NONLINEAR VIBRATIONS OF STRUCTURES

INCLUDING THERMAL LOADING

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(Resubmission of Thesis revised in the light of the Examiner's Report)

CERTIFICATE

This is to certify that the research carried out by Shri Utpal Kumar Mandal for the preparation of this thesis entitled, "Nonlinear Vibrations of Structures Including Thermal Loading" has been supervised by me. The original thesis was submitted to the University of North Bengal in partial fulfillment of the requirements for the Ph. D. (Civil Engineering) degree, and was not been submitted for a degree elsewhere.

However, in the light of the valuable comments and observations made by one of the honourable Examiners of the original thesis, Shri U. K. Mandal has completed revision of the thesis and he is now submitting the revised thesis to the University of North Bengal with reference to the Registrar's letter having Ref. No. 168/Ph. D./C. Engg./1646(R) 04(2) dated 11th August 2004.

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ABSTRACT

This thesis studies the nonlinear dynamic (free vibration) behavior of thin plate and shell structures of different geometrical shapes with or without thermal loading. In many situations, geometric characteristics of structures combined with oppressive operational conditions induce large deflections i. e. deflections that are of the same order as the plate or shell thickness and small compared to the in-plane dimensions of the structures; even within elastic limit of the structural material and thus nonlinear effects come into play. It is worth mentioning that any effort to restrict the deflections appropriate to the linear theory results in uneconomic structures. Further, stress-strain characteristics of structural material with or without thermal loading also induce non-linearity in the behavior of structures. Structural components undergoing large deflections (nonlinear deformations) may exhibit strain-hardening or strain-softening behavior. The advantage of extra strength of strain-hardening as well as desired level of natural frequency of vibrations may be achieved by properly proportioning and designing geometric elements of the structures and its end conditions. These facts warrant to investigate nonlinear behavior of structures or structural components there of in order to achieve efficient and economic utilization of material. For example, the knowledge of nonlinear dynamic behavior of structures will help to attain desired level of frequency of vibrations by properly proportioning and designing geometric elements.

The structures are assumed to be made of homogeneous, isotropic and elastic materials. In some problems material properties of materials are assumed constant and

temperature-independent. In some other problems, materials having temperature-dependent material properties are considered.

Further, continuum model has been used to derive the basic governing differential equations as and whatever required, in the sense of von Karman classical large deflection theory or on the basis of Berger approximations i.e. the energy contribution due to second strain invariant of the middle surface is neglected in the total potential energy expression. The basic governing differential equations for the resulting system has been solved by Galerkin error minimizing technique incorporating prescribed boundary conditions to obtain relative nonlinear frequency of vibrations.

Numerical results to study the effects of different parameters, as occurred in the analysis, on nonlinear dynamic behavior of such structures have been presented. The present study reveals some interesting nonlinear dynamic behavior which may prove useful to the designers. A few of the important observations are mentioned in the following paragraphs.

It is observed that thin plates viz. circular, triangular and parabolic plates with clamped immovable edges exhibit strain-hardening type of non-linearity valid within the proportional limit of the plate material.

It is interesting to note that a triangular plate with the geometric configuration having aspect ratio around 1.0 and skew angle around 30^0 is more prone to develop dynamic instability with or without thermal loading. Similarly, parabolic plate having aspect ratio around 1.4 is more prone to develop dynamic instability with or without thermal loading.

Thermal loading influences the dynamic behavior of thin plates in various ways. Thin plates of various shapes with clamped immovable edges viz. circular, triangular and parabolic plates exhibit similar behavior under thermal loading characterized by constant surface temperatures T_u and T_b, measured from stress free temperature, for upper surface and lower surface respectively. For example, the stiffness and hence the frequency of vibration, both linear as well as nonlinear, decreases with increase of the average surface temperature $\left(\frac{T_u + T_b}{2}\right)$ and after certain stage linear frequency becomes zero i.e. thermal instability of the structure occurs based on linear theory; but nonlinear frequency is still non-zero which makes $\frac{\omega_{NL}}{\omega_L}$ infinity. So, thermal instability is delayed due to additional stiffness associated with large deflection as per nonlinear theory.

Thermal loading does influence the temperature-dependent material properties in various ways; but the temperature dependency of material properties does not alter the dynamic characteristics of thin isotropic circular plates with clamped immovable edges significantly within the range of temperature causing thermal instability. However, the temperature dependency of material properties reduces the nonlinear frequency of vibrations and such reduction is not appreciable compared to the effects of large deflection and direct heating.

Axisymmetric thin shallow spherical shells with clamped immovable edge exhibit both strain-hardening and strain-softening types of behavior. A very shallow thin spherical shell tends to behave like a circular plate and exhibit strain-hardening type of nonlinearity. With increase of nondimensional geometric parameter $\left(\frac{R^2}{2R_0 H_0}\right)$ i.e. ratio of the rise of the shell to the shell thickness at the center of the shell, the behavior of shallow spherical shell changes from strain-hardening type to strain-softening type and relative nonlinear frequency decreases to yield the lowest peak and this peak corresponds to the geometrical configuration of thin shallow spherical shell which is most liable to undergo snap buckling. If the shell geometry matches the transition state configuration nonlinear frequency becomes equal to linear frequency. With further increase of $\frac{R^2}{2R_0H_0}$ the resistance of shallow spherical shell to snap buckling and the relative nonlinear frequency increases and finally it approaches unity. Nonlinear frequency approaches linear frequency as the thickness parameter(τ) decreases algebraically i.e. the thickness of the shell increases towards the edges.

It is worth mentioning that the overall dynamic behavior obtained for thin shallow spherical shell with clamped immovable edges based on Berger's approximation matches approximately with those obtained in the present study based on classical large deflection theory in the von Karman sense.

