Chapter 7

ON RELATIVE DEFECTS OF DIFFERENTIAL MONOMIALS

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7.1 Introduction, Definitions and Notations.

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $n(t, a; f)$ $(\bar{n}(t, a; f))$ the number of a-points (distinct a-points) of f in $|z| \leq t$, where an ∞ -point is a pole of f.We put

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r$$

and

$$\bar{N}(r, a; f) = \int_0^r \frac{\bar{n}(t, a; f) - \bar{n}(0, a; f)}{t} dt + \bar{n}(0, a; f) \log r.$$

The function $N(r, a; f)$ $(\bar{N}(r, a; f))$ are called the counting function of a-points (distinct a-points) of f. In many occasions $N(r, \infty; f)$ and $\bar{N}(r, \infty; f)$ are denoted by $N(r, f)$ and $\bar{N}(r, f)$ respectively.

We also put

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

The results of this chapter have been accepted for publication and to appear in **Wesleyan Journal of Research**, see [17].

where

$$\log^+ x = \log x, \text{ if } x \geq 1$$
$$= 0, \text{ if } 0 \leq x < 1.$$

For $a \in \mathbb{C}$ we denote by $m(r, \frac{1}{f-a})$ the function $m(r, a; f)$ and we mean by $m(r, \infty; f)$ the function $m(r, f)$, which is called the proximity function of f.

The function $T(r, f) = m(r, f) + N(r, f)$ is called the characteristic function of f. If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, a; f)}{T(r, f)}$$

is called the Nevanlinna deficiency of the value 'a'. From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C}\cup\{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf.[22], p.43). If in particular $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

Similarly, the Valiron deficiency $\Delta(a; f)$ of the value 'a' is defined as

$$\Delta(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r\to\infty} \frac{m(r, a; f)}{T(r, f)}.$$

Milloux [35] introduced the concept of absolute defect of 'a' with respect to f'. Later Xiong [48] extended this definition. He introduced the term

$$\delta_R^{(k)}(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f^{(k)})}{T(r, f)}, \quad \text{for } k = 1, 2, 3, ...$$

and called it the relative Nevanlinna defect of 'a' with respect to $f^{(k)}$. Xiong [48]has shown various relations between the usual defects and the relative defects for meromorphic functions. Singh [39] introduced the term relative defect for distinct zeros and poles and established various relations between it, relative defects and the usual defects.

Let f be a transcendental meromorphic function defined in the open complex plane $\mathbb{C}$. Also let n_0,n_1,n_2,...,n_k be non-negative integers such that $\sum_{i=0}^{k} n_i \geq 1$. We call $P[f] = b_0 f^{n_0}(f^{(1)})^{n_1}...(f^{(k)})^{n_k}$ where $T(r, b_0) = S(r, f)$, to be a differential monomial generated by f. The numbers $\gamma_P = \sum_{i=0}^{k} n_i$ and $\Gamma_P = \sum_{i=0}^{k}(i+1)n_i$ are respectively called the degree and weight of $P[f]$ {cf. [8]}.

In this paper we call the terms

$$\delta_A^P(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; P[f])}{T(r, P[f])} = \liminf_{r\to\infty} \frac{m(r, a; P[f])}{T(r, P[f])},$$

the usual Nevanlinna defect or the absolute Nevanlinna defect of the value 'a' with respect to $P[f]$,

$$\Delta_A^P(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a; P[f])}{T(r, P[f])} = \limsup_{r\to\infty} \frac{m(r, a; P[f])}{T(r, P[f])},$$

the usual Valiron defect or the absolute Valiron defect of the value 'a' with respect to $P[f]$,

$$\delta_R^P(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; P[f])}{T(r, f)},$$

the relative Nevanlinna defect of 'a' with respect to $P[f]$ and

$$\Delta_R^P(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a; P[f])}{T(r, f)},$$

the relative Valiron defect of 'a' with respect to $P[f]$ and prove various relations among them.

The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set E of finite linear measure otherwise.

The following definitions are well known.

