+\limsup_{r\to\infty}\chi(r) \leq 6.$$

Thus the theorem follows. ■

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