CHAPTER - 12
SPECTRAL ANALYSIS ON THE RELATIONSHIP BETWEEN EXCHANGE RATE (E_t) AND RELATIVE PRICE LEVEL (P_t)

12.1 Introduction:

The study in Chapters 4 through 11 constitutes the '*Time Domain*' analysis which considers the evolution of a process through time. The fundamental base of the '*Time Domain*' study is the *Autocovariance (or autocorrelation*) function. A complementary analysis for the '*Time Domain*' study is constituted by the '*Frequency Domain*' study which considers the frequency properties of a time series. The '*Spectral Density Function*' constitutes the natural tool for the study. As a matter of fact, inference regarding the '*Spectral Density Function*' is called an analysis in the '*Frequency Domain*'.

'*Spectral Analysis*' is the name given to the methods of estimating the '*Spectral Density Function*' or '*Spectrum*' of a given time series. The study under '*Frequency Domain*' is concerned with estimating the '*Spectrum*' over the whole range of frequencies in order to identify '*hidden periodicities*'. '*Cross-Spectral*' properties are expected to provide us the relation between relative price level and exchange rate over the sub-period 1993:2-2006:1.

12.2 Spectral Estimation: Methodology

12.2.1 Fourier Series:

Traditional '*Spectral Analysis*' is a modified form of '*Fourier Analysis*' and this modification makes it suitable for *stochastic* rather than *deterministic function. Fourier Analysis* (Priestly, 1981) relates to approximating a function by a sum of *sine* and *cosine* terms which are called the '*Fourier Series Representation'*.

Let a function f(t) be defined on (- π, π) satisfying the *Dirichlet Conditions* which ensure that f(t) is reasonably '*well behaved'* such that f(t) is absolutely integrable over the range (-π, π) with a finite number of *discontinuities* and a finite number of maxima and minima. Then f(t) may be approximated by the *Fourier Series.*

$$\frac{a_0}{2} + \sum_{r=1}^{k} (a_r \cos rt + b_r \sin rt) \qquad (12.1)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)dt \qquad (12.2)$$

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos rtdt \qquad (12.3)$$

$$(r = 1,2..........\,..)$$

$$b_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin rtdt \qquad (12.4)$$

$$(r = 1,2..........\,..)$$

The *Fourier Series* then converges to f(t) as k→∞ except at points of discontinuities, where it converges to halfway up the step change.

12.2.2 Fourier Transformations:

Given a function h(t) of a real variable t, the ***Fourier Transform*** of h(t) is

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt \qquad (12.5)$$

provided the integral exists for every real (ω).

A sufficient condition for H(ω) to exist is that

$$\int_{-\infty}^{\infty} | h(t) | dt < \infty \qquad (12.6)$$

Let equation (12.5) be regarded as an integral equation for h(t). Then we may have h(t) from H(ω) such that

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{i\omega t}\, d\omega \qquad (12.7)$$

Then h(t) is called the *Inverse Fourier Transform* of H(ω). The two function h(t) and H(ω) are commonly called a '*Fourier Transform Pair*'.

However, Cox and Miller (1968) find it convenient to put $\frac{1}{2\pi}$ outside the integral in equation (12.5) such that

$$H(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt \qquad (12.8)$$

Consequently, in the *Inverse Fourier Transform*

$$h(t) = \int_{-\infty}^{\infty} H(\omega)\, e^{i\omega t}\, d\omega \qquad (12.9)$$

Time series analysis usually involves the use of the variable $f = \frac{\omega}{2\pi}$ rather than ω. Then the resulting *Fourier Transform Pair* is

$$G(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt \tag{12.10}$$

$$h(t) = \int_{-\infty}^{\infty} G(f) e^{2\pi i f t} df \tag{12.11}$$

12.2.3 The Spectral Distribution Function

Wiener Khintchine Theorem states that for any real valued stationary stochastic process with autocovariance function γ(k), there exists a monotonically increasing function F(ω) such that

$$\gamma(k) = \int_{0}^{\pi} \cos \omega k dF(\omega) \tag{12.12}$$

Equation (12.12) is called the '*Spectral Representation of the autocovarince Function'*.

F(ω) is the contribution to the variance of the series accounted for by the frequencies in the range (0, ω) given that

$$F(\omega) = 0 \text{ for } \omega<0$$

For a discrete time series process measured at unit intervals of time, the highest possible frequency is the *Nyquist Frequency* (π) and so all the variations is accounted for by frequencies less than π. Thus

$$F(\pi) = Var(X_t) = \sigma^2_x \tag{12.13}$$

In between ω = 0 and ω = π, F(ω) is monotonically increasing. So F(ω) is also called the '*Spectral Distribution Function'*.

