C O N C L U S I O N

It is clear from the first chapter of the present thesis that the analysis of the particular plate geometry leads to the solutions of different differential equations for each plate geometry. Thus a comprehensive as well as a comparative study is not at all possible. The method of solutions for individual plate geometry is laborious too. On the other hand, the conformal mapping technique employed in the second chapter is more powerful and advantageous, for if the mapping function is known, solutions of different plates come out from a single differential equation. Thus labour is minimum and a comparative as well as comprehensive study is possible both for linear as well as non-linear plate problems. Also the investigations of the behaviours of irregular shaped plates under different types of loading are possible only by conformal mapping technique and this is certainly a special advantage of this method.

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VIBRATIONS OF POLYGONAL PLATES DUE TO THERMAL SHOCK

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Vibrations of polygonal elastic plates due to thermal shock are investigated through use of complex variable theory. The numerical results are shown graphically for different polygonal plates and results for triangular, square and circular plates are compared with published results.

1. INTRODUCTION

Thermally induced vibrations are of interest in aircraft and machine design, in nuclear engineering and in astronautical engineering.

The general theory of transverse vibrations of a circular plate was obtained by Kirchhoff [1] who gave a full discussion of the results. The problem has also been discussed very elegantly by Lord Rayleigh [2]. In problems of non-stationary quasi-static stresses in plates, the temperature field varies slowly with time, but in cases of sudden heating, or for temperature fields varying harmonically with time at a high frequency, the role of inertia may become important.

Boley and Barbar [3] were the first to investigate thermal shock problems in rectangular plates. Later, Nowacki [4] and Boley and Weiner [5] merely included the above work of Boley and Barbar in their respective monographs. Thermal vibrations of a right angled isosceles triangular plate and of an equilateral triangular plate have been investigated by Biswas [6], who has applied trilinear co-ordinates to obtain the solution. The vibrations of circular plate due to thermal shock have been investigated by Sarkar [7].

All the above investigations were based on the assumption that the non-stationary motion which results from the thermal shock can be written as the sum of a quasi-static displacement W_s and a dynamic displacement W_d. According to the above assumptions the following two differential equations result:

$$\nabla^4 W_s + K_s \nabla^2 \tau = 0, \qquad \nabla^4 W_d + K_d(\ddot{W}_s + \ddot{W}_d) = 0. \tag{1, 2}$$

It would seem, however, that the quasi-static assumption which resulted in the above two equations is not valid for thermal shock problems, nor indeed is it valid in general for any type of impulsive or step-loading. The quasi-static assumption is completely valid when the thermal loading, as any other loading, is in fact applied sufficiently slowly so that the inertial forces associated with the slow displacements are completely negligible. But under thermal shock loading inertial forces are assumed to result from the displacements and must be accounted for. When a thermal shock load is applied such that the induced displacements cause inertial forces of the same order of magnitude as the vibratory inertia forces, the "quasi-steady" assumption would seem to be no more valid for thermal shock of beams and plates than the quasi-steady assumption is valid for the flutter of airplane wings under aerodynamic loading.

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For accurate analysis of thermal shock plate problems there is no need for the quasi-static assumption and there is only one differential equation to be solved for the time dependent plate deflection.

In what follows the displacement of polygonal elastic plates is investigated by solving the differential equation without the usual quasi-static assumption. Complex variable theory has been applied for the solution.

2. ANALYSIS

Consider an elastic plate of uniform thickness h. In a Cartesian co-ordinate system, the z-azis is taken along the thickness of the plate and perpendicular to both the x and y axes. The face $z = +h/2$ is subjected to a sudden heating while the other face $z = -h/2$ together with all edges is insulated (a list of notation is given in the Appendix).

The equation for vibrations of such a plate is given by [4]

$$\nabla^4 W(x, y, t) + (1+\nu)\alpha_t \nabla^2 \tau(x, y, t) + (\rho h/D)\ddot{W}(x, y, t) = 0, \tag{3}$$

where $W(x, y, t)$ is the displacement of the plate and $\tau(x, y, t)$ is the temperature field given by [4]

$$\tau(x, y, t) = \frac{12}{h^2}\int_{-h/2}^{+h/2} zT(z, t)\,dz = \frac{q_1}{2\lambda}\left(1 - \frac{96}{\pi^4}\sum_{j=1,3,\ldots}\frac{1}{j^4}e^{-j^2/3t}\right), \tag{4}$$

where $\lambda\ \partial T/\partial z = q_1$ at $z = h/2$ [4], and $\tau(x, y, t) = (q_1/2\lambda)K(t)$ (say).

