

## **Chapter III**

# **The Principle of Equivalence and the Twin Paradox**

## 3.1 Introduction

The principle of equivalence between acceleration and gravity is considered as a *cornerstone* of Einstein's theory of gravitation or that of general relativity (GR). According to Einstein, the principle states that: "A system in a uniform acceleration is equivalent to a system at rest immersed in a uniform gravitational field"[1]. Text books often introduce GR by first demonstrating that the Equivalence Principle (EP) predicts gravitational redshift, which Einstein viewed as a test of general relativity. However we now regard it as a more basic test of EP and the existence of curved space-time[2]. The phenomenon of gravitational red-shift, which has been tested by precision experiments by Pound-Rebka and Snider in the sixties[3, 4] is also interpreted as that of gravitational slowing down of clocks (GSDC). The GSDC has now been tested with much accuracy by using a hydrogen maser clock with extraordinary frequency stability flown on a rocket to an altitude of about 10,000 Km[2]. In the literature GSDC phenomenon has been found to play an important role in resolving the notorious twin paradox[5].

In the canonical version of the twin paradox, of the two twins initially living on earth (assumed to be an inertial frame), one leaves the earth by a fast rocket to a distant star and then returns to meet her stay-at-home brother to discover that they age differently. This as such is not a paradox since the rocket-bound sibling, on account of her high velocity will suffer relativistic time dilation of her (biological) clock throughout her journey and will therefore return younger with respect to her brother. Indeed with respect to the inertial frame of the stay-at-home twin, the world lines of the twins in the Minkowski diagram are different (although from the description of the problem the end points of these lines i.e the time and the place of departure and that of their reunion, meet) and hence the asymmetry in the aging

can be attributed to the fact that proper time is not integrable[6]. The paradox arises if one naïvely treats the perspectives of the twins symmetrically. For example if the traveller twin considers herself to remain stationary and relate the motion to her brother, she would (erroneously) expect her brother to stay younger by believing that the Lorentz transformation (LT) predicts reciprocal time dilation of moving clocks. Qualitatively the resolution lies in the observation that one of the twins is in an accelerated (non-inertial) frame of reference and hence the postulates of Special Relativity (SR) are not applicable to it and therefore the claim of reciprocity of time dilation between the frames of reference of the twins falls through. Indeed Einstein himself found this sort of argument preferable in dismissing the paradoxical element in the twin problem[7]. However this suggestion should not be construed as a statement that the resolution of the paradox falls outside the purview of SR . On the contrary much of the expositions found in the literature on the subject deal with the problem in the frame work of SR alone<sup>1</sup>, although many tend to believe that the introduction of GR and a gravitational field at the point of acceleration is the right way to understand the asymmetry in the perspectives of the twins. Bohm notes in the context that “ two clocks running at places of different gravitational potential will have different rates”[10]. This suggests that EP can directly be used to explain the asymmetry (difference between the experiences of the rocket-bound and the stay-at-home twin). However, as pointed out by Debs and Redhead[6] and also others[11], that since in the twin problems one deals with flat space-time, any reference of GR in this context is quite confusing.

Coming back to the issue of acceleration, one finds often that the direct role of acceleration of the rocket-bound twin in causing the differential aging has been

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<sup>1</sup>Very extensive treatment is available in Special Relativity Theory-Selected Reprints[8], (see also Ref.[9]). For newer expositions see for example Ref.[6] and references therein.

much criticized although it is quite clear that in order to have twice intersecting trajectories of the twins (this is necessary since the clocks or ages of the twins have to be compared at the same space-time events) one cannot avoid acceleration.

In an interesting article Gruber and Price[12] dispel the idea of any direct connection between acceleration and asymmetric aging by presenting a variation of the paradox where although one twin is subjected to undergo an arbitrarily large acceleration, no differential aging occurs. That the acceleration per se cannot play a role is also evident from the usual calculation of the age difference from the perspective of the inertial frame of the stay-at-home twin if one notes that the duration of the turn-around process of the rocket can be made arbitrarily small in comparison to that for the rest of the journey and hence the final age difference between the twins can then be understood in terms of the usual relativistic time dilation of the traveller twin during essentially the unaccelerated segment of her journey<sup>2</sup>. One is thus caught in an ambivalent situation that, on the one hand the acceleration does not play any role, on the other hand the paradox is not well posed unless there is a turn-around (acceleration) of the traveller twin<sup>3</sup>.