12.2.4 The Spectral Density Function (SPECTRUM)

F(ω) is usually a continuous (monotone bounded) function in [0, π]. Therefore, F(ω) may be differentiated with respect to ω in [0, π] such that

$$f(\omega) = \frac{dF(\omega)}{d\omega} \tag{12.14}$$

f(ω) is the '*Spectral Density Function*' or '*Spectrum*'

If f(ω) exists , then equation (12.12) can be expressed as

$$\gamma(k) = \int_0^{\pi} \cos \omega k f(\omega) \quad d\omega \tag{12.15}$$

Now putting k = 0, we have

$$\gamma(0) = \sigma^2{}_x = \int_0^{\pi} f(\omega) d\omega = F(\pi) \tag{12.16}$$

Thus f(ω) dω represents the contribution to variance of components with frequencies in the range (ω, ω + dω).

For a continuous purely indeterministic stationary process, X(t), the *autocovariance function*, $\gamma(\tau)$, is defined for all τ and the *spectral density function*, f(ω) is defined for all positive ω. Then

$$f(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \gamma(\tau) e^{-i\omega\tau} d\tau \tag{12.17}$$

$$= \frac{2}{\pi} \int_0^{\infty} \gamma(\tau) e^{-i\omega\tau} d\tau \tag{12.18}$$

For 0 < ω < ∞, the *inverse transformation* gives

$$\gamma(\tau) = \int_0^{\infty} f(\omega) \cos \omega\tau d\omega \tag{12.19}$$

12.2.5 The Cross Spectrum

In *Time Domain* analysis the tool for examining the relationship between time series is the *Cross-Correlation Function.* In *Frequency Domain* analysis, a complementary function used as a tool for the same purpose in the *Cross Spectral Density Function* or the *Cross-Spectrum.*

Let X and Y be two discrete-time stationary processes, measured at unit intervals of time. Let $\gamma_{xy}(k)$ represent the relevant *Cross Covariance Function.* Then the *Fourier Transform* of the *cross-covariance function* $\gamma_{xy}(k)$ is

$$f_{xy}(\omega) = \frac{1}{\pi}[\sum_{k=-\infty}^{\infty} \gamma_{xy}(k) e^{-i\omega k}] \tag{12.20}$$

over the range $0 < \omega < \pi$.

Here $f_{xy}(\omega)$ is called the '*Cross- Spectral Density Function*' or the '*Cross-Spectrum*'.

If the *Cross-Spectrum'* is defined over the range $[-\pi,\pi]$, then

$$f_{xy}(\omega) = \frac{1}{2\pi}[\sum_{k=-\infty}^{\infty} \gamma_{xy}(k) e^{-i\omega k}] \tag{12.21}$$

Consequently, the '*inverse transformation'* gives

$$f_{xy}(k) = \int_{-\pi}^{\pi} e^{i\omega k} f_{xy}(\omega) d\omega \tag{12.22}$$

12.2.6 Co-spectrum and Quadrature Spectrum

$f_{ry}(\omega)$, the *Cross –spectrum,* is a complex function since $\gamma_{xy}(k)$ is not an even function. The real part of the '*Cross-spectrum'* is called the '*Co-Spectrum*' which is given by

$$\begin{aligned} c(\omega) &= \frac{1}{\pi}[\sum_{k=-\infty}^{\infty} \gamma_{xy}(k) \cos \omega t] \\ &= \frac{1}{\pi}\{\gamma_{xy}(0) + \sum_{k=1}^{\infty}[\gamma_{xy}(k) + \gamma_{yx}(k)] \cos \omega k \end{aligned} \tag{12.23}$$

The complex part of the *Cross-Spectrums* is called the *Quadrature Spectrum* and it is given by

$$q(\omega) = \frac{1}{\pi}[\sum_{k=-\infty}^{\infty} \gamma_{xy}(k) \sin \omega k]$$

$$= \frac{1}{\pi}\{\sum_{k=1}^{\infty}[\gamma_{xy}(k) - \gamma_{yx}(k)] \sin \omega k\} \tag{12.24}$$