Definition 7.1.1. *[50] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $n(r, a; f \mid= 1)$ the number of simple zeros of $f - a$ in $|z| \leq r$. $N(r, a; f \mid= 1)$ is defined in terms of $n(r, a; f \mid= 1)$ in the usual way. Also we put*

$$\delta_1(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f \mid= 1)}{T(r, f)}.$$

Yang [49] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which

$$\delta_1(a; f) > 0 \text{ and } \sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4.$$

Definition 7.1.2. *The quantity $\Theta(\infty; f)$ is defined as*

$$\Theta(\infty; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, f)}{T(r, f)}.$$

Definition 7.1.3. *The order ρ_f and lower order λ_f of a meromorphic function f are defined as follows*

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

7.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 7.2.1. *[32] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Then*

$$\lim_{r \to \infty} \frac{T(r, P[f])}{T(r, f)} = \Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f).$$

Lemma 7.2.2. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Then for any α,*

$$\Delta_R^P(\alpha; f) = \{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty; f)\} + \limsup_{r \to \infty} \frac{m(r, \alpha; P[f])}{T(r, f)}$$

and

$$\delta_R^P(\alpha; f) = \{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty; f)\} + \liminf_{r \to \infty} \frac{m(r, \alpha; P[f])}{T(r, f)}.$$

Proof. In view of Lemma 7.2.1, we obtain that

$$\begin{aligned}
\Delta_R^P(\alpha; f) &= 1 - \liminf_{r\to\infty} \frac{N(r,\alpha;P[f])}{T(r,f)} \\
&= 1 - \liminf_{r\to\infty} \frac{N(r,\alpha;P[f])}{T(r,P[f])}. \lim_{r\to\infty} \frac{T(r,P[f])}{T(r,f)} \\
&= 1 - \liminf_{r\to\infty} \frac{N(r,\alpha;P[f])}{T(r,P[f])}.\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\} \\
&= \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}\{1 - \liminf_{r\to\infty} \frac{N(r,\alpha;P[f])}{T(r,P[f])}\} \\
&\quad +\{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty; f)\} \\
&= \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\} \limsup_{r\to\infty} \frac{m(r,\alpha;P[f])}{T(r,P[f])} \\
&\quad +\{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty; f)\} \\
&= \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\} \\
&\quad .\{\limsup_{r\to\infty} \frac{m(r,\alpha;P[f])}{T(r,f)}. \lim_{r\to\infty} \frac{T(r,f)}{T(r,P[f])}\} \\
&\quad +\{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty; f)\} \\
&= \{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty; f)\} + \limsup_{r\to\infty} \frac{m(r,\alpha;P[f])}{T(r,f)}.
\end{aligned}$$

This proves the first part of the lemma. Similarly we can prove the second part of the lemnma. ■

The following lemma is due to Milloux {p.55, [22]}.

Lemma 7.2.3. *{p.55, [22]} Let k be any positive integer and $\psi = \sum_{i=0}^{k} a_i f^{(i)}$, where a_i are meromorphic functions, such that $T(r, a_i) = S(r, f)$, for $i = 0, 1, 2, ..., k$. Then $m(r, \frac{\psi}{f}) = S(r, f)$.*

7.3 Theorems.

In this section we present the main results of the chapter.

Theorem 7.3.1. *Let f be a transcendental meromorphic function of finite order ρ_f and 'a' be any non-zero finite complex number. Then*

$$\delta(0; f) + \Delta_R^P(\infty; f) + \delta(a; f) \leq (2\gamma_P - 1)\Delta(\infty; f) + \Delta_R^P(0; f).$$

Proof. Let us consider the following identity

$$\frac{a}{f} = 1 - \frac{f-a}{P[f]}.\frac{P[f]}{f}.$$

Since $m(r, \frac{1}{f}) \leq m(r, \frac{a}{f}) + O(1)$, in view of Lemma 7.2.3 we get from the above identity that

$$\begin{aligned} m(r, \frac{1}{f}) &\leq m\left(r, \frac{f-a}{P[f]}\right) + m\left(r, \frac{P[f]}{f^{\gamma_P}}.f^{\gamma_P - 1}\right) \\ \text{i.e., } m(r, \frac{1}{f}) &\leq m(r, \frac{f-a}{P[f]}) + (\gamma_P - 1)m(r, f) + S(r, f). \quad (7.1) \end{aligned}$$