In order to get out of this dichotomy it is enough to note that from the point of view of the traveller twin, the acceleration (or the change of reference frame in the abrupt turn-around scenario) is important. The consideration of this acceleration only has the ability to explain that the expectation of symmetrical time dilation of the stationary twin from the point of view of the rocket-bound twin is incorrect.

In an interesting paper A.Harpaz[5] tries to explain the twin paradox by calculat-

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<sup>2</sup>In such a calculation the time dilation is also calculated during the acceleration phase (assuming the clock hypothesis to be true[6]) and is shown to contribute arbitrarily small value in the age offset if the duration of the acceleration phase is assumed to tend to zero.

<sup>3</sup>Here we are considering the standard version of the paradox and the variation where the twins live in a cylindrical universe[13, 14] has been kept out of the present scope.

ing the age difference from the perspective of the traveller twin directly by applying EP i.e by introducing GSDC. From the previous discussions it may seem unnecessary (or even confusing) to invoke gravity in the essentially special relativistic problem. However the fact is, Harpaz's approach apparently provides an alternate explanation for the differential aging from the traveller's perspective.

The author of the pedagogical article observes that although the special relativistic approach can correctly account for the age difference between the twins, "it does not manifest the 'physical agent' responsible for the creation of such a difference"[5]. It is held that EP provides such an agent and that is gravity. But how does gravity find way into the problem? Gravity enters through EP and its connection with the resolution of the paradox can symbolically be written as

$$\textit{Acceleration} \xrightarrow{EP} \textit{Gravity} \rightarrow \textit{Gravitational red-shift} \rightarrow \textit{GSDC} \rightarrow \textit{Extra aging},$$

where the last item of the flow diagram indicates that with respect to the rocket-bound twin, GSDC provides the extra aging of the stay-at-home one, explaining the asymmetrical aging of the problem.

However while there is as such no harm in understanding the twin problem from a different perspective (here, this is in terms of GSDC), Harpaz's approach suffer from two fold conceptual difficulties which we will elaborate in the next section. These difficulties include the fact that the calculations are only approximate. The other difficulty will be seen to be of more fundamental in nature. The aim of the present study (reported in this chapter) is to remove these difficulties and give an *accurate* account of the asymmetric aging from the perspective of the rocket-bound twin directly in terms of a time-offset between the siblings which is introduced due to the pseudo-gravity experienced by the traveller twin.

## 3.2 GSDC and Extra Aging

In the standard version of the twin paradox the differential aging from the perspective of the stay-at-home (inertial) observer  $A$  can easily be calculated assuming that for the most parts of the journey of the traveller twin  $B$ , the motion remains uniform except that there is a turn-around acceleration of the rocket so that finally the siblings are able to meet and compare their ages. In the Minkowski diagram the whole scenario is characterized primarily by three events: (1) Meeting of the world lines of  $A$  and  $B$  when the voyage starts taking place, (2) the turn around of  $B$  and (3) meeting of the world lines when  $A$  and  $B$  reunite. For the paradox it is not necessary that at events (1) and (2), the relative velocity between  $A$  and  $B$  has to be zero, since ages or clocks can be compared at a point even if the observers are in relative motion, therefore the analysis of the problem can be done by considering the acceleration only during the turn-around. The duration of the acceleration phase can be considered to be arbitrarily small compared to the time it takes during its forward and return journeys and hence the age difference occurs due to the usual relativistic time dilation of a clock for its uniform motion. This is clearly given by

$$\text{Age difference} = 2t_A(1 - \gamma^{-1}) \approx 2t_A v^2/c^2, \quad (3.1)$$

where  $2t_A$  is the time the rocket takes for its entire journey (up and down) in uniform speed  $v$  and  $\gamma = (1 - v^2/c^2)^{-1/2}$  is the usual Lorentz factor.

The paradox is resolved if one can show that  $B$  also predicts the same difference in spite of the fact that the time dilation effect is reciprocal. Clearly some new considerations (that were absent in arriving at Eq.(3.1)) must offset this reciprocal time dilation and also this must provide some extra aging to  $A$  from the point of view of  $B$  so that the age difference remains independent of the two perspectives.

One of these new considerations, as has already been pointed out, is the one of a synchronization gap that  $B$  discovers due to her change of inertial frame during her entire voyage. This has been clearly demonstrated by Bondi[15] in the context of Lord Hulsbery's three brother approach[6] to understanding the twin paradox.