Finally, it can be mentioned that this study reveals some interesting dynamic behavior of thin isotropic plate and shell structures which may be useful to the designers, For example, the knowledge of dynamic behavior of such structures will help to achieve desired level of frequency of vibrations by properly proportioning the geometric elements. This will also help to utilize the advantage of extra strength due to strain-hardening and to reduce the failure probability by taking proper precaution against structural instability.

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CHAPTER I

INTRODUCTION

1.1 Scope

In many fields of engineering structures viz. high speed spacecrafts, nuclear power plants, chemical plants, offshore and ship structures, storage structures , building structures etc., plates and shells find wide applications as integral structural components. Such structural components are likely to be subjected to different kinds of static loads or excitations such as mechanical, seismic, blast, hydrodynamic, aerodynamic etc. with or without thermal loading. Engineers and Scientists all over the world are exerting relentless effort to design and construct economic and efficient structures with very low failure probability. In view of this, for proper modeling, analyzing and designing structural engineers should be well acquainted with the dynamic behavior of such structural elements with different boundary and loading conditions.

The linear theory of analyzing these structural components viz. plates and shells is based on the assumption that the deflections are small in comparison with the thickness of the elements. This assumption follows from the hypothesis that the components of strain ε_{11}, ε_{22} and ε_{12} of the deflected middle surface have negligible magnitude. Based on linear theory, enormous volume of works on static and dynamic analysis of thin plate and shell structures with different boundary and loading conditions have been carried out by many researchers.

In many situations, geometric characteristics of structures combined with oppressive operational conditions induce large deflections i. e. deflections that are of the same order as the plate or shell thickness and small compared to the in-plane dimensions of the structures; even within the elastic limit (proportional limit) of the structural material and thus nonlinear effects come into play. It is worth mentioning that any effort to restrict the deflections appropriate to the linear theory results in uneconomic structures. Thus, when the deflections are no longer small in comparison with thickness, but small compared to the in-plane dimensions, the middle plane strains must be considered in deriving the differential equations of thin plate and shell structures. It may be mentioned that structural components undergoing large deflections (nonlinear deformations) may exhibit strain-hardening or strain-softening behavior. The advantage of extra strength due to strain-hardening as well as desired level of natural frequency of vibrations may be achieved by properly proportioning and designing geometric elements of the structures and its end conditions. These facts warrant investigation of the effects of nonlinear behavior of structures or structural components in order to achieve efficient and economic utilization of material.

Further, the materials properties viz. thermal conductivity, modulus of elasticity, coefficient of thermal deformations, Poisson's ratio etc. do change at elevated temperature. In the presence of thermal gradient the material properties of homogeneous materials become functions of space variables. Hence, under such situation, determination of dynamic characteristics of continuous elastic system must be based on nonhomogeneous elastic theory.

Besides, the complexities of such problems are manifold due to (i) stress-strain behavior of structural material viz. nonlinear elastic, elasto-plastic etc., (ii) type of plate and shell structures viz. isotropic, anisotropic, composite etc. and (iii) irregular geometric configurations etc.

1.2 Previous works

This field of structural mechanics has drawn attention of many engineers, scientists and researchers. As a result a sizable amount of research works on nonlinear(large deflection) static and dynamic analysis of thin plates and shell structures with different boundary and loading conditions have been published.

The coupled nonlinear partial differential equations for analysis of isotropic plates undergoing moderately large deflections were initially established by von Karman[1910]. Marguerre, K.(1938) developed von Karman type nonlinear differential equations for initially deflected plates or generally shallow shells by use of energy method. These equations are coupled in nature and hence, are difficult to deal with particularly for plates and shells of complex geometric shapes. Approximate analytical methods viz. double Fourier series, Ritz, Galerkin, perturbation etc. and numerical methods viz. finite difference, finite element, boundary element, etc. are adopted for the analysis of different plate and shell structures. To analyze and study large deflection behavior of isotropic, anisotropic and laminated composite plate and shell structures, von Karman's equations have been employed by many outstanding research workers such as Levy, S.(1942a, 1942b, 1947), Sundara Raja Iyengar, K. T. and Naqvi, M. M.(1966), Kennedy, J. B. and Simon, N. G.(1967), Schmidt, R.(1968), Chia, C. Y.(1972), Neyogi, A. K.(1973), Chia, C. Y. and Prabhakara, M. K.(1975, 1976), Prabhakara, M. K. and Chia, C. Y.(1973, 1974, 1975, 1977b), Zaghloul, S. A. and Kennedy, J. B.(1975a, 1975b), Chandra, R.(1976), Little, G. H. (1987a, 1987b, 1989), etc.

Many investigators studied successfully large amplitude vibrations of plate and shell structures based on von Karman type large deflection theory. Chu, H. N. and Hermann, G.(1956) studied influence of large amplitudes on free flexural vibrations of rectangular elastic plates. Yamaki, N.(1961) investigated large amplitude vibrations of rectangular plates due to harmonic force for various edge conditions. Nowinski, J. L.(1963a, 1963b) studied large amplitude free vibrations of orthotropic circular plates and orthotropic cylindrical shells. Nowinski, J. L.(1964) used Marguerre's equations to investigate large amplitude oscillations of oblique panels with initial curvature. Bauer, H. F.(1968) analyzed for nonlinear response of rectangular and circular plates due to pulse excitations for various edge conditions. Kung, G. C. and Pao, Y. H.(1972) presented nonlinear response of a circular plate subjected to sinusoidally varying axisymmetric load. Vendhan, C. P., and Dhoopar, B. L. (1973) performed analysis for large amplitude vibrations of orthotropic right angled and isosceles triangular plates with clamped edges. Ramachandran, J. (1973b) furnished solution for nonlinear vibrations of a simply supported isotropic rectangular plate carrying concentrated mass. Ramachandran, J.(1976) also studied large amplitude vibrations of a shallow spherical shell with concentrated mass. Satyamoorthy, M. and Pandalai, K. A. V.(1970, 1972) investigated nonlinear free flexural vibrations of orthotropic rectangular and skew plates.