Now by the relation $T(r, f) = T(r, \frac{1}{f}) + O(1)$ and by Lemma 7.2.3 it follows from (7.1) that

$$\begin{aligned} m(r, \frac{1}{f}) &\leq T(r, \frac{f-a}{P[f]}) - N\left(r, \frac{f-a}{P[f]}\right) \\ &\quad + (\gamma_P - 1)m(r, f) + S(r, f) \\ i.e., \ m(r, \frac{1}{f}) &\leq T(r, \frac{P[f]}{f-a}) - N\left(r, \frac{f-a}{P[f]}\right) \\ &\quad + (\gamma_P - 1)m(r, f) + S(r, f) + O(1) \\ i.e., \ m(r, \frac{1}{f}) &\leq N(r, \frac{P[f]}{f-a}) + m(r, \frac{P[f]}{(f-a)^{\gamma_P}}) + (\gamma_P - 1)m(r, f-a) \\ &\quad - N\left(r, \frac{f-a}{P[f]}\right) + (\gamma_P - 1)m(r, f) + S(r, f) + O(1) \\ i.e., \ m(r, \frac{1}{f}) &\leq N(r, \frac{P[f]}{f-a}) - N\left(r, \frac{f-a}{P[f]}\right) \\ &\quad + 2(\gamma_P - 1)m(r, f) + S(r, f) + O(1). \quad (7.2) \end{aligned}$$

In view of {p.34,[22]} it follows from (7.2) that

$$\begin{aligned} m(r, \frac{1}{f}) &\leq N(r, P[f]) + N(r, \frac{1}{f-a}) - N(r, f-a) \\ &\quad - N\left(r, \frac{1}{P[f]}\right) + 2(\gamma_P - 1)m(r, f) + S(r, f) + O(1) \end{aligned}$$

$$
\begin{aligned}
i.e., \liminf_{r\to\infty} \frac{m(r,\frac{1}{f})}{T(r,f)} &\leq \liminf_{r\to\infty}\{\frac{N(r,P[f])}{T(r,f)} - \frac{N(r,f)}{T(r,f)} - \frac{N\left(r,\frac{1}{P[f]}\right)}{T(r,f)}\} \\
&\quad + \limsup_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)} + 2(\gamma_P - 1)\limsup_{r\to\infty} \frac{m(r,f)}{T(r,f)} \\
i.e., \liminf_{r\to\infty} \frac{m(r,\frac{1}{f})}{T(r,f)} &\leq \liminf_{r\to\infty} \frac{N(r,P[f])}{T(r,f)} - \liminf_{r\to\infty} \frac{N(r,f)}{T(r,f)} \\
&\quad - \liminf_{r\to\infty} \frac{N\left(r,\frac{1}{P[f]}\right)}{T(r,f)} + \limsup_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)} \\
&\quad +2(\gamma_P - 1)\limsup_{r\to\infty} \frac{m(r,f)}{T(r,f)}
\end{aligned}
$$

$$
\begin{aligned}
i.e., \delta(0;f) &\leq \{1 - \Delta_R^P(\infty;f)\} - \{1 - \Delta(\infty;f)\} - \{1 - \Delta_R^P(0;f)\} \\
&\quad +\{1 - \delta(a;f)\} + 2(\gamma_P - 1)\Delta(\infty;f)
\end{aligned}
$$

$$
i.e., \delta(0;f) + \Delta_R^P(\infty;f) + \delta(a;f) \leq (2\gamma_P - 1)\Delta(\infty;f) + \Delta_R^P(0;f).
$$

This proves the theorem. ■

Remark 7.3.1. *The sign '≤' in Theorem 7.3.1 cannot be replaced by '<' only. This is evident from the following example.*

Example 7.3.1. *Let* $f = \exp z$, $n_0 = 1$, $n_1 = \ldots = n_k = 0$ *and* $b_0 = 1$.

Then

$$
P[f] = f, \Delta(\infty;f) = \Delta_R^P(0;f) = \Delta_R^P(\infty;f) = 1
$$

and

$$
\delta(0;f) = \delta(\infty;f) = 1.
$$

So $\delta(a;f) = 0$. *Also* $\gamma_P = 1$.