The other way of understanding the same thing is the consideration of pseudo-gravity experienced by  $B$  because of its turn-around. In order to demonstrate how EP plays the role in the analysis, Harpaz uses the gravitational red-shift formula, which can be obtained heuristically (using the EP) as

$$\Delta\nu = \nu_0(1 + gh/c^2), \quad (3.2)$$

where  $g$  is the acceleration due to (pseudo) gravity and  $\Delta\nu$  represents the change of frequency of light observed from a distance  $h$  from the source where the frequency of the same light is seen to be  $\nu_0$ . Interpreting this red-shift effect in terms of GSDC, the formula can be written as

$$t_1 = t_2(1 + \Delta\Phi/c^2), \quad (3.3)$$

where  $t_1$  and  $t_2$  are times measured by clocks located at two points  $P_1$  and  $P_2$  (say) and  $\Delta\Phi = gh$ , is the potential difference between these points. It has been shown that with respect to  $B$  the acceleration plays a role by providing an extra time difference between  $B$  and  $A$ , because of the integrated effect of GSDC during the (arbitrarily) short duration of  $B$ 's acceleration. This time difference more than offsets the age difference calculated by  $B$  solely assuming the reciprocal time dilation so much so that finally  $B$  ages less by the correct amount. As pointed out earlier there are two conceptual difficulties in understanding the treatment. First, in an effort to find a "physical agent" responsible for the extra aging, Harpaz relies on some approximate formulae including that of the gravitational red-shift because of his assumption,  $v^2/c^2 \ll 1$  inherent in the analysis, and therefore, the

pseudo-gravitational effect has the ability to resolve the paradox only approximately. Clearly there is no valid reason to make any such small velocity approximation for the problem. One might of course argue that for the author's stated purpose it would be enough to show that the "physical agent" i.e. gravity is at work when  $B$ 's point of view is considered. However, it will be shown that such an argument would also not hold good and the reason for it concerns the second difficulty. The explanations based on SR relies on the fact that during the direction reversing acceleration, the travelling twin changes from one reference frame to another and the lack of simultaneity of one reference frame with respect to the other provides the "missing time" which constitutes the reason for the differential aging[6]. Now the lack of agreement in simultaneity is a special relativistic concept without any classical analogue, on the other hand in many standard heuristic derivations of the gravitational red-shift formula (see for example[16, 17, 18]) which is also followed by the author of Ref.[5], one finds that no reference to SR is made. Indeed the well-known formula for the red-shift parameter  $Z = gh/c^2$  is only approximate and is derived by making use of the classical Doppler effect for light between the source of light and a detector placed at a distance  $h$  along the direction of acceleration  $g$  of an Einstein elevator[5]. According to EP an observer within the elevator will "attribute his observations in the elevator, to the existence of a uniform gravitational field in a rest system of reference"[5]. Thus the equivalence of gravity and acceleration in terms of gravitational red-shift or GSDC therefore turns out to be as if a purely classical (Newtonian) concept in this approximation! How then is GSDC able to account for an effect, viz. the lack of simultaneity which is essentially a standard relativistic phenomenon?

In the next section we will show that indeed the EP can explain the twin paradox exactly provided the connection of EP and GSDC is obtained using the full

machinery of SR.

### 3.3 EP and the Gravitational Time Offset

In an interesting paper Boughn[19] presents a variation of the twin paradox where two twins  $A$  and  $B$  on board two identical rockets (with equal amount of fuel), initially at rest a distance  $x_0$  apart in an inertial frame  $S$ , get identical accelerations for some time in the direction  $AB$  ( $x$ -direction say), and eventually come to rest (when all their fuel has been expended) with respect to another inertial frame  $S'$  moving with velocity  $v$  along the positive  $x$ -direction with respect to  $S$ . From the simple application of Lorentz transformation Boughn obtains a very surprising result that after the acceleration phase is over, the age of  $A$  becomes less than that of  $B$ .

The result is counter-intuitive by virtue of the fact that the twins throughout have identical local experiences but their presynchronised (biological) clocks go out of synchrony. The amount of this time offset turns out to be

$$\Delta t' = -\gamma v x_0 / c^2. \quad (3.4)$$

The result follows from the simple application of LT which one may write for time as

$$t_k' = \gamma(t_k - v x_k / c^2), \quad (3.5)$$

where  $t_k$  and  $x_k$  denote the time and space coordinates of the observer  $k$  ( $k$  stands for  $A$  or  $B$ ) with respect to  $S$  and the prime refers to the corresponding coordinates in  $S'$ .

