R E P R I N T S

Indian Journal of Technology
Vol. 21, November 1983, pp. 453-455

Deflections of Orthotropic Polygonal Plates under Concentrated Load at the Centre

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Received 21 *October* 1982; *accepted* 25 *June* 1983

Deflections of clamped edged orthotropic polygonal plates under the action of concentrated load at the centre have been investigated following complex variables theory. Expressions for the deflections of different plates are obtained and numerical results are presented.

The dynamic behaviour of different isotropic plates has been investigated by Laura[1] who has applied conformal mapping technique to get the solutions. These solutions are approximate as these have been obtained by error minimising method. The deflections of such irregular plates under uniform load have been studied by Mansfield[2]. But no paper has apparently been devoted to investigation of orthotropic plates of irregular shape under concentrated loading.

The object of this paper is to study the deflections of polygonal plates of orthotropic materials, such as reinforced concrete, under the action of a concentrated load at the centre. Conformal mapping technique has been applied to get the desired solutions of such polygonal plates. The solutions thus obtained are sufficiently accurate for practical purpose and these are given in tabular form both for weaker and stronger orthotropy.

The results of square and circular isotropic plates deduced from this study are found to be in excellent agreement with the results obtained by Timoshenko and Woinowsky-Krieger[3].

Differential Equations and Method of Solution

Consider a clamped edged orthotropic plate under the action of concentrated load P at the centre. Following Timoshenko and Woinowsky-Krieger[3], we can write the differential equation for the deflection of an orthotropic plate except at the load point in rectangular co-ordinates as

$$D_x\frac{\partial^4 w}{\partial x^4}+2H\frac{\partial^4 w}{\partial x^2\partial y^2}+D_y\frac{\partial^4 w}{\partial y^4}=0 \quad \ldots (1)$$

where w is the deflection in the z-direction,

$$D_x=\frac{E_x'h^3}{12},\ D_y=\frac{E_y'h^3}{12},$$

$$H=\frac{E''h^3}{12}+\frac{Gh^3}{6},$$

h = thickness of the plate and E_x', E_y', E'' and G are elastic constants of the material.

For an orthotropic material such as reinforced concrete

$H=(D_xD_y)^{1/2}$.

Eq. (1) reduces to

$$\frac{\partial^4 w}{\partial z_1^2\partial \bar{z}_1^2}=0 \quad \ldots (2)$$

with the substitution

$z_1=x+\rho y$.

where ρ is the root of equation $D_y\rho^4+2H\rho^2+D_x=0$.

Clearly $\rho=i\omega$ where $\omega^2=(D_x/D_y)^{1/2}$.

The solution of Eq. (2) will have two parts

$$w=w_0+w_1 \quad \ldots (3)$$

where w_0 is the particular solution of Eq. (2) at the load point and w_1 is the general solution of Eq. (2).

In case of single load P at the centre, the solution w_0 will be of the form

$$w_0=\frac{P}{16\pi D_x}Z_1\bar{Z}_1\log\frac{Z_1\bar{Z}_1}{L^2} \quad \ldots (4)$$

where L is a dimension of the plate.

The general solution of Eq. (2) will be of the form

$$w_1=Z_1\varphi(Z_1)+Z_1\varphi(\bar{Z}_1)+\chi(Z_1)+\chi(\bar{Z}_1) \quad \ldots (5)$$

Eqs (4) and (5) together gives the complete solution of Eq. (2). Now the expression for Z_1 in terms of $Z=x+iy$ is

$$Z_1=\lambda_1 Z+\lambda_2\bar{Z} \quad \ldots (6)$$

where $\lambda_1=\frac{1+\omega}{2}$ and $\lambda_2=\frac{1-\omega}{2}$

Let $Z=L(\sigma+\delta\sigma^5)$ be the mapping function which maps the domain under consideration on to a unit circle where $\sigma=e^{i\theta}$.