Then

$$
\begin{aligned}
\delta(0;f) + \Delta_R^P(\infty;f) + \delta(a;f) &= 2 \\
\text{and } (2\gamma_P - 1)\Delta(\infty;f) + \Delta_R^P(0;f) &= 2.
\end{aligned}
$$

Theorem 7.3.2. *If* f *be a transcendental meromorphic function with* $\rho_f < \infty$, $\delta(\infty;f) = 1$ *and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$, *then*

$$
\begin{aligned}
1 + \Delta_R^P(\infty;f) + \delta(0;f) &\leq \Delta_R^P(0;f) + \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty;f)\} \\
&\quad .\Delta_A^P(\infty;f) + \gamma_P.
\end{aligned}
$$

Proof. Since $f = P[f].\dfrac{f}{P[f]}$ we get that

$$m(r,f) \le m\left(r,P[f]\right) + m\left(r,\frac{f}{P[f]}\right). \tag{7.3}$$

Now by the relation $T\left(r,f\right) = T\left(r,\dfrac{1}{f}\right) + O(1)$ and by Lemma 7.2.3 we obtain from (7.3) that

$$\begin{aligned}
m(r,f) &\le m\left(r,P[f]\right) + T\left(r,\frac{f}{P[f]}\right) - N\left(r,\frac{f}{P[f]}\right)\\
i.e.,\ m(r,f) &\le m\left(r,P[f]\right) + T\left(r,\frac{P[f]}{f}\right) - N\left(r,\frac{f}{P[f]}\right) + O(1)\\
i.e.,\ m(r,f) &\le m\left(r,P[f]\right) + N\left(r,\frac{P[f]}{f}\right) + m\left(r,\frac{P[f]}{f^{\gamma_P}}\right)\\
&\quad +(\gamma_P-1)m(r,f) - N\left(r,\frac{f}{P[f]}\right) + O(1).
\end{aligned} \tag{7.4}$$

Now in view of {p.34,[22]} it follows from (7.4) that

$$\begin{aligned}
m(r,f) &\le m\left(r,P[f]\right) + N\left(r,P[f]\right) + N\left(r,\frac{1}{f}\right) - N(r,f)\\
&\quad -N\left(r,\frac{1}{P[f]}\right) + (\gamma_P-1)m(r,f) + S(r,f) + O(1)
\end{aligned}$$

$$\begin{aligned}
i.e.,\ \liminf_{r\to\infty}\frac{m(r,f)}{T(r,f)} &\le \liminf_{r\to\infty}\left\{\frac{N(r,P[f])}{T(r,f)} - \frac{N(r,f)}{T(r,f)} - \frac{N\left(r,\frac{1}{P[f]}\right)}{T(r,f)}\right\}\\
&\quad + \limsup_{r\to\infty}\left\{\frac{N(r,\frac{1}{f})}{T(r,f)} + \frac{m(r,P[f])}{T(r,f)}\right\}\\
&\quad + \limsup_{r\to\infty}\left\{(\gamma_P-1)\frac{m(r,f)}{T(r,f)}\right\}
\end{aligned}$$

$$\begin{aligned}
i.e.,\ \liminf_{r\to\infty}\frac{m(r,f)}{T(r,f)} &\le \liminf_{r\to\infty}\frac{N(r,P[f])}{T(r,f)} - \liminf_{r\to\infty}\frac{N(r,f)}{T(r,f)}\\
&\quad - \liminf_{r\to\infty}\frac{N\left(r,\frac{1}{P[f]}\right)}{T(r,f)} + \limsup_{r\to\infty}\frac{N(r,\frac{1}{f})}{T(r,f)}\\
&\quad + \limsup_{r\to\infty}\frac{m(r,P[f])}{T(r,f)} + (\gamma_P-1)\limsup_{r\to\infty}\frac{m(r,f)}{T(r,f)}.
\end{aligned} \tag{7.5}$$