Now, changing equation (3) into complex co-ordinates by the transformation $Z = x + iy$ and $\bar{Z} = x - iy$, one has,

$$16(\partial^4 W/\partial Z^2\,\partial\bar{Z}^2) + 4(1+\nu)\alpha_t(\partial^2\tau/\partial Z\,\partial\bar{Z}) + (\rho h/D)\ddot{W} = 0. \tag{5}$$

Let

$$Z = f(\xi) \approx (L\xi + \delta\xi^5) \tag{6}$$

be the mapping function which maps the domain under consideration on to a unit circle in the complex plane, where $\xi = r\,e^{i\theta}$. With this transformation one gets

$$16\left[\frac{\partial^4 W}{\partial\xi^2\,\partial\bar{\xi}^2}\frac{dZ}{d\xi}\frac{d\bar{Z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial\bar{\xi}^2\,\partial\xi}\frac{d^2 Z}{d\xi^2}\frac{d\bar{Z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial\xi^2\,\partial\bar{\xi}}\frac{d^2\bar{Z}}{d\bar{\xi}^2}\frac{dZ}{d\xi} + \frac{\partial^2 W}{\partial\xi\,\partial\bar{\xi}}\frac{d^2\bar{Z}}{d\bar{\xi}^2}\frac{d^2 Z}{d\xi^2}\right]$$
$$+\frac{\rho h}{D}\left(\frac{dZ}{d\xi}\right)^3\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^3\ddot{W} = -4(1+\nu)\alpha_t\frac{\partial^2\tau}{\partial\xi\,\partial\bar{\xi}}\left(\frac{dZ}{d\xi}\right)^2\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2. \tag{7}$$

Let

$$W = B_0(t)[1 - 2P\xi\bar{\xi} + Q\xi^2\bar{\xi}^2], \tag{8}$$

which clearly satisfies the simply supported edge condition when $P = (3+\nu)/(5+\nu) = 0{\cdot}6226$ $(\nu = 0{\cdot}3)$ and $Q = (1+\nu)/(5+\nu) = 0{\cdot}2453$. Also let

$$\tau = (q_1/2\lambda)K(t)[1 - \xi\bar{\xi}]. \tag{9}$$

This states that τ vanishes at the boundary $r = 1$. Substituting equations (8), (9) and (6) in equation (7) one gets the error function $\varepsilon(r, \theta)$; Galerkin's procedure requires that

$$\int_c \varepsilon(r, \theta)Wr\,dr\,d\theta = 0. \tag{10}$$

After evaluating the integrals in expression (10) one gets the required differential equation determining $B_0(t)$.

This equation, on simplification, takes the form

$$\ddot{B}_0(t)+w^2B_0(t)=C^2K(t), \tag{11}$$

where

$$w^2=\frac{16A_{11}}{\psi_4}\frac{D}{\rho h},\qquad C^2=\frac{4(1+\nu)\alpha_t\psi_5}{\psi_4}\frac{D}{\rho h}\frac{q_1}{2\lambda},$$

$$A_{11}=(4Q\psi_1-800Q\,\delta^2\psi_2+400\,\delta^2\psi_3),$$

$$\psi_1=(0{\cdot}5-0{\cdot}5P+Q/6)L^2+(2{\cdot}5-\tfrac{25}{6}P+\tfrac{25}{14}Q)\delta^2,$$

$$\psi_2=0{\cdot}1-\tfrac{1}{6}P+\tfrac{1}{14}Q,$$

$$\psi_3=-0{\cdot}25P+0{\cdot}4P^2-\tfrac{5}{6}PQ+0{\cdot}4Q+\tfrac{2}{7}Q^2,$$

$$\begin{aligned}\psi_4=&(0{\cdot}5-P+\tfrac{1}{3}Q-0{\cdot}5PQ+\tfrac{2}{3}P^2+0{\cdot}1Q^2)L^6\\&+(\tfrac{1}{26}-\tfrac{1}{7}P+\tfrac{1}{15}Q-\tfrac{1}{8}PQ+\tfrac{2}{15}P^2+\tfrac{1}{34}Q^2)15\,625\delta^6\\&+(0{\cdot}1-\tfrac{1}{3}P+\tfrac{1}{7}Q-0{\cdot}25PQ+\tfrac{2}{7}P^2+\tfrac{1}{18}Q^2)75L^4\delta^2\\&+(\tfrac{1}{18}-0{\cdot}2P+\tfrac{1}{11}Q-\tfrac{1}{6}PQ+\tfrac{2}{11}P^2+\tfrac{1}{26}Q^2)1875L^2\delta^4,\end{aligned}$$

$$\psi_5=(0{\cdot}5-0{\cdot}5P+\tfrac{1}{6}Q)L^4+50(0{\cdot}1-\tfrac{1}{6}P+\tfrac{1}{14})L^2\delta^2+625(\tfrac{1}{18}-0{\cdot}1P+\tfrac{1}{22}Q)\delta^4.$$