From Eq.(3.5) it follows that

$$t_B' - t_A' = \gamma[(t_B - t_A) - v(x_B - x_A)/c^2]. \quad (3.6)$$

Assuming the clocks of the observers  $A$  and  $B$  are initially synchronized in  $S$ , i.e assuming  $t_B - t_A = 0$  and also noting that  $x_B - x_A = x_0$  remains constant throughout their journeys, the time offset between these clocks is given by the expression (3.4) provided  $\Delta t'$  is substituted for  $t_B' - t_A'$ .

The paradox however can be explained by noting that for spatially separated clocks the change of relative synchronization cannot be unequivocally determined. The clocks can only be compared when they are in spatial coincidence. For example, when in  $S'$  either of the observers can slowly walk towards the other or both the observers can walk symmetrically (with respect to  $S'$ ) towards the other and compare their clocks (ages) when they meet. However in that case one can show[20] that they do not have identical local experiences— thus providing the resolution of the paradox.

While the paradoxical element of the problem goes away, the fact remains that the result (3.4) is correct and this time offset remains unchanged even if they slowly walk towards each other and compare their clocks (ages) when they meet[21].

This temporal offset effect of identically accelerated clocks gives an important insight into the behaviour of clocks in a uniform gravitational field, for, according to EP “...all effects of a uniform gravitational field are identical to the effects of a uniform acceleration of the coordinate system”[17]. This suggests, as correctly remarked by Boughn that two clocks at rest in a uniform gravitational field are in effect perpetually being accelerated into the new frames and hence the clock at the higher gravitational potential (placed forward along the direction of acceleration)

runs faster. With this insight we write Eq.(3.4) as

$$t - t_0 = -\gamma(t)v(t)x_0/c^2 = -f(t), \quad (3.7)$$

where now  $t$  and  $t_0$  are the readings of two clocks at higher and lower potentials respectively and also  $f(t)$  stands for the right hand side of Eq.(3.4) without the minus sign

$$f(t) = \gamma(t)v(t)x_0/c^2. \quad (3.8)$$

In terms of differentials one may write Eq.(3.7) as

$$\delta t - \delta t_0 = -f(\dot{t})\delta t, \quad (3.9)$$

where the time derivative  $f(\dot{t}) = gx_0/c^2$ , with  $g = d(\gamma v)/dt$  is the *proper acceleration*.

We may now replace  $\delta t$  and  $\delta t_0$  by  $n$  and  $n_0$ , where the later quantities corresponds to the number of ticks (second) of the clocks at their two positions. We therefore have,

$$(n - n_0)/n_0 = -f(\dot{t}), \quad (3.10)$$

or in terms of frequency of the clocks

$$-\delta\nu/\nu_0 = f(\dot{t}), \quad (3.11)$$

where  $\delta\nu$  refers to the frequency shift of an oscillator of frequency  $\nu_0$ . The slowing down parameter for clocks,  $-\delta\nu/\nu_0$  in Eq.(3.11) is nothing but the so called red-shift parameter  $Z$  for which we obtain the well-known formula<sup>4</sup>

$$Z = gx_0/c^2. \quad (3.12)$$

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<sup>4</sup>In terms of *ordinary* acceleration  $\bar{g} = dv/dt$ , measured with respect to  $S$  the formula comes out to be  $Z = (\bar{g}\gamma x_0/c^2)(1 - v^2\gamma^2/c^2)$  which for small velocities can also be written as  $Z = \bar{g}x_0/c^2$ .

One thus observes that the time-offset relation (3.7) of Boughn's paradox can be interpreted as the accumulated time difference between two spatially separated clocks because of the pseudo-gravity experienced by the twins.<sup>5</sup> We shall see the importance of the time-offset relation (3.7) in accounting for the asymmetrical aging of the standard twin paradox from the perspective of the traveller twin. However before that, in the next section we show that the connection of the time-offset and GSDC is purely relativistic in nature.

### 3.4 Boughn's Paradox in the Classical World

The origin of Boughn's paradox can be traced to the space dependent part in the time transformation of LT. The existence of this term is indeed the cause of relativity of simultaneity in SR.