Since $\delta(\infty; f) = 1$, then $\Delta(\infty; f) = 1$. So by Lemma 7.2.1 we obtain from (7.5) that

$$\begin{aligned}\delta(\infty; f) \leq & \{1 - \Delta_R^P(\infty; f)\} - \{1 - \Delta(\infty; f)\} - \{1 - \Delta_R^P(0; f)\} \\ & + \{1 - \delta(0; f)\} + \limsup_{r\to\infty} \frac{m(r, P[f])}{T(r, P[f])} \cdot \lim_{r\to\infty} \frac{T(r, P[f])}{T(r, f)} \\ & + (\gamma_P - 1)\Delta(\infty; f)\end{aligned}$$

$$\begin{aligned} & i.e., \delta(\infty; f) + \Delta_R^P(\infty; f) + \delta(0; f) \leq \gamma_P.\Delta(\infty; f) \\ & + \Delta_R^P(0; f) + \Delta_A^P(\infty; f)\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\} \\ & i.e., 1 + \Delta_R^P(\infty; f) + \delta(0; f) \leq \Delta_R^P(0; f) \\ & + \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}\Delta_A^P(\infty; f) + \gamma_P.\end{aligned}$$

Thus the theorem is established. ■

Remark 7.3.2. *If we omit the condition $\delta(\infty, f) = 1$ of Theorem 7.3.2 and the other conditions remaining the same, using the first part of Lemma 7.2.2 we may establish the next theorem.*

Theorem 7.3.3. *Let f be a transcendental meromorphic function of finite order ρ_f and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. Then*

$$\begin{aligned} & \delta(\infty, f) + \delta(0; f) + (\Gamma_P - \gamma_P)\Theta(\infty; f) + 1 \\ \leq\ & \gamma_P.\Delta(\infty; f) + \Delta_R^P(0; f) + \Gamma_P.\end{aligned}$$

Proof. Using the first part of Lemma 7.2.2 and the inequality (7.5) it follows that

$$\begin{aligned}\delta(\infty; f) \leq & \{1 - \Delta_R^P(\infty; f)\} - \{1 - \Delta(\infty; f)\} - \{1 - \Delta_R^P(0; f)\} \\ & + \{1 - \delta(0; f)\} + \Delta_R^P(\infty; f) \\ & - \{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty; f)\} \\ & + (\gamma_P - 1)\Delta(\infty; f)\end{aligned}$$

$$\begin{aligned} i.e., \ \delta(\infty, f) + \delta(0; f) + (\Gamma_P - \gamma_P)\Theta(\infty; f) + 1 \leq\ & \gamma_P.\Delta(\infty; f) \\ & + \Delta_R^P(0; f) + \Gamma_P.\end{aligned}$$

Thus the theorem is proved. ■

Theorem 7.3.4. *Let $a, b \neq 0, \infty$ be any two distinct complex numbers. Then for any transcendental meromorphic function f of finite order ρ_f,*

$$2\delta(a;f) + \delta(b;f) + 2\Delta_R^P(\infty;f) \leq (3\gamma_P - 1)\Delta(\infty;f) + 2\Delta_R^P(0;f).$$

Proof. Considering the identity

$$\frac{b-a}{f-a} = \frac{P[f]}{f-a}\left\{\frac{f-a}{P[f]} - \frac{f-b}{P[f]}\right\},$$

we obtain in view of Lemma 7.2.3 that

$$\begin{aligned} m\left(r, \frac{b-a}{f-a}\right) &\leq m\left(r, \frac{f-a}{P[f]}\right) + m\left(r, \frac{f-b}{P[f]}\right) \\ &\quad + m\left(r, \frac{P[f]}{(f-a)^{\gamma_P}}\right) \\ &\quad + (\gamma_P - 1)m(r, f-a) \\ \text{i.e.}, m\left(r, \frac{b-a}{f-a}\right) &\leq T\left(r, \frac{f-a}{P[f]}\right) - N\left(r, \frac{f-a}{P[f]}\right) \\ &\quad + T\left(r, \frac{f-b}{P[f]}\right) - N\left(r, \frac{f-b}{P[f]}\right) \\ &\quad + (\gamma_P - 1)m(r, f) + S(r, f) \end{aligned} \tag{7.6}$$