Satyamoorthy, M.(1978) studied effects of transverse shear and rotatory inertia on large amplitude vibration of plates. Prathap, G. and Varadhan, T. K.(1978, 1979) carried out large amplitude vibrations of rectangular plates and anisotropic skew plates. Karmakar, B.(1979) studied large amplitude vibrations of clamped isotropic elliptic plates carrying a concentrated mass. Datta, S. and Banerjee, B.(1979b) presented large amplitude vibration of thin elastic plates by the method of conformal transformation (Laura, P. A. et al., 1967, Laura, P. A. and Shahady, P. A., 1969). The von Karman equations have been further developed and extended to investigate nonlinear dynamic behavior of composite laminated plate and shell structures by Chandra, R. and Basava Raju, B.(1975), Chia, C. Y. and Prabhakara, M. K.(1978), Prahbakara, M. K. and Chia, C. Y.(1977a) etc. Chia, C. Y.(1980) has published an excellent book entitled "Nonlinear Analysis of Plates" in which problems on large deflection and nonlinear vibration of isotropic, anisotropic and laminated plates have been analyzed in addition to other problems with extensive bibliography of other related works. Biswas, P. and Kapoor, P.(1986a) investigated large amplitude free vibrations of axisymmetric orthotropic circular plates. Biswas, P.(1991) presented analytical formulation for large amplitude free vibration of a thin shallow spherical shell of variable thickness. Librescu, L. and Lin, W.(1997b) presented von Karman type large deflection vibration analysis for flat and curved plates resting on nonlinear elastic foundation considering transverve shear deformations. Banerjee, M. M. et al.(1995), Banerjee, M. M.(1997) and Chakraborty and Banerjee, M. M. (1999a, 1999b) have studied nonlinear vibration of plate and shell structures of various shapes in the von Karman sense using the method of constant deflection contours, introduced by Mazumdar, J.(1970, 1971) and Jones, R. and Mazumdar, J.(1974).

In case of thermal problems, Gossard, M. L. et al.(1952) studied large deflection behavior of initially imperfect rectangular plates subjected to tent like temperature distributions by Raleigh-Ritz and Galerkin method. They incorporated thermal stresses and initial imperfections into von Karman classical large deflection theory. Similarly, Forray, M. and Newman, M.(1962) investigated large deflection behavior of isotropic rectangular plates heated symmetrically about two orthogonal centerlines and Mahayni, M. A.(1966) studied large deflection behavior of shallow cylindrical shells under axially parabolic temperature field. von Karman's equations, extended to thermal loading in the static and dynamic cases, have been quoted by Nowacki, W.(1962) Jones, D. J.(1965), Parkas, H.(1976), Nowinski, J. L.(1978) and Chia, C. Y.(1980) in their respective books. Bailey, C. D. (1973) investigated the free vibrations of thermally-stressed plates with various boundary conditions. Prabhu, M. S. S. and Druvasula, S.(1975, 1976) studied post-buckling behavior of heated isotropic skew plates with restrained edges. Biswas, P. (1981b) derived nonlinear governing differential equations for the large deflection of heated orthotropic plates in the von Karman sense and applied the same to a rectangular plate subjected to stationary temperature distribution. Mansfield, E. H.(1982) has successfully employed von Karman's equations extended to thermal loading for investigating large amplitude free vibrations of heated elliptic plates. These equations have further been extended to skew plates at large amplitude vibrations including thermal loading by Dalamangas, A.(1984). Biswas, P. and Kapoor, P.(1984a) carried out nonlinear free vibrations and buckling analysis of isotropic circular plates with clamped immovable edges subjected to temperature distribution, not mentioned specifically, assumed to vary radially. Biswas, P. and Kapoor, P.(1984b) performed nonlinear free vibration analysis of orthotropic circular plates with clamped movable and immovable edges subjected to temperature field linearly varying in the radial direction. Biswas, P. and Kapoor, P.(1986b) also investigated nonlinear free vibration of polygonal plates with clamped immovable edges at elevated temperature with the help of complex variable theory and conformal mapping technique. Raju, K. K. and Rao, G. V.(1989) analyzed thermal postbuckling behavior of thin simply supported orthotropic square plates by Raleigh-Ritz procedure incorporating nonlinear strain displacement relation in the von Karman sense. Huang, N. N. and Tauchert, T. R.(1988) utilized total potential energy to attain equilibrium configuration to study large deformation behavior of antisymmetric angle-ply laminates subjected to nonnuiform thermal loading using conjugate direction method with incremental thermal loading. Huang, N. N. and Tauchert, T. R.(1991) studied large deflection of laminated cylindrical and doubly-curved panels under thermal loading by finite element method based on total potential energy. Birman, V.(1990) studied dynamic behavior of reinforced composite cylinders subjected to thermal loading using Donnell shell theory incorporating geometric nonlinearities in the sense of von Karman. Birman, V. and Bert, C. W.(1993) studied post buckling behavior of composite plates and shells subjected to elevated temperature. Singh, G. et al.(1993) performed analysis for thermal post-buckling behavior of simply supported rectangular antisymmetric cross-ply composite plates with immovable edges by Raleigh-Ritz method. Librescu, L. and Souza, M. A..(1993) and Libresco et al.(1995) developed a von Karman type of large-deflection theory for plates made of transversely isotropic materials considering the effects of transverse shear deformations under combined thermal and in-plane edge loads to study the effects of shear deformation and in-plane edge boundary conditions on post buckling behavior of geometrically imperfect flat and curved panels. Similarly, Librescu, L. et. al.(1996a, 1996b) and Librescu, L. and Lin, W.(1997a) investigated dynamic behavior of geometrically imperfect panels subjected to thermal and mechanical loads. Shen, H. S. et. al.(1996), Shen, H. S. and Williams, F. W.(1997a, 1997b) presented large deflection postbuckling analysis of heated composite laminated plate resting on elastic foundation considering initial geometrical imperfection. Shen, H. S. (2000) studied large deflection behavior of a simply supported shear deformable composite laminated plate subjected to a uniform lateral pressure under thermal loading and resting on two parameter elastic foundation based on higher order shear deformation plate theory (Reddy, J. N., 1997).