Consequently,

$$f_{xy}(\omega) = c(\omega) - q(\omega) \tag{12.25}$$

12.2.7 Cross Amplitude Spectrum and Phase Spectrum

The *Cross Spectrum* can be expressed as

$$f_{xy}(\omega) = \alpha_{xy}(\omega) e^{i\phi xy(\omega)} \tag{12.26}$$

where $$\alpha_{xy}(\omega) = \sqrt{c^2(\omega) + q^2(\omega)} \tag{12.27}$$

Here $\alpha_{xy}(\omega)$ is the *Cross- Amplitude Spectrum.*

$$\phi_{xy}(\omega) = \tan^{-1}[\frac{-q(\omega)}{c(\omega)}] \tag{12.28}$$

is the *Phase Spectrum.*

12.2.8 Coherency Spectrum and Gain Spectrum

From the Equations (12.23) and (12.26) we obtain

$$c(\omega) = [c^2(\omega) + q^2(\omega) / f_x(\omega) f_y(\omega)]$$

$$= \sigma^2_{xy}(\omega) / f_x(\omega) f_y(\omega) \tag{12.29}$$

where $f_x(\omega)$, $f_y(\omega)$ are the *power spectra* of the individual processes, $\{x_t\}$ and $\{Y_t\}$ such that

$$0 \le c(\omega) \le 1$$

c(ω) is called the *Coherency Spectrums.*

The estimate of c(ω) measures the square of the linear correlation between the two components of the bivariate process at frequencies ω and it is analogous to the square of the usual correlation coefficient.

The *Gain Spectrums* is given by

$$G_{xy}(\omega) = \sqrt{[f_y(\omega)c(\omega)]} / f_x(\omega)$$

$$= \frac{\alpha_{xy}(\omega)}{f_x(\omega)}$$

This is essentially the regression coefficient of the process Y_t on the process X_t at frequency ω.

A second *Gain Spectrum* can also be defined by

$$G_{yx}(\omega) = \frac{\alpha_{xy}(\omega)}{f_y(\omega)}$$

This is the regression coefficient of the process X_t on the process Y_t at the frequency ω.

12.3 The 'Cospectral Densities' of E_t and P_t

The '*Cospectral Density by Frequency*' for E_t and P_t is given by the Figure 12.1 while the '*Cospectral Density by Period*' for E_t and P_t is given by the Figure 12.2.

Figure 12.1

Cospectral Density By Frequency Exchange Rate (E_t) and Relative Price Level (P_t)

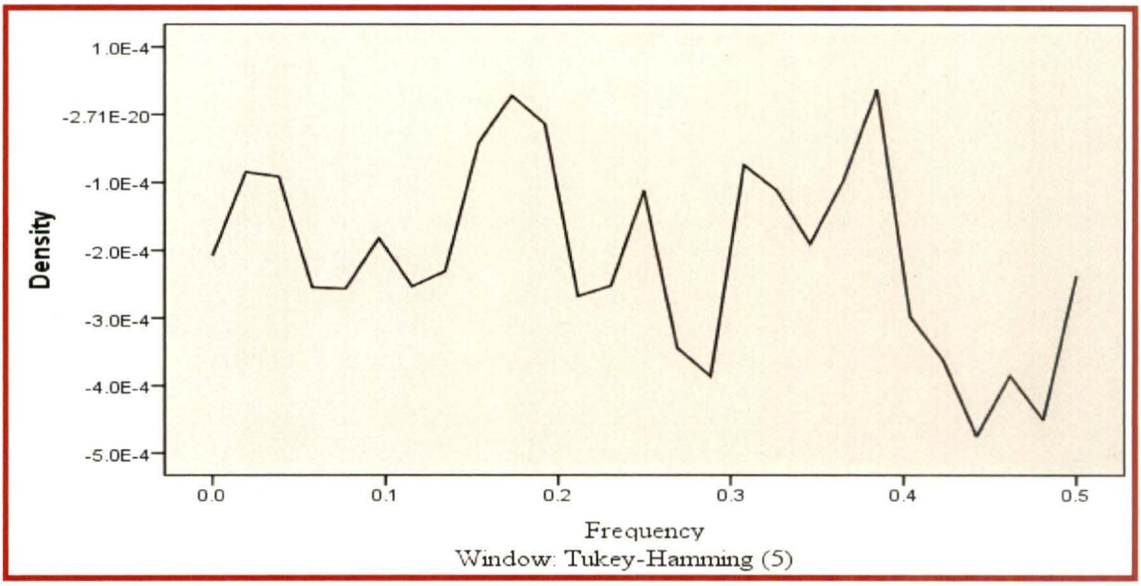
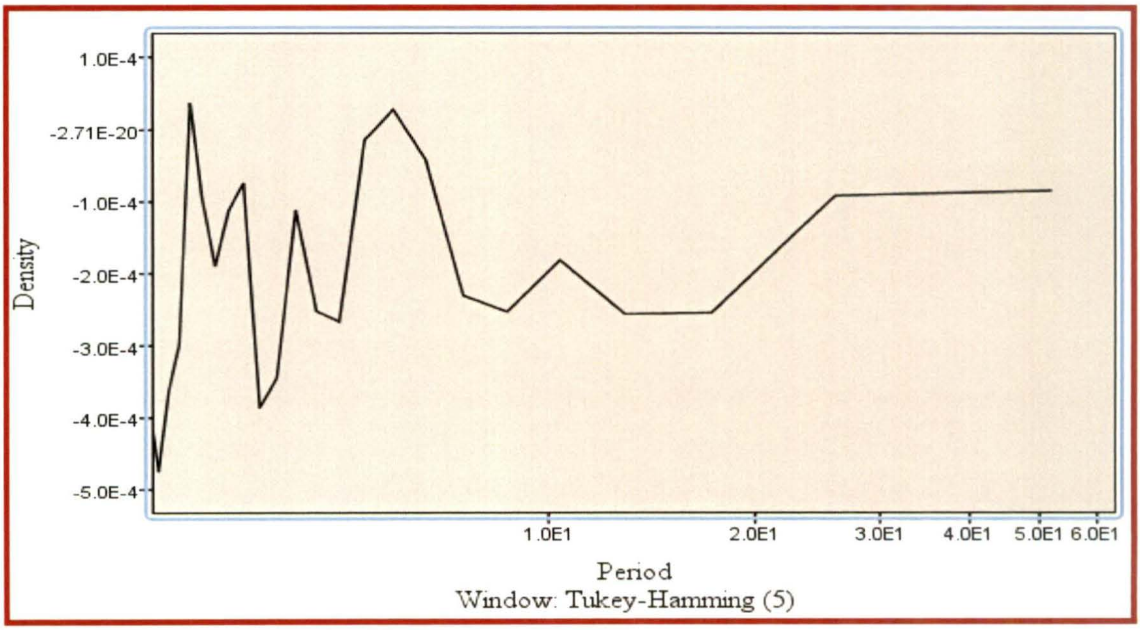