Let us assume

$$\varphi(Z_1)=\sum_{n=1}^{\infty} a_n\sigma^n,\ \varphi(\overline{Z}_1)=\sum_{n=1}^{\infty}\bar{a}_n\bar{\sigma}^n,$$

$$\chi(Z_1)=\sum_{n=0}^{\infty} b_n\sigma^n,\ \chi(\overline{Z}_1)=\sum_{n=0}^{\infty}\overline{b_n}\bar{\sigma}^n$$

and

$$\chi'(\overline{Z}_1)=\sum_{n=0}^{\infty}\overline{E}_n\bar{\sigma}^n$$

For the clamped edge we have on the boundary

$$w=0 \quad \ldots (7)$$

$$\text{and}\quad \frac{\partial w}{\partial \overline{Z}_1}=0 \quad \ldots (8)$$

Now putting the values of $\varphi(Z_1)$, $\varphi'(\overline{Z}_1)=\frac{d}{d\overline{Z}_1}\{\varphi(\overline{Z})\}$, $\chi(Z_1)$, Z_1 we have from Eqs (3), (4), (5) and (8)

$$\sum_{n=1}^{\infty} a_n\sigma^n+\frac{(\lambda_1L\sigma+\lambda_1L\delta\sigma^5+\lambda_2L\bar{\sigma}+\lambda_2L\delta\sigma^5)}{L(1+5\delta\bar{\sigma}^4)\lambda_1}$$

$$\sum_{n=1}^{\infty} n\bar{a}_n\bar{\sigma}^{n-1}+\sum_{n=0}^{\infty}\overline{E}_n\bar{\sigma}^n$$

$$=-\frac{PL}{16\pi D_x}[(\lambda_1\sigma+\lambda_1\delta\sigma^5+\lambda_2\bar{\sigma}$$
$$+\lambda_2\delta\bar{\sigma}^5)\log\{(\lambda_1^2+\lambda_2^2).(1+\delta\sigma^4+\delta\bar{\sigma}^4+\delta^2)+$$
$$\lambda_1\lambda_2(\sigma^2+\bar{\sigma}^2+2\delta\sigma^6+2\delta\bar{\sigma}^6+\delta^2\sigma^{10}$$
$$+\delta^2\bar{\sigma}^{10})\}+\lambda_1\sigma+\lambda_1\delta\sigma^5+\lambda_2\bar{\sigma}+\lambda_2\delta\bar{\sigma}^5] \quad \ldots (9)$$

Equating the coefficients of different powers of σ on both sides and since the coefficients of different powers of σ on right hand side are purely real, we have

$$a_1=\bar{a}_1=-\frac{PL}{32\pi D_x}\left[\lambda_1\{1+A_1+\log(\lambda_1^2+\lambda_2^2)\}\right.$$
$$-\lambda_1\delta\frac{4Q_1}{1-5\delta^2}+\lambda_2\frac{B_1-5\delta F_1}{1-5\delta^2}$$
$$\left.+\lambda_2\delta\frac{F_1-5\delta K_1}{1-5\delta^2}\right] \quad \ldots (10)$$

$$a_3=\bar{a}_3=-\frac{PL}{16\pi D_x}\left[\lambda_1\frac{B_1(1+\delta)}{1+3\delta}+\lambda_2\frac{Q_1+\delta I_1}{1+3\delta}\right] \quad \ldots (11)$$

$$a_5=\bar{a}_5=-\frac{PL}{16\pi D_x}[\lambda_1\delta\{1+A_1+\log(\lambda_1^2+\lambda_2^2)\}$$
$$+\lambda_1Q_1+\lambda_2(F_1+\delta K_1)]-\delta a_1 \quad \ldots (12)$$

$a_2=\bar{a}_2=a_4=\bar{a}_4=0$ and so on.

Here

$$A_1=-R^2\left(1+4\delta^2+\delta^4+\frac{\delta^4}{2R^2}\right)$$
$$B_1=R(1-\delta-3\delta^2)$$
$$Q_1=\left(\delta-\delta^3-\frac{R^2}{2}-2R\delta^2-2R^2\delta^3\right) \quad \ldots (13)$$
$$F_1=R(\delta-3\delta^3)$$
$$I_1=-\left(\frac{\delta^2}{2}+2R^2\delta+R^2\delta^2\right)$$
$$K_1=-R(\delta^2+\delta^4)$$
$$\text{and } R=\frac{\lambda_1\lambda_2}{\lambda_1^2+\lambda_2^2}$$

Thus $\varphi(Z_1)$ and $\varphi(\overline{Z}_1)$ are completely known.