The notion of relativity of simultaneity however can also be imported to the classical world. By classical or Galilean world we mean a kinematical world endowed with a preferred frame (of ether)  $S$  with respect to which the speed of light  $c$  is isotropic and moving rods and clocks do not show any length contraction and time dilation effects. However the speed of light measured in any other inertial frame  $S'$  moving with velocity  $v$  with respect to  $S$  will change and will depend on direction. The synchronization of spatially separated clocks is generally not an issue in this world as clocks can be transported freely without having to worry about time

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<sup>5</sup>The connection between gravity with this temporal offset through EP was first pointed out by Barron and Mazur[22], who derived the approximate formula for the "clock rate difference" mentioned in the previous foot-note.

dilation, therefore all clocks can be synchronized at one spatial point and then may be transported with arbitrary speed to different locations. (The process is generally forbidden in SR). Clearly one uses the Galilean transformation (GT) to compare events in different inertial frames. Using GT one can show that the two way speed (TWS) of light  $\overleftrightarrow{c}$  in  $S'$  along any direction  $\theta$  with respect to the  $x$ -axis (direction of relative velocity between  $S$  and  $S'$ ) is given by

$$\overleftrightarrow{c}(\theta) = c(1 - \beta^2)/(1 - \beta \sin^2 \theta)^{1/2}. \quad (3.13)$$

According to GT this TWS is not the same as the one-way speed (OWS) of light, for example, along the  $x$ -axis it is  $c - v$  and  $c + v$  in the positive and negative  $x$ -directions respectively, while the two way speed, i.e the average round-trip speed of light along the  $x$ -direction is given by  $c(1 - v^2/c^2)$ . However, in a playful spirit one may choose to synchronize the clocks in  $S'$  such that the one way speeds, to and fro are, the same as  $\overleftrightarrow{c}$ . This is similar to Einstein's stipulation in SR which is commonly known as the standard synchrony. In the Galilean world the synchrony is somewhat an awkward one but none can prevent one in adopting such a method. For this synchrony GT changes to the following transformations<sup>6</sup>

$$\begin{aligned} x' &= (x - vt), \\ t' &= \gamma^2(t - vx/c^2), \end{aligned} \quad (3.14)$$

which was first obtained by E. Zahar and is therefore known as the Zahar transformation (ZT)[23, 24, 25, 26]. The transformations have been successfully used to clarify some recently posed counter-intuitive problems in SR[27, 28]. The presence of the phase term and  $\gamma^2$  in Eq.(3.14) distinguishes the ZT from GT. Clearly the appearance of these terms is just an artifact of this synchrony.

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<sup>6</sup>See chapter II for a derivation of the transformation equations following conventionality of simultaneity thesis in the classical world.

One is thus able to recast Boughn's paradox using the above transformations and extending the arguments leading to the Eq.(3.4), one obtains for the differential aging,

$$\Delta t' = -\frac{\gamma^2 v x_0}{c^2}. \quad (3.15)$$

The above expression for the differential aging between two spatially separated twins is also therefore an artifact of the synchrony.

Let us note that ZT has many interesting features which include the existence of apparent time dilation and length contraction effects as observed from an arbitrary reference frame  $S'$ . (With respect to the preferred frame however there are no such effects). We have already pointed out that the temporal offset between clocks cannot have any unequivocal meaning unless it corresponds to measurement at one spatial point.

One may therefore define without much ado the reality of the temporal offset effect due to Boughn (hereafter referred to as Boughn-effect), provided the clocks are finally compared when they are brought together. In the relativistic world a clock is slowly transported towards the other in order to minimize the time dilation effect in the process. In this world if one of the presynchronized spatially separated clocks is brought to the other in an arbitrarily slow motion, it can be seen that when they are compared at the position of the second clock, they remain synchronized. In other words if two clocks have an initial temporal offset between them (due to Boughn-effect or otherwise) when separated, the value for this offset will remain unchanged when they are brought together for comparison. Boughn-effect is thus a real effect (according to the definition) in the relativistic world. In the classical world the situation is different. Below we calculate the effect of clock transport from ZT.

From ZT between a preferred frame  $S_0$  and an arbitrary frame  $S$ , one may write

the transformation equation between any inertial frames  $S_i$  and  $S_k$  as,

$$x_i = \gamma_k^2 \left(1 - \frac{v_i v_k}{c^2}\right) x_k - (v_i - v_k) t_k, \quad (3.16)$$

$$t_i = \gamma_i^2 \left[ \left(1 - \frac{v_i v_k}{c^2}\right) t_k - \frac{\gamma_k^2}{c^2} (v_i - v_k) x_k \right], \quad (3.17)$$

where the suffixes  $i$  and  $k$  of coordinates  $x$ ,  $t$  and  $v$  refer to the coordinates in  $S_i$  and  $S_k$  and velocities of the concerned frames with respect to  $S_0$  respectively. Also  $\gamma_i = \left(1 - \frac{v_i^2}{c^2}\right)^{-1/2}$  and  $\gamma_k = \left(1 - \frac{v_k^2}{c^2}\right)^{-1/2}$ .