Figure 12.2

'Cospectral Density' by Period for Exchange Rate(E_t) and Relative Price Level (P_t)



The Figure 12.1 shows that the '*Cospectral Density by Frequency*' for E_t and P_t

i. is not a horizental straight line.
ii. is marked by the presence of structural ups and down.
iii. exhibits sharp peaks at frequencies 0.25, 0.35 and 0.4 (approximately).

The Figure 12.2 shows that the *Cospectral Density by Period* for E_t and P_t

i. is far from being a horizontal straight line.
ii. exhibits several prominent ups and down, and
iii. is marked by the presence of prominent peaks at periods 2,3,4,(approximately).

These features of the '*Cospectral Density*' for E_t and P_t indicate that

i. there did exist significant covariations of E_t and P_t over the period of study (1993:2-2006:1).
ii. these co-movements were marked by some 'periodicities'.
iii. there did exist dominant periodicities at periods 2,3 and 4 (approx).

All these observations testify that over the period 1993:2 – 2006:1

i. E_t and P_t were cointegrated, and
ii. the long-run relationship between these variables was 'stable'.

12.4 Features of the 'Gain Spectrum' of Exchange Rate (E_t) and Relative price Level (P_t)

The '*Gain Spectrum*' by *frequency* for E_t and P_t is being presented through the Figure 12.3 while th Figure 12.4 presents the corresponding '*Gain Spectrum*' by *period.*

Figure –12.3

'Gain Spectrum' By Frequency For Exchange Rate(E_t) and Relative Price Level (P_t)