Since $m\left(r, \dfrac{1}{f-a}\right) \leq m\left(r, \dfrac{b-a}{f-a}\right) + O(1)$ and $T(r,f) = T(r, \frac{1}{f}) + O(1)$, it follows from (7.6) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &\leq T\left(r, \frac{P[f]}{f-a}\right) - N\left(r, \frac{f-a}{P[f]}\right) \\ &\quad + T\left(r, \frac{P[f]}{f-b}\right) - N\left(r, \frac{f-b}{P[f]}\right) \\ &\quad + (\gamma_P - 1)m(r, f) + S(r, f) + O(1) \end{aligned}$$

$$\begin{aligned}
i.e., m\left(r, \frac{1}{f-a}\right) \leq\ & N\left(r, \frac{P[f]}{f-a}\right) + m\left(r, \frac{P[f]}{(f-a)^{\gamma_P}}\right) \\
& +(\gamma_P - 1)m(r, f-a) - N\left(r, \frac{f-a}{P[f]}\right) \\
& +N\left(r, \frac{P[f]}{f-b}\right) + m\left(r, \frac{P[f]}{(f-b)^{\gamma_P}}\right) \\
& +(\gamma_P - 1)m(r, f-b) - N\left(r, \frac{f-b}{P[f]}\right) \\
& +(\gamma_P - 1)m(r, f) + S(r, f) + O(1)
\end{aligned}$$

$$\begin{aligned}
i.e., m\left(r, \frac{1}{f-a}\right) \leq\ & N\left(r, \frac{P[f]}{f-a}\right) - N\left(r, \frac{f-a}{P[f]}\right) \\
& +N\left(r, \frac{P[f]}{f-b}\right) - N\left(r, \frac{f-b}{P[f]}\right) \\
& +3(\gamma_P - 1)m(r, f) + S(r, f) + O(1). \quad (7.7)
\end{aligned}$$

In view of {p.34, [22]} we get from (7.7) that

$$\begin{aligned}
m\left(r, \frac{1}{f-a}\right) \leq\ & N(r, P[f]) + N\left(r, \frac{1}{f-a}\right) - N(r, f-a) \\
& -N\left(r, \frac{1}{P[f]}\right) + N(r, P[f]) + N\left(r, \frac{1}{f-b}\right) \\
& -N(r, f-b) - N\left(r, \frac{1}{P[f]}\right) \\
& +3(\gamma_P - 1)m(r, f) + S(r, f) + O(1)
\end{aligned}$$

$$\begin{aligned}
i.e., m\left(r, \frac{1}{f-a}\right) \leq\ & 2N(r, P[f]) - 2N(r, f) - 2N\left(r, \frac{1}{P[f]}\right) \\
& +N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) \\
& +3(\gamma_P - 1)m(r, f) + S(r, f) + O(1)
\end{aligned}$$

$$
\begin{aligned}
i.e., \liminf_{r\to\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} &\le 2\liminf_{r\to\infty}\left\{\frac{N(r,P[f])}{T(r,f)} - \frac{N(r,f)}{T(r,f)} - \frac{N\left(r,\frac{1}{P[f]}\right)}{T(r,f)}\right\} \\
&\quad + \limsup_{r\to\infty}\left\{\frac{N(r,\frac{1}{f-a})}{T(r,f)} + \frac{N\left(r,\frac{1}{f-b}\right)}{T(r,f)}\right\} \\
&\quad + \limsup_{r\to\infty}\left\{3(\gamma_P-1)\frac{m(r,f)}{T(r,f)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
i.e., \liminf_{r\to\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} &\le 2\{\liminf_{r\to\infty}\frac{N(r,P[f])}{T(r,f)} - \liminf_{r\to\infty}\frac{N(r,f)}{T(r,f)} \\
&\quad - \liminf_{r\to\infty}\frac{N\left(r,\frac{1}{P[f]}\right)}{T(r,f)}\} + \limsup_{r\to\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)} \\
&\quad + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-b}\right)}{T(r,f)} + 3(\gamma_P-1)\limsup_{r\to\infty}\frac{m(r,f)}{T(r,f)}
\end{aligned}
$$

$$
\begin{aligned}
i.e., \delta(a;f) &\le 2\{1-\Delta_R^P(\infty;f)\} - 2\{1-\Delta(\infty;f)\} - 2\{1-\Delta_R^P(0;f)\} \\
&\quad +\{1-\delta(a;f)\} + \{1-\delta(b;f)\} + 3(\gamma_P-1)\Delta(\infty;f)
\end{aligned}
$$

$$
i.e., 2\delta(a;f) + \delta(b;f) + 2\Delta_R^P(\infty;f) \le (3\gamma_P-1)\Delta(\infty;f) + 2\Delta_R^P(0;f).
$$