As already mentioned, in using the von Karman equations, being in the coupled form, it has always been a very difficult task for investigators to obtain even an approximate solution. In an attempt to ease such problems Berger, H. M.(1955) proposed a pair of quasi-linear partial differential equations for analyzing the large deflection of isotropic plates. These equations are in the decoupled form and have obvious advantage for the solution of problems. Berger's method is based on the neglect of the second strain invariant of the middle surface strains in the expression of the total potential energy of the system. An application of the variational principle to this simplified energy expression yields the equations of equilibrium of the plate in the decoupled form. Although no complete explanation of this method has not been set forth, yet the results obtained by him and by other investigators for simply supported and clamped plates are in good agreement with those obtained from more precise analysis.

Nash, W. and Modeer, J.(1959) extended Berger's method to investigate the nonlinear dynamic behavior of elastic plates and moderately large deflection analysis of shallow spherical shells and obtained solutions which are in excellent agreement with those obtained from classical equations. Subsequently, many investigators have used Berger's method to analyze and study large deflection behavior of plate and shell structures viz. Basuli, S. (1961,1964), Sinha, S. N.(1963), Nowinski J. L. and Ismail, I. A.(1964), Bera, R.(1965), Banerjee, B.(1967b), Bannerjee, M. M.(1967, 1976b, 1977), Datta, S.(1975), Kamiya, N.(1976a, 1976b, 1977), Datta S. and Bannerjee, B. (1979a), Sathyamoorthy, M.(1981), Biswas, P.(1981c, 1983b) etc.

Wah, T.(1963a and 1963b) studied large amplitude vibration of circular and rectangular plates. Gajendar, N.(1967) presented large amplitude vibration of elastic plates resting on elastic foundation. Banerje, B.(1967a, 1968) discussed large amplitude free vibration of elliptic plates and isosceles right angled triangular plates. Satyamoorthy, M and Pandalai, K. A. V.(1973a, 1973b, 1974) extended Berger's method to large amplitude free vibration of skew plates. Ramachandran, J.(1973a) studied large amplitude vibration of isotropic rectangular plates with one pair of edges elastically restrained. Banerjee, M. M..(1974, 1976a, 1981) investigated large amplitude vibrations of plates of different geometric shapes. Datta, S.(1976b) presented large amplitude vibrations of irregular plates resting on elastic foundation in a unified way. Choudhury, S. K.(1981) presented large amplitude vibration of equilateral triangular plate using tri-linear coordinates. Choudhury, S. K.(1984) studied large amplitude dynamic behavior of a diagonally line-loaded rectangular plate. Wu, C. I. and Vinson, J. R.(1969a) studied influence of transverse shear deformation and rotatory inertia on large amplitude vibration of transversely isotropic plates. Wu, C. I. and Vinson, J. R.(1969b, 1971) extended Berger mmethod to study nonlinear oscillations of composite plates. Satyamoorthy, M.(1977, 1981a, 1981b) also studied influence of transverse shear deformation and rotatory inertia on large amplitude vibration of different types of plates. Karmakar, B(1979) applied Berger method to analyze large amplitude vibration of sandwich plates.

However, Nowinski, J. L. and Ohnabe, H.. (1972) and Prathap, G.(1979) pointed out certain inaccuracies in the Berger's equations and concluded that this method lead to quite meaningless and absurd results for plates with movable edge conditions. This is due to the fact that the neglect of second strain invariant of the middle surface strains for movable edges of the plates, fails to imply the freedom of rotation in the meridian plane where the meridian stress exists. For movable edges the in-plane displacement u is never zero and thus Berger's equations leads to absurd results. For immovable edges u = 0 on the boundary and Berger's equations give rise to satisfactory results. Vendhan, C. P(1975), Banerjee, M. M.(1977), Banerjee, M. M. and Sarkar, P. K.(1981), Banerjee, M. M. and Das, J. N.(1991) etc. also studied Berger's approximation in this direction.

To overcome difficulties involved in the von Karman and Berger's methods, Datta, S. and Banerjee, B.[1981] suggested a modified energy expression by introduction of a new expression for second strain invariant(e_2) in the potential energy expression of the system and derived a new set of differential equations, also in the decoupled form. They observed accuracy of these equations for circular plates with movable and immovable edge conditions. Their proposition was extended to the large deflection static as well as dynamic analysis of elastic plates, spherical and cylindrical shells by Sinharay, G. C. and Banerjee, B.(1985a, 1985b, 1985c,1986a, 1986b). Following the same approach Banerjee, B.(1983) presented large deflection analysis of circular plates of variable thickness and Ghosh, S. K.(1986) carried out large amplitude vibration of clamped circular plate of variable thickness. Further, Datta, S. and Banerjee, B.(1991) extended the method to large deflection analysis of sandwich plates.