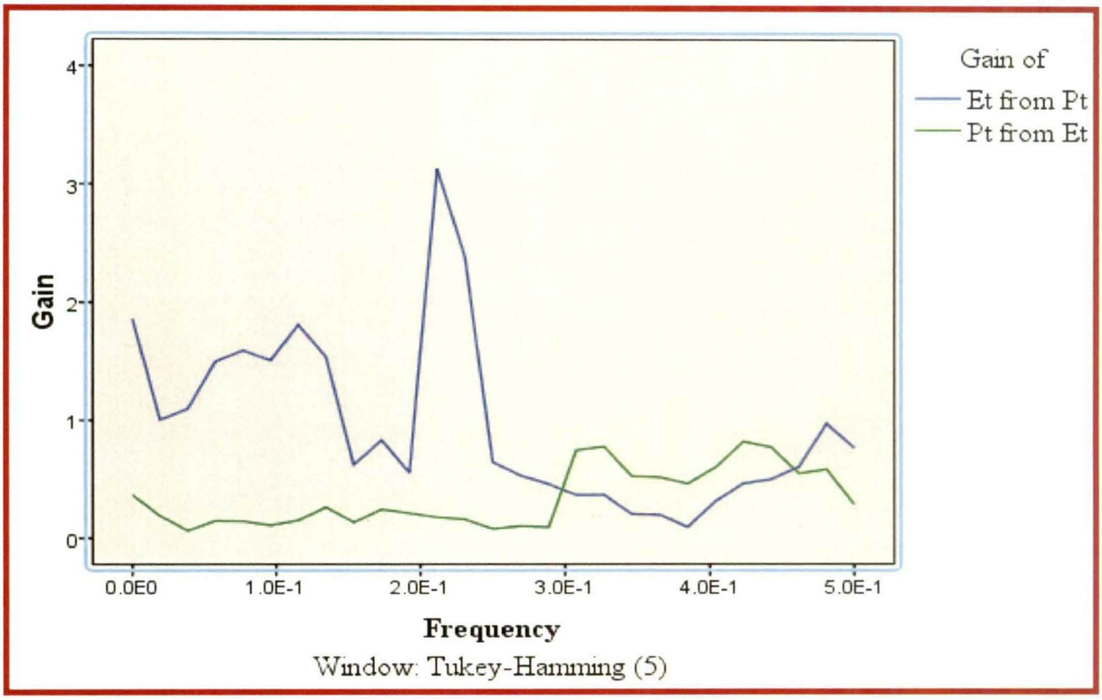
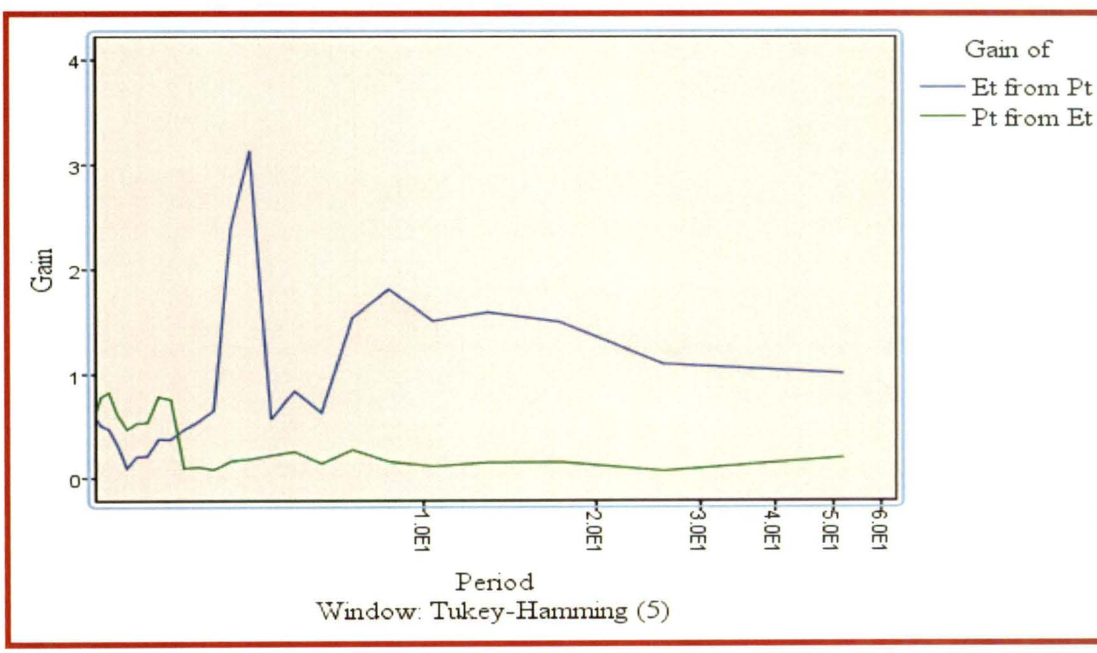


Figure 12.4

'Gain Spectrum' By Period For Exchange Rate (E_t) and Relative Price Level(P_t)