Taking the Laplace transformation of equation (11) one has

$$s^2\bar{B}_0(t)-sB_0(0)-\dot{B}_0(0)+w^2\bar{B}_0(t)=C^2\bar{K}(t). \tag{12}$$

In the case of a sudden heating of the plate $W(x, y, 0)=0$, $\dot{W}(x, y, 0)=0$ and hence $B_0(0)=\dot{B}_0(O)=0$. Thus, from equation (12), one gets

$$\bar{B}_0(t)=C^2\bar{K}(t)/(s^2+w^2). \tag{13}$$

Performing the inverse transformation one gets

$$\begin{aligned}B_0(t)=&\frac{C^2}{w}\int_0^t K(t-t')\sin wt'\,\mathrm{d}t'=\frac{C^2}{w^2}\Bigg[(1-\cos wt)\\&-\frac{96\beta^2}{\pi^4}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{j^4\beta^2+w^2}\left\{\frac{w}{j^2\beta}\sin wt-\frac{w^2}{j^4\beta^2}\cos wt+\frac{w^2}{j^4\beta^2}\mathrm{e}^{-j^2\beta t}\right\}\Bigg].\end{aligned} \tag{14}$$

Thus W is completely determined.

Now, from equation (14), one has the maximum displacement as

$$(W)_{\max}=|B_0(t)|_{\max}=2C^2/w^2. \tag{15}$$

Neglecting the inertia term in equation (11) one gets the maximum displacement for the quasi-static part as

$$(W_s)_{\max}=C^2/w^2. \tag{16}$$

This value of $(W_s)_{\max}$ is in good agreement with that given in reference [4].

Unfortunately there appears to be a printing mistake in reference [4] in the solution for the dynamic displacement: that solution should read correctly as

$$\begin{aligned}B_{mn}(t)=&\frac{16q_1(1+\nu)\alpha_t}{2\lambda\alpha_n\beta_n(\alpha_n^2+\beta_n^2)ab}\Bigg\{\frac{12\beta}{\pi^2 w}\sin wt-\cos wt\\&-\frac{96\beta^2}{\pi^4}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{j^4\beta^2+w^2}\left[\frac{w}{j^2\beta}\sin wt-\frac{w^2}{j^4\beta^2}\cos wt-\mathrm{e}^{-j^2\beta t}\right\}\Bigg].\end{aligned} \tag{17}$$

Thus the maximum value of $B_{mn}(t)$ in reference [4], which is also the maximum dynamic displacement, is given by

$$(W_d)_{\max}=|B_{mn}(t)|_{\max}=\frac{16q_1(1+\nu)\alpha_t}{2\lambda\alpha_n\beta_n(\alpha_n^2+\beta_n^2)ab}\simeq\frac{C^2}{w^2}\text{(in the present notation).}$$

3. NUMERICAL RESULTS

Table 1 contains the first term coefficient of the mapping function for elastic plates of different shapes.

TABLE 1

Mapping function $Z=f(\xi)\simeq L\xi$

Polygon of side $2a$	L
Equilateral triangle	$1\cdot1352\,a$
Square	$1\cdot08\,a$
Pentagon	$1\cdot0526\,a$
Hexagon	$1\cdot0376\,a$
Heptagon	$1\cdot0279\,a$
Octagon	$1\cdot0219\,a$
Circle of radius a	a

In the present study $W_{\max}$ is the total displacement and in Table 2 it is compared with the quasi-steady total displacement [4] $W_{\max}$ (quasi-steady) = $W_{d\max}$ (quasi-steady) + $W_{s\max}$ (quasi-steady). For $B=0$ the two separate solutions yield the same result for any shape of the plate.

The numerical results for different plates have been plotted to show the variation of $|(W_{\max}/K'a^2)\times10^2|$ *vs.* B in Figure 1. A graph showing variation of $|(W_{\max}/K'a^2)\times10^2|$ *vs.* the number of sides of the regular polygons for $B=0$ is shown in Figure 2.