Clearly a clock stationary with respect to  $S_k$  will suffer a time "dilation" according to

$$\Delta t_i = \frac{1 - v_i v_k / c^2}{1 - v_i^2 / c^2} \Delta t_k, \quad (3.18)$$

where  $\Delta t_k$  refers to the proper time between two events at the same point of  $S_k$  and  $\Delta t_i$  is the corresponding time measured by observers in  $S_i$ .

Consider now two synchronized clocks are spatially separated by a distance  $x$  in  $S_i$  and a third clock attached to  $S_k$  slowly covers the distance. The time taken by the clock to cover this distance in  $S_i$  is given by

$$\Delta t_i = \frac{x}{w}, \quad (3.19)$$

where  $w$  is the relative velocity of  $S_k$  with respect to  $S_i$ . The corresponding time measured by the third clock ( $S_k$  - clock) may be obtained from Eq.(3.18).

From ZT the relative velocity formula is obtained as

$$w = \frac{\left(1 - \frac{v_i^2}{c^2}\right)(v_k - v_i)}{1 - \frac{v_i v_k}{c^2}}. \quad (3.20)$$

Using Eqs.(3.18), (3.19) and (3.20) one obtains for the difference of these two times

$$\delta t' = \Delta t_k - \Delta t_i = \frac{v_i x}{c^2} \gamma_i^2. \quad (3.21)$$

This non-vanishing integrated effect of the time dilation in the classical world due to clock transport is independent of the speed ( $v_k$ ) at which the clock is transported.

In contrast, in the relativistic world one finds different values for the effect for different velocities and in particular the value is zero when the speed is vanishingly small.

If now the two stationary (with respect to  $S_i$ ) clocks refer to two Boughn's observers  $A$  and  $B$ , they have precisely this amount (Eq.(3.21)) of temporal offset with a negative sign and hence if the observer  $A$  walks towards  $B$  no matter whether slow or fast, the result will be the zero time difference between the clocks when compared at one spatial point. This observation demonstrates that although Boughn's paradox can be recast in the Galilean world the time-offset effect is just an artifact and not real according to our definition of "reality" of the effect. Thus GSDC cannot be obtained from this Boughn's effect in the classical world via EP. Conversely Boughn's temporal offset may be regarded as an integrated effect of GSDC while in the classical world if it exists is just an artifact of the synchrony.

### 3.5 Resolution

Let us now move on to the details of the arguments leading to Eq.(3.1). The outward trip of the traveler twin  $B$  from the point of view of the earth twin is composed of two phases. In the first phase, the rocket moves a distance  $L_A$  in time  $t_{A1}$  with uniform velocity  $v$  which is given by

$$t_{A1} = \frac{L_A}{v}, \quad (3.22)$$

and in the second phase, which corresponds to the deceleration phase of the rocket which finally stops before it takes the turn-around, the time  $t_{A2}$  taken by  $B$  is given

by

$$t_{A2} = \frac{\gamma v}{g}, \quad (3.23)$$

where the proper acceleration  $g$  has been assumed to be uniform with respect to the earth frame. In the present analysis this term does not contribute since we consider the abrupt turn-around scenario where  $t_{A2}$  tends to zero as  $g \rightarrow \infty$ ; however for the time being we keep it. Therefore the total time elapsed in  $S$  for the entire journey is given by

$$T_A = \frac{2L_A}{v} + 2t_{A2}. \quad (3.24)$$

Now we compute this time as measured in  $B$ 's clock by taking the time dilation effect from the point of view of  $A$ . For phase 1 this time  $t_{B1}$  may be computed as

$$t_{B1} = \gamma^{-1}t_{A1} = \frac{\gamma^{-1}L_A}{v}, \quad (3.25)$$

where we have applied the simple time dilation formula. For phase 2 however this time-dilation formula is differentially true as the speed is not a constant i. e one may write

$$dt_{B2} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt_{A2} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \frac{1}{g} d(\gamma v). \quad (3.26)$$