The 'Gain Spectrum' in Figures 12.3 and 12.4 for E_t and P_t for the period 1993:2 – 2006:1 show that

i. *the ' Gains of exchange rate (E_t) from relative price level (P_t) lie over the 'Gains of relative price level' (P_t) from exchange rate across almost all the frequency levels and periods.*
ii. *the 'Gain of E_t from P_t attains the highest value (3.25) at the frequency level 0.22 i.e. at period 4 (approx).*
iii. *the ' Gains of P_t from E_t' hardly exceeded 0.25 beyond period 2 i.e. within the frequency range [0,0.3].*
iv. *the 'Gain of P_t from E_t' was close to unity at period 2 i.e. at frequency 0.325.*

All these observations indicate that

i. *the regression coefficients, in case of regressions of exchange rate on relative price level, on the basis of period or frequencies, exceeded those when relative price level series was regressed on exchange rate series across different periods or frequencies.*

This testifies for the fact that exchange rate variations were 'Granger Caused' by those in relative price level, On the contrary, variations in relative price level displayed no significant relation with those in exchange rate.

ii. *the coefficient of regression of exchange rate on relative price level appeared to be singularly significant at period 4 or at frequency level 0.22.*

This observation again supplants and confirms the predominant significance of P_{t-4} in the vector of regressors for E_t in the '*Unrestricted*' and '*Restricted*' VAR systems.

12.5 Study with the 'Coherency Spectrum' of Exchange Rate (E_t) and Relative Price Level(P_t) by Frequency and by Period.

The '*Coherency Spectrum*' for E_t and P_t by frequency is being presented by the Figure 12.5 while the Figure 12.6 presents the corresponding '*Coherency Spectrum*' by period.

Figure 12.5

'Coherency Spectrum' By Frequency for Exchange Rate (E_t) and Relative Price Level (P_t)

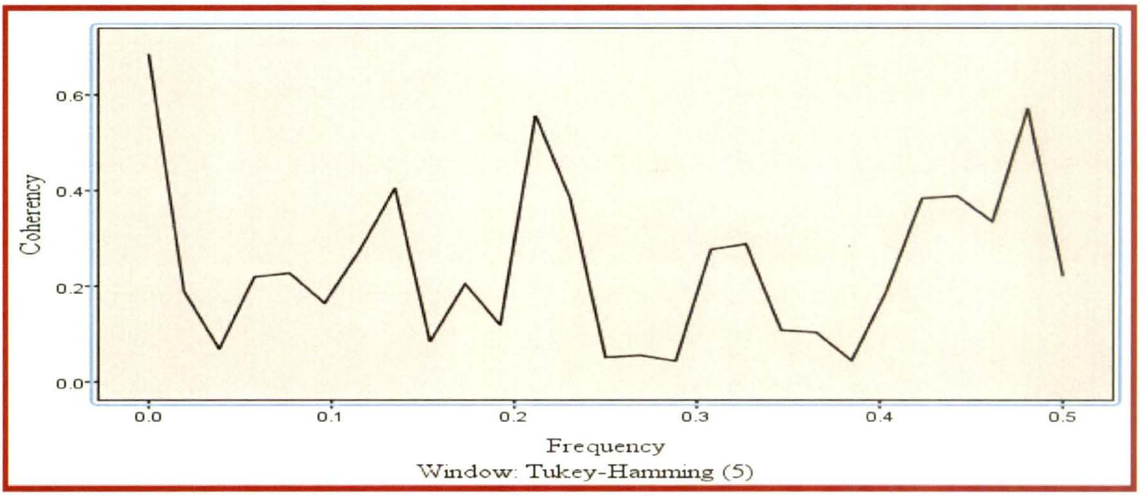
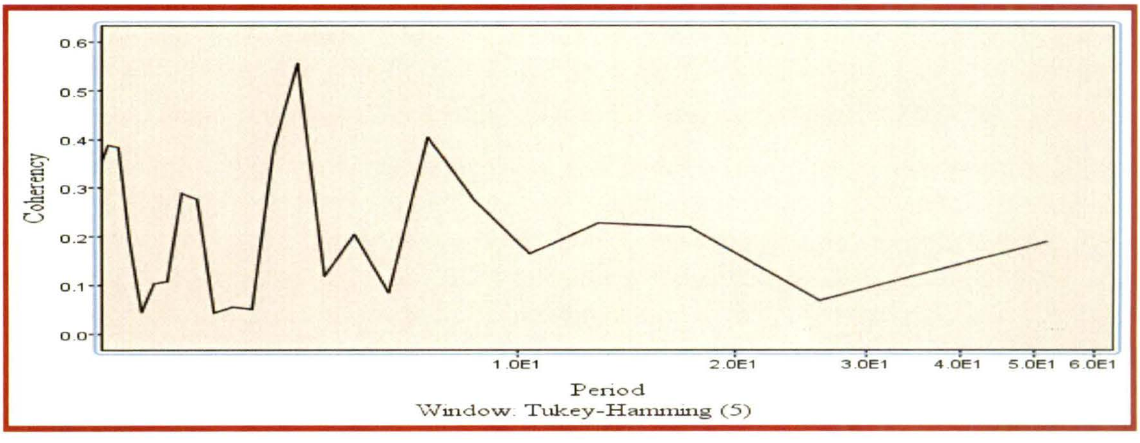