This proves the theorem. ■

Remark 7.3.3. *The condition $a, b \ne 0, \infty$ in Theorem 7.3.4 is essential which is evident from the following four examples.*

Example 7.3.2. *Let $f = \exp z$, $n_0 = 1$, $n_1 = \ldots = n_k = 0$ and $b_0 = 1$.*

Then $P[f] = \exp z$ and $\Delta(\infty; f) = \Delta_R^P(0; f) = \Delta_R^P(\infty; f) = 1$.

Also $\delta(0; f) = \delta(\infty; f) = 1$ and $\gamma_P = 1$.

Let $a = 0$ and $b = 0$. Then

$$
\begin{aligned}
2\delta(0;f) + \delta(0;f) + 2\Delta_R^P(\infty;f) &= 5 \\
\text{and } (3\gamma_P - 1)\Delta(\infty;f) + 2\Delta_R^P(0;f) &= 4.
\end{aligned}
$$

Example 7.3.3. *Let $f = \exp z$, $n_0 = 1$, $n_1 = \ldots = n_k = 0$ and $b_0 = 1$.*

Then $P[f] = \exp z$ *and* $\Delta(\infty; f) = \Delta_R^P(0; f) = \Delta_R^P(\infty; f) = 1$.
Also $\delta(0; f) = \delta(\infty; f) = 1$ *and* $\gamma_P = 1$.
Let $a = \infty$ *and* $b = \infty$. *Then*

$$\begin{aligned} 2\delta(\infty; f) + \delta(\infty; f) + 2\Delta_R^P(\infty; f) &= 5 \\ \text{and } (3\gamma_P - 1)\Delta(\infty; f) + 2\Delta_R^P(0; f) &= 4. \end{aligned}$$

Example 7.3.4. *Let* $f = \exp z$, $n_0 = 1$, $n_1 = ... = n_k = 0$ *and* $b_0 = 1$.
Then $P[f] = \exp z$ *and* $\Delta(\infty; f) = \Delta_R^P(0; f) = \Delta_R^P(\infty; f) = 1$.
Also $\delta(0; f) = \delta(\infty; f) = 1$ *and* $\gamma_P = 1$.
Let $a = 0$ *and* $b = \infty$. *Then*

$$\begin{aligned} 2\delta(0; f) + \delta(\infty; f) + 2\Delta_R^P(\infty; f) &= 5 \\ \text{and } (3\gamma_P - 1)\Delta(\infty; f) + 2\Delta_R^P(0; f) &= 4. \end{aligned}$$

Example 7.3.5. *Let* $f = \exp z$, $n_0 = 1$, $n_1 = ... = n_k = 0$ *and* $b_0 = 1$.
Then $P[f] = \exp z$ *and* $\Delta(\infty; f) = \Delta_R^P(0; f) = \Delta_R^P(\infty; f) = 1$.
Also $\delta(0; f) = \delta(\infty; f) = 1$ *and* $\gamma_P = 1$.
Let $a = \infty$ *and* $b = 0$. *Then*

$$\begin{aligned} 2\delta(\infty; f) + \delta(0; f) + 2\Delta_R^P(\infty; f) &= 5 \\ \text{and } (3\gamma_P - 1)\Delta(\infty; f) + 2\Delta_R^P(0; f) &= 4. \end{aligned}$$

Theorem 7.3.5. *Let 'a' be a finite complex number and* b, c *be two distinct non-zero complex numbers. Then for any transcendental meromorphic function* f *with finite order* ρ_f *and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$,

$$\begin{aligned} &\delta(a; f) + \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}.\{\delta_A^P(b; f) + \delta_A^P(c; f)\} \\ \leq\ &(\gamma_P - 1)\Delta(\infty; f) + 2\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}. \end{aligned}$$