Berger's quasi-linear partial differential equations have been extended to thermal loading by many researchers and scientists to study nonlinear static and dynamic behavior of different plate and shell structures. Gajendar, N(1964) investigated large deformation behavior of a heated rectangular plate having a pair of opposite edges simply supported and other pair clamped. Basuli, S.(1968) presented large deflection analysis of plates under stationary temperature fields as well as under normal pressure. Pal, M. C.(1969, 1970, 1973) studies nonlinear static and dynamic behavior of circular plates under non-uniform temperature field both across the surface and through the thickness. Several problems on large deflection analysis of heated plates, isotropic as well as orthotropic, of various shapes are investigated by Biswas, P.(1974, 1975, 1976a, 1976b, 1976c, 1978a, 1978b, 1979, 1980, 1981a, 1983a, 1983c). Das, S. C.(1977) performed large deflection analysis of annular plates, made of nonhomogeneous material, under variable normal pressure and heating. Jones, R. et al.(1980) carried out extensive investigation to study linear as well as nonlinear behavior of both elastic and visco-elastic plates at elevated temperature and discussed thermal buckling criteria also. Biswas, P. and Kapoor, P.(1984c) studied large amplitude free vibration of simply supported heated right angled isosceles and equilateral triangular plates. Also, Biswas, P. and Kapoor, P.(1986c) presented large amplitude free vibration of heated parabolic elastic plate with clamped immovable edges. Chang, W. P. and Jen, S. C.(1986) investigated large amplitude free vibration of heated orthotropic rectangular plates. Irschik, H.(1986) investigated large deflection behavior of thermally loaded initial curved elastic plates in the light of Berger-type approximation.

Further, Biswas, P. and Kapoor, P.(1985) used modified Berger's method(Datta, S. and Banerjee, B., 1981) to study nonlinear free vibration and buckling of elastic plates at elevated temperature. Similarly, Paliwal, D. W. and Rai, R. N.(1987) studied nonlinear behavior of clamped shallow spherical shell resting on Pasternak foundation under elevated temperature. Banerjee, M. M. et al.(1993) extended modified Berger's method to investigating the large amplitude free vibration of shallow spherical shell subjected to thermal gradient including effects of temperature-dependent modulus of elasticity of material and criticized modified Berger's method severely.

Ray, A. et al.(1993) extended modified Berger method proposed by Datta S. and Banerjee B.(1991) to heated sandwich plates. Similarly, Pal, A. and Bera, R. K.(1996) studied large deflection of heated sandwich plates. Kamiya, N.(1978a,1978b) modified Berger's method to analyze large thermal bending of sandwich plates and shells by incorporating average face strain in place of membrane strains. In this context, it is worth mentioning that Ohnabe, H.(1995) investigated nonlinear vibrations of heated orthotropic sandwich plates and shallow shells based on governing equations derived by Hamilton's principle in combination with Reisner-Hellinger variational principle.

Most of the investigations on nonlinear static and dynamic behavior of plate and shell structures under thermal loading mentioned in the previous paragraphs deals with temperature-independent material properties. In reality, the thermal and mechanical properties of materials viz. thermal conductivity, modulus of elasticity, coefficient of thermal deformations, Poisson's ratio etc. do change at elevated temperature [Hoff, N. (1958), Spinner, S. (1961), Garrick, I. E. (1963)]. Tanigawa, Y. et al.(1996) pointed out that the temperature dependency of elastic coefficients of material play very important role in thermal stress distribution. In presence of temperature gradient the thermal and mechanical properties of materials become functions of space variables. Hence, under such situation, determination of dynamic characteristics of continuous elastic system must be based on nonhomogeneous elastic theory. Many investigations of dynamic behavior of plate and shell structures subjected to thermal loading considering temperature-dependent elastic coefficients of material have been carried out on the basis of linear theory viz. Fauconneau, G. and Margangani, R. D.(1970), Kameswara Rao C. and Satyanarana, B.(1975), Dhotrad M. S. and Ganesan, N.(1978), Ganesan, N.(1978), Massalas, C., et al.(1981), Tomar, J. S. and Tiwari, A. K.(1981), Tomar, A. K. and Gupta, A. K.(1984a, 1984b), Birman, V.(1991a, 1991b), Chandrashekhara, K. and Bhimaraddi, A.(1994) etc. Very few investigations in this direction, based on large deflection theory, have been reported viz. Das, S. C.(1977), Chen, L. W. and Chen, L. Y.(1991), Banerjee, M. M. et al.(1993) etc.

There has been remarkable progress in the analytical methods to predict the nonlinear static and dynamic behavior of plate and shell structures subjected to different types of mechanical loading or excitations with or without thermal loading. Though such methods fail to provide solutions when applied to plate and shell structures with complex geometry and boundary conditions, may be applied efficiently and successfully to deal with a wide class of plate and shell structures. To deal with such plate and shell structures having complex geometry and boundary conditions numerical methods like finite element method, boundary element method etc. have been used by many researchers and scientists. A few of the worth-mentioning recent research works in this field are Kamiya, N. et. al.(1983), Raju, K. K. and Rao, G. V.(1988), Chen, L. W. and Chen, L. Y.(1989 and 1991), Huang, N. N. and Tauchert, T. R.(1991), Singh, G. et al.(1993b), Madency, E. and Barut, A.(1994), Liu, C. F. and Huang, C. H.(1996), Noor, A. K. and Peters, J. M.(1996, 1997), Dano, M. L. and Hyer, M. W.(1998), Barut, A. et. al.(2000) etc.

1.3 Outline of the present work

The present work studies the nonlinear dynamic behavior(free vibrations) of thin plate and shell structures of different geometrical shapes and boundary conditions with or without thermal loading. The structures are assumed to be made of homogeneous, isotropic and elastic materials. In some problems elastic properties of materials are assumed constant and temperature-independent. Thin isotropic circular plates, made of materials having temperature-dependent material properties, are also considered.