Figure 12.6

'Coherency Spectrum' By Period for Exchange Rate (E_t) and Relative Price Level (P_t)



The '*Coherency Spectrum*' in the Figure 12.5 shows that

i. the '*coherency*' for the variables E_t and P_t was as high as 0.6 (approx) at frequency 0.46 (approx).
ii. the '*coherency*' in 0.5 (approx) at frequency 0.22 (approx).

The '*Coherency Spectrum*' in the Figure 12.6 correspondingly shows that

i. the '*coherency*' was as high as 0.6 (approx) at period 4 while.
ii. the '*coherency*' was 0.5 (approx) at period 8.

These observations confirm that

i. *there did exist high degree of co-movements (association) between the variables E_t and P_t over the period of study.*
ii. *there did exist a 'stable' relationship between the variable concerned.*
iii. *there did exist significant periodicity at frequency 0.46 or at period 4.*

12.6 The 'Phase Spectrum' for Exchange Rate (E_t) and Relative Price Level (P_t)

The '*Phase Spectrum*' for E_t and P_t by frequency is shown by the Figure 12.7 and the Figure 12.8 shows the corresponding '*Phase Spectrum*' of the variables concerned by period.

Figure 12.7

Phase Spectrum By Frequency For Exchange Rate (E_t) and Relative Price Level (P_t)

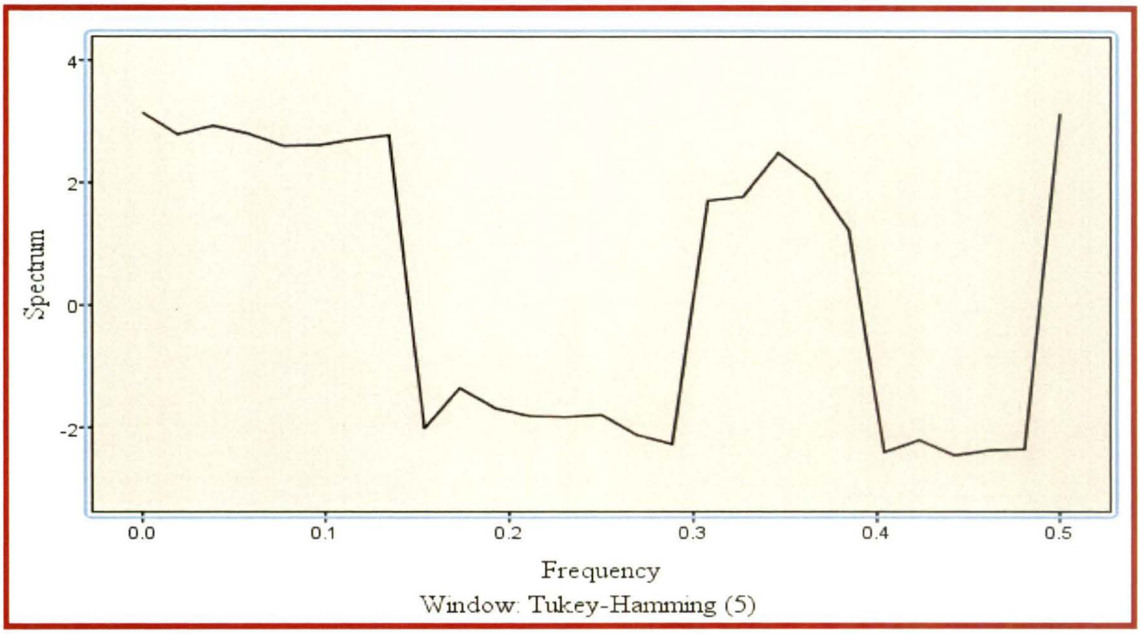
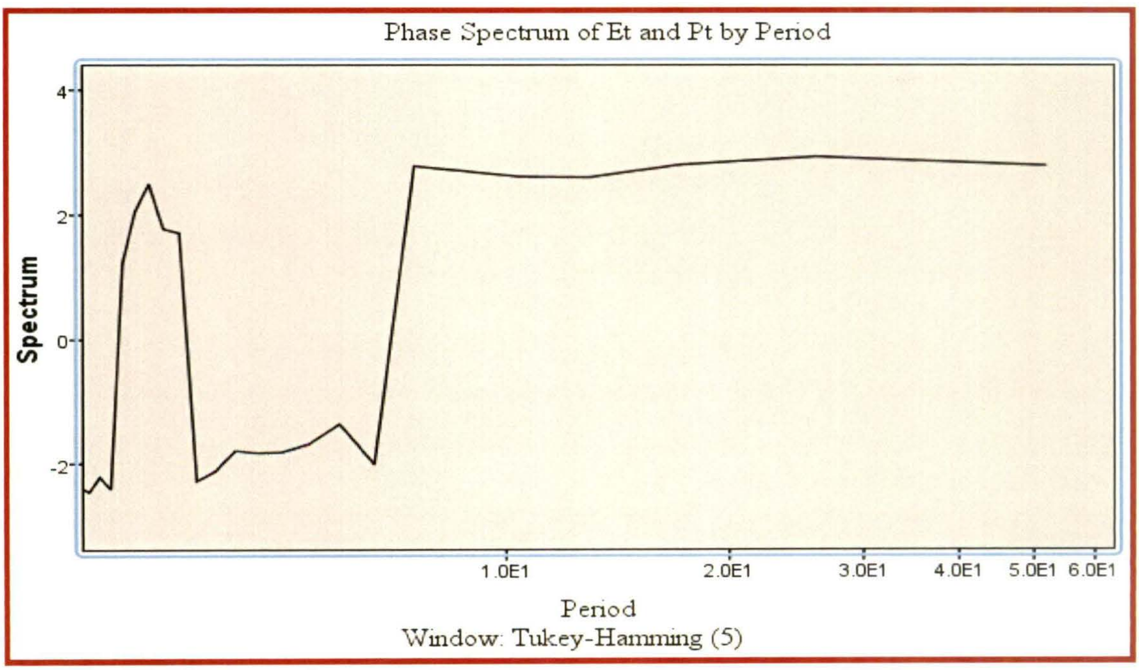