Now from Eqs (3), (4), (5) and (7), we have

$$(\lambda_1\overline{Z}+\lambda_2Z)\sum_{n=1}^{\infty}a_n\sigma^n+(\lambda_1Z+\lambda_2\overline{Z})\sum_{n=1}^{\infty}\bar{a}_n\bar{\sigma}^n$$
$$+\sum_{n=0}^{\infty}b_n\sigma^n+\sum_{n=0}^{\infty}\overline{b_n}\bar{\sigma}^n$$
$$=-\frac{P}{16\pi D_x}(\lambda_1Z+\lambda_2\overline{Z})(\lambda_1\overline{Z}+\lambda_2Z)$$
$$\log\frac{(\lambda_1Z+\lambda_2\overline{Z})(\lambda_1\overline{Z}+\lambda_2Z)}{L^2} \quad \ldots (14)$$

From Eq. (14), after putting the values of Z and $\overline{Z}$ and then equating the coefficients of different powers of σ on both sides, we have

$$b_0+\overline{b_0}=-\frac{PL^2}{16\pi D_x}[(1+\delta^2)(\lambda_1^2+\lambda_2^2)\{A_1+\log(\lambda_1^2+\lambda_2^2)$$
$$+2\lambda_1\lambda_2B_1+2\delta(\lambda_1^2+\lambda_2^2)Q_1+2\delta^2\lambda_1\lambda_2K_1$$
$$+4\delta\lambda_1\lambda_2F_1]-2\lambda_1L(a_1+\delta a_5) \quad \ldots (15)$$

$$b_2=-\frac{PL^2}{16\pi D_x}[(1+\delta)^2(\lambda_1^2+\lambda_2^2)B_1$$
$$+\lambda_1\lambda_2\{A_1+\log(\lambda_1^2+\lambda_2^2)$$
$$+(1+2\delta)Q_1+2\delta I_1\}]$$
$$-\lambda_1L\{(1+\delta)a_3+\delta a_7\} \quad \ldots (16)$$

$b_1=b_3=0$ and so on.

Thus $\chi(Z_1)$ and $\chi(Z_1)$ are known where A_1, B_1, $C_1\ldots$, etc. are given by relations (13).

Table 1—Mapping Function $Z=L(\sigma+\delta\sigma^5)$

Polygons of side a	Mapping function coefficients		Orthotropic plates: Maximum deflection $\left(\frac{W}{h}\right)_{max;orth}=\alpha\frac{Pa^2}{D_x h}$		Isotropic plates: Maximum deflection $\left(\frac{W}{h}\right)_{max;iso}=\alpha\frac{Pa^2}{Dh}$	Known results from
	L	δ	Weak orthotropy α	Strong orthotropy α	α	α
Square	0.54a	−0.102	0.0069	0.0118	0.005615	0.0056
Pentagon	0.5265a	—	0.00656	0.01128	—	—
Hexagon	0.519a	—	0.006377	0.01096	—	—
Heptagon	0.514a	—	0.006255	0.01075	—	—
Octagon	0.5109a	—	0.006179	0.0106	—	—
Circle of radius a	a	0	0.0237	0.041	0.02	0.02

Hence the complete solution of Eq. (2) is known. The maximum deflection at the centre is given by

$$(w)_{max}=b_0+\bar{b}_0 \quad \dots (17)$$

From Eqs (17) and (15) the maximum deflection for isotropic plate can be deduced putting $D_x=D$, $\lambda_1=1$ and $\lambda_2=0$. Thus

$$(w)_{max;iso}=\frac{PL^2}{16\pi D}\left[(1+\delta^2)+2\delta Q_1' - \frac{4Q_1'\delta}{1-5\delta^2}(1-\delta^2)-2\delta Q_1'\right] \quad \dots (18)$$

where $Q_1'=\delta(1-\delta^2)$

Numerical Results

The maximum deflections of different polygonal plates both for weak and strong orthotropy have been calculated. The maximum deflections for isotropic square and circular plates have also been calculated and given in Table 1. For weak orthotropy

$$\lambda_1=1.095,\ \lambda_2=-0.095\left(\frac{E_y'}{E_x'}=\frac{1}{2}\right)$$

For strong orthotropy,

$$\lambda_1=1.557 \text{ and } \lambda_2=-0.557\left(\frac{E_y'}{E_x'}=\frac{1}{20}\right)$$

Conclusion

For calculating the maximum deflection of orthotropic plates of more than 4 sides, only one term of mapping function has been considered. In case of square and circular isotropic plates the results of this study are in excellent agreement with the known results.

The deflections of orthotropic plates are greater than those of isotropic plates. In case of orthotropic plates the deflections decrease as the number of sides of the polygon increases. The deflection is maximum for a circular plate.

Acknowledgement

Thanks are due to Dr B Banerjee, Department of Mathematics, Jalpaiguri Government Engineering College, for guidance.

References

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