Hence after integration one obtains[29]

$$t_{B2} = \frac{c}{2g} \ln\left(\frac{1 + v/c}{1 - v/c}\right). \quad (3.27)$$

However once again this tends to zero as  $g \rightarrow \infty$ . In any case we shall however not need this expression any more. Therefore the total elapsed time measured in  $B$ 's clock for the complete journey is given by

$$T_B = \frac{2\gamma^{-1}L_A}{v} + 2t_{B2}. \quad (3.28)$$

The differential aging from the point of view of  $A$  is thus

$$\delta T_A = T_A - T_B = \frac{2L_A}{v}(1 - \gamma^{-1}) + 2(t_{A2} - t_{B2}). \quad (3.29)$$

From the point of view of  $B$  the stay-at-home observer  $A$  is moving in the opposite direction and as before one may divide the relative motion of  $A$  into two phases, phase I and phase II, where the later corresponds to the acceleration phase. The phase II may be interpreted as turning on of a gravitational field. When this field is switched off (marking the end of the acceleration phase), the phase I starts where the stay-at-home observer  $A$  moves with a velocity  $-v$  up to a distance  $L_B$  which on account of the Lorentz contraction of  $L_A$  is given by,

$$L_B = L_A \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}, \quad (3.30)$$

and the corresponding elapsed time  $t_{B1}$  is given by,

$$t_{B1} = \frac{L_B}{v} = \frac{\gamma^{-1} L_A}{v}. \quad (3.31)$$

This obviously comes out to be the same as  $t_{B1}$  since the result is obtained from considerations with respect to the inertial observer  $A$ . Similarly  $t_{BII}$  i.e.  $B$ -clock's time during phase II should be the same as  $t_{B2}$  during which the gravitational field is turned on, i.e

$$t_{BII} = t_{B2}, \quad (3.32)$$

and hence the total time

$$\tau_B = 2t_{B1} + 2t_{BII} = \frac{2\gamma^{-1} L_A}{v} + 2t_{BII} = T_B. \quad (3.33)$$

The corresponding time of  $A$ 's clock by taking into account the time dilation effect is

$$t_{A1} = \gamma^{-1} t_{B1} = \frac{\gamma^{-2} L_A}{v}. \quad (3.34)$$

Writing  $A$ -clock's time during phase II from  $B$ 's perspective as  $t_{AII}$ , one may write for  $A$ 's clock time for the entire journey as

$$\tau_A = 2t_{A1} + 2t_{AII} = \frac{2\gamma^{-2} L_A}{v} + 2t_{AII}. \quad (3.35)$$

The difference of these times of clocks  $A$  and  $B$  as interpreted by the observer  $B$ , is given by,

$$\delta T_B = \tau_A - \tau_B = \frac{2\gamma^{-1}L_A}{v}(\gamma^{-1} - 1) + 2(t_{AII} - t_{BII}). \quad (3.36)$$

Note that at the moment we do not know the value of  $t_{AII}$ , since it refers to the time measured by  $A$  as interpreted by  $B$  when it is in its acceleration phase. The paradox is resolved if

$$\delta T_A = \delta T_B. \quad (3.37)$$

In other words using Eqs.(3.29) and (3.36) one is required to have,

$$t_{AII} = \frac{L_A}{v}(1 - \gamma^{-2}) + t_{A2} = \frac{L_A v}{c^2} + t_{A2}. \quad (3.38)$$

In the abrupt turn-around scenario, as we have already observed  $t_{A2} = 0$ , one therefore must have

$$t_{AII} = \frac{L_A v}{c^2} = \frac{\gamma L_B v}{c^2}. \quad (3.39)$$

The resolution of the twin paradox therefore lies in accounting for this term. It is interesting to note that the term is independent of the acceleration in phase II. This is possibly the implicit reason why the role of acceleration in the explanation of the twin paradox is often criticized in the literature. However we shall now see how, we can interpret this term as an effect of the direction reversing acceleration (or the pseudo-gravity) experienced by the traveller twin.