To achieve this goal analytical as well as numerical techniques may be used. The numerical techniques, using finite elements, may prove most applicable approach to deal with wide range of structures including complicated structures such as structures which include arbitrary openings or irregular boundary etc. To achieve satisfactory accuracy, a structural component has to be divided into several elements; therefore size of the structure may be some what restrictive. However, due to revolutionary development in the computational technology this problem can be eliminated. Further, these methods involve lot of computational effort, specially to get complete picture of the nonlinear dynamic characteristics of structures. On the other hand analytical techniques viz. Ritz method, Galerkin method etc., may also be used efficiently and successfully to deal with a wide class of plate and shell structures, with some limitations to structures having complex geometrical configurations viz. arbitrary openings, irregular boundary etc. These latter methods have certain advantages, namely (i) yield sufficiently accurate results for engineering purposes, (ii) require less computational effort and (iii) yield closed form solutions. These facts make one's job easier to perform parametric study and to understand the behavior of such structures with less effort. This type of closed-form solution is very useful to have a feel of the dynamics of such structures with less computational efforts as well as to properly interpret and verify results of finite element type numerical analysis.

With this view to use approximate analytical method, continuum model has been used to derive basic governing differential equations as and whatever required, in the sense of von Karman classical large deflection theory or on the basis of Berger approximations i.e. ignoring the energy contribution due to the second strain invariant of the middle surface. In doing so inplane inertia and shear deformations are neglected. The governing differential equations for the resulting system has been solved by Galerkin `error minimizing technique incorporating prescribed boundary conditions to obtain relative nonlinear frequency of vibrations. To obtain solution single mode approach is adopted instead of multimode approach which leads to highly coupled nonlinear ordinary differential equations for the time functions; however former approach gives good approximations for isotropic plate and shell structures.

Numerical results to study the effects of different parameters, as occurred in the analysis, on nonlinear dynamic behavior of such structures have been presented graphically and compared with the available published results or observed behavior wherever possible in some cases.

1.4 Organization

This thesis has been divided into five chapters. Chapter I is introduction which includes scope, previous works, outline of present work and organization. Chapter II studies nonlinear free vibrations of axisymmetric thin isotropic circular plates with clamped immovable edges subjected to thermal loading. Basic governing differential equations in the von Karman sense have been derived in terms of displacement components with the inclusion of thermal loading. These equations are solved by Galerkin's error minimizing technique incorporating clamped immovable edge conditions. A parametric study of such structures is also presented.

Chapter III presents nonlinear free vibrations of thin elastic plates of various geometric shapes viz. triangular and parabolic plates with clamped immovable edges in presence of thermal loading. Decoupled nonlinear governing differential equations on the basis of Berger approximation (i. e. neglecting second strain invariant e_2) have been used. These equations have been have been solved by Galerkin's error minimizing technique. This Chapter also include parametric studies.

Chapter IV investigates nonlinear free vibrations of axisymmetric thin isotropic circular plates with clamped immovable edges, made of homogeneous and elastic material, under thermal loading considering non-homogeneity arising due to temperature-dependent material properties viz. thermal conductivity, modulus of elasticity, coefficient of linear thermal deformation and Poisson's ratio. In this Chapter basic governing differential equations in the von Karman sense have been derived in terms of displacement components with the inclusion of thermal loading. These equations are solved by Galerkin's error minimizing technique incorporating clamped immovable edge conditions. Parametric studies of such structures are presented to study the influence of temperature dependency of material properties viz. thermal conductivity, modulus of elasticity, coefficient of linear thermal deformation and Poisson's ratio.

Chapter V studies the nonlinear free vibrations of axisymmetric thin shallow elastic spherical shells with clamped immovable edges using both coupled nonlinear governing differential equations derived in the von Karman sense in terms of displacement components as well as decoupled nonlinear governing differential equations on the basis of Berger approximation (i. e. neglecting second strain invariant e_2) derived from energy expression applying Hamilton's principle and Euler's variational equations. The governing differential equations are solved by Galerkin's error minimizing technique incorporating clamped immovable edge conditions. Parametric studies of such structures are also presented.

in respect of discussion of results and explanation of observed behavior. As such there arises some scope for further research on this topic.

This Chapter intends to study large amplitude vibrations of axisymmetric thin isotropic circular elastic plates subjected to a steady-state temperature field characterized by constant surface temperatures T_u and T_b, measured from stress free temperature, for the upper and lower surfaces respectively. The material properties are assumed temperature-independent. The basic governing differential equations have been derived in the von Karman sense in terms of displacement components under thermal loading. These equations are solved by Galerkin's method satisfying prescribed boundary conditions. Numerical results have been presented to understand influence of thermal as well as other parameters which appear in the analysis considering clamped immovable boundary condition. The present study reveals some interesting nonlinear dynamic behavior which may prove useful to the designers.

2.2 Temperature Distribution

A thin axisymmetric circular plate of uniform thickness H and radius R is considered. The middle surface of the plate coincides with the r-θ plane of the polar coordinate system having origin at the center of the plate. The z direction is normal to the r-θ plane.

A steady-state thermal field is considered. The upper surface and the lower surface are subjected to constant temperatures, measured from stress free temperature, denoted by T_u and T_b respectively. Heat loss through the edges is assumed negligible. So, heat flow will takes place along the thickness direction i.e. z direction only. The

Summary and conclusions are presented in chapter VI. Mathematical notations and definitions have been defined where they appear first and are also furnished in "NOTATIONS AND DEFINITIONS" section. All the references mentioned in this thesis are documented in the section "REFERENCES".