TABLE 2

Comparison of results for total displacement (all results in units of $K'a^2$)

	Equilateral triangular plate		Square plate		Circular plate	
B	$W_{\max}$ (present solution)	$W_{\max}$ (quasi-steady solution)	$W_{\max}$ (present solution)	$W_{\max}$ (quasi-steady solution)	$W_{\max}$ (present solution)	$W_{\max}$ (quasi-steady solution)
0	0·3282	0·3282	0·2927	0·2927	0·2548	0·2548
0·5	0·2376	0·2134	0·2150	0·1932	0·1843	0·1656
1·0	0·1846	0·1706	0·1671	0·1545	0·1433	0·1325
1·5	0·1735	0·1668	0·1570	0·1510	0·1318	0·1296
2·0	0·1697	0·1658	0·1536	0·1501	0·1289	0·1287
2·5	0·1680	0·1655	0·1521	0·1499	0·1276	0·1285
3·0	0·1672	0·1653	0·1514	0·1497	0·1270	0·1283
3·5	0·1668	0·1652	0·1510	0·1496	0·1267	0·1282
4·0	0·1666	0·1651	0·1508	0·1495	0·1265	0·1281

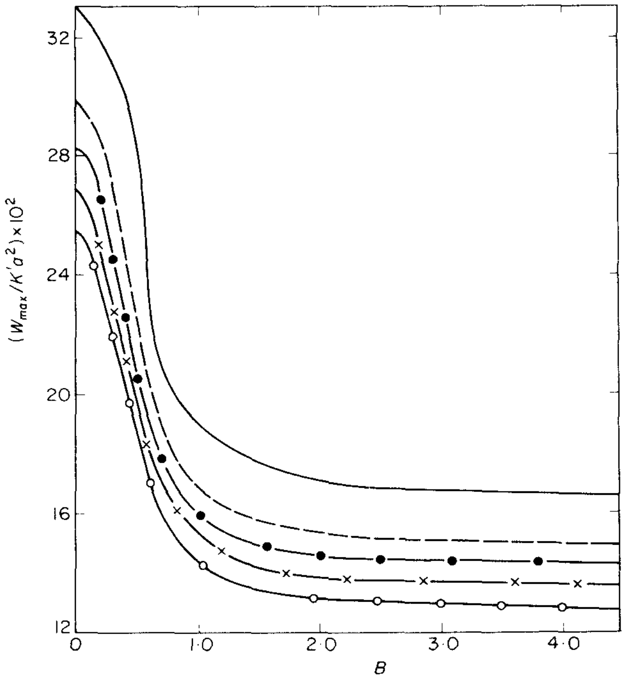

Figure 1. $(W_{max}/K'a^2)$ *vs.* B. ——, Equilateral triangle; – – –, square; —●—, pentagon; —×—, heptagon; —○—, circle.

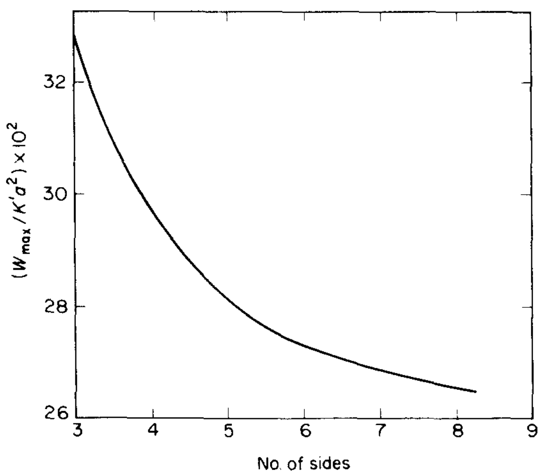

Figure 2. $(W_{max}/K'a^2)$ *vs.* number of sides of regular polygons, for $B = 0$.

For all the calculations $K' = q_1(1+\nu)\alpha_t$. Only one term of the mapping function has been taken, for simplicity of calculation, and that has yielded sufficiently accurate results.

4. DISCUSSION

From Table 2 it is evident that the role of inertia is much greater than that given in reference [4], and this is to be expected when thermal shock loading is applied to any elastic plate. It is also interesting to note from Table 2 that the maximum displacement obtained from the quasi-steady solution rapidly falls, whereas the maximum displacement as obtained here only steadily decreases. This is due to the fact that in the quasi-steady solution the true role of the inertia has been affected.

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APPENDIX: NOTATION

W	normal displacement
W_s	normal displacement for quasi-static part
W_d	normal displacement for dynamic part
D	flexural rigidity of the plate, $=Eh^3/12(1-\nu^2)$
h	plate thickness
ρ	density of the plate material
α_t	coefficient of linear expansion of solid
E	Young's modulus
ν	Poisson's ratio
β	$\kappa\eta^2/h^2$
κ	conductivity coefficient
η	$\sqrt{\rho h/D}$
a	dimension of the plate
K'	$q_1(1+\nu)\alpha_t/\lambda$
λ	coefficient of internal heat conduction
q_1	$\lambda\ \partial T/\partial z$ at $z=h/2$
B	$h/2a\sqrt{\kappa\eta}$