Proof. Since $\dfrac{1}{f-a} = \dfrac{P[f]}{f-a}.\dfrac{1}{P[f]}$, by Lemma 7.2.3 we obtain that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &\leq m\left(r, \frac{1}{P[f]}\right) + m\left(r, \frac{P[f]}{(f-a)^{\gamma_P}}\right) \\ &\quad + (\gamma_P - 1)m(r, f) \\ \text{i.e.,} m\left(r, \frac{1}{f-a}\right) &\leq m\left(r, \frac{1}{P[f]}\right) \\ &\quad + (\gamma_P - 1)m(r, f) + S(r, f). \end{aligned} \tag{7.8}$$

Applying the relation $T(r,f) = T\left(r,\frac{1}{f}\right) + O(1)$ we get from (7.8) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq T(r,P[f]) - N\left(r,\frac{1}{P[f]}\right) \\ &+(\gamma_P - 1)m(r,f) + S(r,f) + O(1). \qquad (7.9) \end{aligned}$$

Now by Nevanlinna's second fundamental theorem it follows from (7.9) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq \overline{N}\left(r,\frac{1}{P[f]}\right) + \overline{N}\left(r,\frac{1}{P[f]-b}\right) + \overline{N}\left(r,\frac{1}{P[f]-c}\right) \\ &-N\left(r,\frac{1}{P[f]}\right) + (\gamma_P - 1)m(r,f) \\ &+S(r,f) + O(1). \qquad (7.10) \end{aligned}$$

Since

$$\overline{N}\left(r,\frac{1}{P[f]}\right) - N\left(r,\frac{1}{P[f]}\right) \leq 0,$$

we obtain from (7.10) in view of Lemma 7.2.1 that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq \overline{N}\left(r,\frac{1}{P[f]-b}\right) + \overline{N}\left(r,\frac{1}{P[f]-c}\right) \\ &+(\gamma_P - 1)m(r,f) + S(r,f) + O(1) \\ i.e., m\left(r,\frac{1}{f-a}\right) &\leq N\left(r,\frac{1}{P[f]-b}\right) + N\left(r,\frac{1}{P[f]-c}\right) \\ &+(\gamma_P - 1)m(r,f) + S(r,f) + O(1) \\ i.e., m\left(r,\frac{1}{f-a}\right) &\leq T\left(r,\frac{1}{P[f]-b}\right) + T\left(r,\frac{1}{P[f]-c}\right) \\ &-m\left(r,\frac{1}{P[f]-b}\right) - m\left(r,\frac{1}{P[f]-c}\right) \\ &+(\gamma_P - 1)m(r,f) + S(r,f) + O(1) \\ i.e., m\left(r,\frac{1}{f-a}\right) &\leq 2T(r,P[f]) - m\left(r,\frac{1}{P[f]-b}\right) \\ &-m\left(r,\frac{1}{P[f]-c}\right) + (\gamma_P - 1)m(r,f) \\ &+S(r,f) + O(1) \end{aligned}$$

$$i.e., \liminf_{r\to\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} \leq 2\liminf_{r\to\infty}\frac{T(r,P[f])}{T(r,f)} - \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{P[f]-b}\right)}{T(r,f)}$$
$$-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{P[f]-c}\right)}{T(r,f)}$$
$$+(\gamma_P-1)\limsup_{r\to\infty}\frac{m(r,f)}{T(r,f)}$$

$$i.e., \liminf_{r\to\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} \leq 2\liminf_{r\to\infty}\frac{T(r,P[f])}{T(r,f)}$$
$$-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{P[f]-b}\right)}{T(r,P[f])}.\lim_{r\to\infty}\frac{T(r,P[f])}{T(r,f)}$$
$$-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{P[f]-c}\right)}{T(r,P[f])}.\lim_{r\to\infty}\frac{T(r,P[f])}{T(r,f)}$$
$$+(\gamma_P-1)\limsup_{r\to\infty}\frac{m(r,f)}{T(r,f)}$$

$$i.e., \ \delta(a;f) \leq 2\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\}$$
$$-\delta_A^P(b;f).\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\}$$
$$-\delta_A^P(c;f).\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\}$$
$$+(\gamma_P-1)\Delta(\infty;f)$$

$$i.e., \ \delta(a;f)+\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\}.\{\delta_A^P(b;f)+\delta_A^P(c;f)\}$$
$$\leq (\gamma_P-1)\Delta(\infty;f)+2\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\}.$$

Thus the theorem is established. ■

$-----X-----$