Lastly, it is important to note that some equations have been repeated in the text of some Chapters as and whenever occurred with the same equation number.

CHAPTER II

NONLINEAR VIBRATIONS OF AXISYMMETRIC THIN CIRCULAR ELASTIC PLATES UNDER THERMAL LOADING

2.1 Introduction

Thin circular plates are frequently used as structural components in many structures viz. chemical plants, nuclear plants, offshore and ship-structures, spacecrafts, storage structures etc. and are often subjected to thermal fields. The study of nonlinear dynamic behavior with or without thermal loading is of technical interest to a designer. Nonlinear vibrations of heated circular plates have been studied by many investigators on the basis of Berger's approximation i.e. ignoring the energy contribution due to second strain invariant (e_2) viz. Pal, M. C. (1970a, 1973), Jones, R. et al. (1980) etc.

Subsequently, Biswas, P. and Kapoor, P. (1984a, 1984b) performed nonlinear free vibration analysis of circular elastic plates, isotropic as well as orthotropic, including thermal loading based on coupled nonlinear differential equations derived in the von Karman sense, the classical large deflection theory which yields more accurate results. In their investigation, clamped movable and immovable boundary conditions have been considered and temperature field is assumed to vary linearly in the radial direction without satisfying heat conduction equation leaving some scope for further work in this direction. Their research works seems to having some lacks in the sense of completeness in respect of discussion of results and explanation of observed behavior. As such there arises some scope for further research on this topic.

This Chapter intends to study large amplitude vibrations of axisymmetric thin isotropic circular elastic plates subjected to a steady-state temperature field characterized by constant surface temperatures T_u and T_b, measured from stress free temperature, for the upper and lower surfaces respectively. The material properties are assumed temperature-independent. The basic governing differential equations have been derived in the von Karman sense in terms of displacement components under thermal loading. These equations are solved by Galerkin's method satisfying prescribed boundary conditions. Numerical results have been presented to understand influence of thermal as well as other parameters which appear in the analysis considering clamped immovable boundary condition. The present study reveals some interesting nonlinear dynamic behavior which may prove useful to the designers.

2.2 Temperature Distribution

A thin axisymmetric circular plate of uniform thickness H and radius R is considered. The middle surface of the plate coincides with the r-θ plane of the polar coordinate system having origin at the center of the plate. The z direction is normal to the r-θ plane.

A steady-state thermal field is considered. The upper surface and the lower surface are subjected to constant temperatures, measured from stress free temperature, denoted by T_u and T_b respectively. Heat loss through the edges is assumed negligible. So, heat flow will takes place along the thickness direction i.e. z direction only. The temperature field $T(r,\theta,z)$, also denoted by T, may be represented by three dimensional heat conduction equation for isotropic material with a constant conductivity in the cylindrical coordinate system [Sachdeva, R. C., 1988] as

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}.\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q}{k_c}=\frac{1}{\eta'}\frac{\partial T}{\partial t} \tag{2.1}$$

where, q = the internal heat generation per unit time·per unit volume; k_c = thermal conductivity; $\eta'=\frac{k_c}{\rho c_p}$ = thermal diffusivity; ρ = density of plate material; c_p = specific heat.

For one dimensional steady-state of heat conduction along z-direction without heat generation within the plate

$$\frac{\partial T}{\partial t}=0;\quad \frac{q}{k_c}=0;\quad \frac{\partial^2 T}{\partial r^2}+\frac{1}{r}.\frac{\partial T}{\partial r}=0 \text{ and } \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}=0 \tag{2.2}$$

By virtue of Equation (2.2) Equation (2.1) reduces to

$$\frac{\partial^2 T}{\partial z^2}=0 \tag{2.3}$$

The solution of Equation (2.3) is

$$T=C_1^T z+C_2^T \tag{2.4}$$

The constants C_1^T and C_2^T are determined from the thermal boundary conditions of the plate (i) $T=T_u$ at $z=\frac{H}{2}$ and (ii) $T=T_b$ at $z=-\frac{H}{2}$ as

$$C_1^T=\frac{T_u-T_b}{H} \tag{2.5a}$$

$$C_2^T = \frac{T_u + T_b}{2} \tag{2.5b}$$

The Equation (2.4) becomes

$$T(r,\theta,z) = \frac{T_u + T_b}{2} + \frac{T_u - T_b}{H} z \tag{2.6}$$

The plate is subjected to the temperature field $T(r,\theta,z)$ defined by the Equation (2.6) which is independent of r and θ coordinates and varies linearly along the thickness direction i.e. z coordinate only.

2.3 Governing Differential Equations

The strain-displacement relationships at a point of the middle surface of a geometrically nonlinear isotropic plate in cylindrical polar coordinates according to the large displacement theory(i.e. in the von Karman sense) are given by [Chia, C. Y., 1980]

$$\varepsilon_r^0 = u,_r + \frac{1}{2} w,_r^2 \tag{2.7a}$$

$$\varepsilon_\theta^0 = \frac{1}{r}\left(u + u_\theta,_r\right) + \frac{1}{2r^2} w,_\theta^2 \tag{2.7b}$$

$$\varepsilon_{r\theta}^0 = \frac{1}{r}\left(u,_\theta - u_\theta\right) + u_\theta,_r + \frac{1}{r} w,_r\, w,_\theta \tag{2.7c}$$

in which ε_r^0 and ε_θ^0 are strains of the middle surface in the radial and circumferential directions respectively; $\varepsilon_{r\theta}^0$ is shearing strain of the middle surface in the $r\theta$ plane; u, u_θ and w are displacements of the middle surface in the radial, circumferential and normal (to the middle surface) directions respectively.