Now recall the Boughn-effect of temporal offset between two identically accelerated observers. To be specific, consider an inertial frame of reference  $S$  attached to the observer  $B$  when it is in the uniform motion phase (phase I). Suppose now there is another observer  $B'$  at rest in  $S$  at a distance  $L_B$  behind  $B$  and both of them get identical deceleration and eventually come to rest with respect to  $A$  in the frame of reference  $S'$ , which is moving with velocity  $-v$  in the  $x$ -direction with

respect to  $S$ . According to Boughn-effect then the clocks of these two observers get desynchronized and the amount of this desynchronization is given by the expression (4) only with the sign changed, that means

$$desync = \frac{\gamma v L_B}{c^2}, \quad (3.40)$$

which is nothing but  $t_{AII}$ . It has already been pointed out that this Boughn-effect may be interpreted as the effect of pseudo-gravity (in this case as experienced by the observer  $B$ ) according to EP. In terms of the pseudo acceleration due to gravity the above expression can also be obtained as

$$desync = \frac{g \Delta t_B L_B}{c^2}. \quad (3.41)$$

Note that  $g \Delta t_B$  is finite (equal to  $\gamma v$ ) even if  $g \rightarrow \infty$ .

The observer  $B'$  which is  $L_B$  distance away from  $B$  is spatially coincident with  $A$ , hence, in calculating the clock time of  $A$  from  $B$ 's perspective this time-offset due to Boughn-effect must be taken into account. This effect is ignored when the twin paradox is posed by naïvely asserting the reciprocal time-dilation effect for the stay-at-home and the rocket-bound observers. Clearly the paradox is resolved if the Boughn-effect or the pseudo gravitational effect is taken into consideration.

### 3.6 Concluding Remarks: Test of Boughn-Effect

We have seen that the Boughn-effect can be interpreted as the integrated effect of GSDC. The experimental test of GSDC or the gravitational red-shift is therefore a test of a differential Boughn-effect in a way. On the contrary one may directly measure the integrated effect by the following means:

First two atomic clocks may be compared (synchronized) at the sea level, then one of the clocks may be slowly transported to a hill station of altitude  $h$  and then kept there for some time  $T$ . In this time these two atomic clocks according to Boughn scenario are perpetually accelerated from a rest frame  $S$  to a hypothetical inertial frame  $S'$  moving with velocity  $v$ , with proper acceleration  $g$  so that  $\gamma v = gT$ . Boughn-effect therefore predicts a temporal offset (see Eqs.(3.40) and (3.41)),

$$\Delta t_{offset} = \frac{ghT}{c^2}. \quad (3.42)$$

This offset can be checked by bringing the hill station clock down and then comparing its time with the sea level one. Any error introduced in the measurement due to transport of clocks can be made arbitrarily small compared to  $\Delta t_{offset}$  by increasing  $T$ . As a realistic example for  $h = 7000\text{ft}$  (altitude of a typical hill station in India), and  $T = 1$  year and taking the average  $g$  to be about  $9.8\text{m/sec}^2$ , the Boughn-effect comes out to be in the micro-second order:

$$\Delta t_{offset} = 7.3\mu\text{s}, \quad (3.43)$$

which is easily measurable without requiring sophisticated equipments, such as those used in Pound-Rebka type experiments.

It is interesting to note that from the empirical point of view the effect is not entirely unknown. For example Rindler[16], in seeking to cite an evidence for the GSDC effect, remarks: "Indeed, owing to this effect, the US standard atomic clock kept since 1969 at the National Bureau of standards at Boulder, Colorado, at an altitude of 5400ft. gains about five microseconds each year relative to a similar clock kept at the Royal Greenwich Observatory, England, ...". However one can consciously undertake the project with all seriousness, for the accurate determination of the time-offset (with the error bars and all that), not merely to prove GSDC but to verify the Boughn-effect of SR. It is worth while to note that the

empirical verification of this time-offset as a function of  $T$  would not only test the Boughn-effect and the integral effect of GSDC but it would also provide empirical support for the relativity of simultaneity <sup>7</sup> of SR. So far no experimental test has been claimed to be the one verifying the relativity of simultaneity. Indeed SR is applicable in the weak gravity condition of the earth so that gravity can be thought of as a field operating in the flat (Minkowskian) background of the space-time[30]. Clearly because of EP, the earth with its weak gravity has the ability to provide a convenient Laboratory to test some special relativistic effects like the relativity of simultaneity or the Boughn-effect.

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<sup>7</sup>In the light of the CS-thesis however “relativity of simultaneity” loses its absolute meaning, since for example if absolute synchrony is used, there is no lack of synchrony between two spatially separated events as observed from different inertial frames, however, the differential aging or the temporal offset will pop up as a time dilation effect in the absolute synchrony set-up when the clocks are brought together by slow transport. The details of this issue is a subject matter of another paper by the authors in preparation.

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