Figure 12.8

Phase Spectrum By Period For Exchange Rate (E_t) and Relative Price Level (P_t)



The *'Phase Spectrum'* in the Figures (12.7) - (12.8) show that the phase difference is negative over almost all the frequency levels barring frequency ranges (0.08 - 0.16) and (0.3 - 0.38) or corresponding period ranges (1.5-3) and (6-16) respectively. *Relative Price level* (P_t), therefore, was in '*Lead*' position and *Exchange Rate* (E_t) was in '*Lag*' position across almost all frequency levels. However, the 'lag position' of *'Exchange Rate'* implies that variation in *'Relative Price Level'* was an important source of variation in *'Exchange Rate'*.

These futures of the *'Phase Spectrum'* indicate that variations in *Relative Price Level* (P_t) occurred first and these variations then led to variations in *Exchange Rate* (E_t). *Consequently, the Spectral analysis confirms the Time Domain findings of 'Uni-directional Causality' from Relative Price Level (P_t) to Exchange Rate (E_t) over the Sub-period 1993:2 - 2006:1.*

12.7 Overview of Findings From the Spectral Analysis:

In the *'Spectral Analysis'* of time series dataset for Exchange Rate (E_t) and Relative Price level (P_t) over the sub - period 1993:2 - 2006:1.

i. *the Cospectrum for E_t and P_t exhibits dominant periodicities at 2, 3, 4 periods (or 0.25, 0.35 and 0.4 frequencies). This confirms the Time Domain finding that E_t and P_t were 'Cointegrated' and the long-run relationship between these variables was 'stable'.*

ii. *the 'Gain Spectrum' for E_t ant P_t confirms the Time Domain finding of the existence of 'Uni-directional Causality' running from Relative Price Level (P_t) to Exchange Rate (E_t) and the 'Gain' was more pronounced at the frequency level 0.22 (or at period 4).*

iii. *the 'Coherency Spectrum' for E_t and P_t confirms the presence of strong 'Coherence' in the joint variation of these variables. The maximum 'Coherence' was observed at period 4 (or at frequency level 0.44).*

iv. *the 'Phase Spectrum' for the variables further confirmed the Time Domain finding that Relative Price Level (P_t). 'Granger Caused' Exchange rate.*