Noting that u_r and w are independent of θ coordinate as well as $u_\theta = 0$ for axisymmetric circular plate one can write the Equations (2.7) as

$$\varepsilon_r^0 = u,_r + \frac{1}{2} w,_r^2 \tag{2.8a}$$

$$\varepsilon_\theta^0 = \frac{u}{r} \tag{2.8b}$$

$$\varepsilon_{r\theta}^0 = 0 \tag{2.8c}$$

The strain-displacement relationships at a distance z from the middle surface of plate are given by

$$\varepsilon_{rr} = u,_r + \frac{1}{2} w,_r^2 - z w,_{rr} \tag{2.9a}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} - z \frac{w,_r}{r} \tag{2.9b}$$

$$\varepsilon_{r\theta} = 0 \tag{2.9c}$$

The stresses at a distance z from the middle surface of the plate subjected to thermal gradient along the z-direction measured from the stress free temperature (T_0) are:

$$\sigma_{rr} = \frac{E_c}{(1 - \nu_c^2)} \left[\left(\varepsilon_{rr} - \int_{T_0}^{T} \alpha_c dT \right) + \nu_c \left(\varepsilon_{\theta\theta} - \int_{T_0}^{T} \alpha_c dT \right) \right] \tag{2.10a}$$

$$\sigma_{\theta\theta} = \frac{E_c}{(1 - \nu_c^2)} \left[\left(\varepsilon_{\theta\theta} - \int_{T_0}^{T} \alpha_c dT \right) + \nu_c \left(\varepsilon_{rr} - \int_{T_0}^{T} \alpha_c dT \right) \right] \tag{2.10b}$$

$$\sigma_{r\theta} = 0 \tag{2.10c}$$

in which T_0 is stress free reference temperature.

The Equations (2.9) are substituted in Equations (2.10) and the resulting Equations are integrated with respect to z coordinate to yield the stress resultants and the stress couples:

$$N_{rr} = \left\{\frac{E_c H}{(1-\nu_c^2)}\right\}\left\{u,_r + 0.5w,_r^2 + \nu_c\left(\frac{u}{r}\right)\right\} - N_T \tag{2.11a}$$

$$N_{\theta\theta} = \left\{\frac{E_c H}{(1-\nu_c^2)}\right\}\left\{\frac{u}{r} + \nu_c\left(u,_r + 0.5w,_r^2\right)\right\} - N_T \tag{2.11b}$$

$$M_{rr} = -D_c\left\{w,_{rr} + \left(\frac{\nu_c w,_r}{r}\right)\right\} - M_T \tag{2.12a}$$

$$M_{\theta\theta} = -D_c\left\{\nu_c w,_{rr} + \frac{w,_r}{r}\right\} - M_T \tag{2.12b}$$

where

$$D_c = \frac{E_c H^3}{12\left(1-\nu_c^2\right)}; \tag{2.13}$$

$$N_T = \frac{\alpha_c E_c}{\left(1-\nu_c\right)}\int_{-H/2}^{H/2}\left(\int_{T_0}^{T} dT\right)dz; \tag{2.14a}$$

$$M_T = \frac{\alpha_c E_c}{\left(1-\nu_c\right)}\int_{-H/2}^{H/2} z\left(\int_{T_0}^{T} dT\right)dz; \tag{2.14b}$$

D_c = constant flexural rigidity of the plate; E_c = constant modulus of elasticity of the plate material; M_{rr} = bending moment per unit length in the radial direction; $M_{\theta\theta}$ = bending moment per unit length in the circumferential direction; M_T = thermal stress couple (bending moment) per unit length; N_{rr} = stress resultant in radial direction per unit length; $N_{\theta\theta}$ = stress resultant in circumferential direction per unit length; N_T =

thermal stress resultant per unit length; α_c = constant coefficient of linear thermal deformation of the plate material; "," indicates derivative of the subscripted variable with respect to the subscript and ν_c = constant Poisson's ratio of the plate material.

Neglecting the inplane inertia, the equilibrium equations of nonlinear free vibrations for axisymmetric thin circular plate [Chia, C. Y., 1980] are

$$-(rN_{rr})_{,r} + N_{\theta\theta} = 0; \tag{2.15a}$$

$$-(rM_{rr})_{,r} + M_{\theta\theta} + rQ = 0 \tag{2.15b}$$

and

$$-(rw_{,r} N_{rr})_{,r} - (rQ)_{,r} = -\rho H r\ddot{w} \tag{2.15c}$$

where Q = Transverse shear force per unit length on circumferential plane ; $\ddot{w}$ represents double derivative of w with respect to time; ρ = density of the plate material.

Combining Equations (2.11), (2.12) and (2.15) one obtains

$$r^2 u_{,rr} + ru_{,r} - u = \frac{(\nu_c - 1)}{2} rw_{,r}^2 - r^2 w_{,r} w_{,rr} + \frac{r^2(1-\nu_c^2)}{E_c H}(N_T)_{,r} \tag{2.16}$$

and

$$D_c \nabla^4 w + \rho H \ddot{w} + \nabla^2 M_T = \frac{1}{r}(rN_{rr} w_{,r})_{,r} \tag{2.17}$$

where the Laplace operator $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}$ in polar coordinates and $\nabla^4 = \nabla^2 . \nabla^2$.

These two Equations are to be used to study the static and dynamic nonlinear behavior of axisymmetric circular plate structures at elevated temperature.

2.4 Boundary Conditions

A thin axisymmetric circular plate with clamped immovable edges is considered. So the boundary conditions (at r = R) are

(i) $w = 0$ (deflection along z-direction is zero)

(ii) $w,_r = 0$ (slope is zero in r-z plane)

(iii) u = 0 (radial displacement is zero for immovable edges)

2.5 Solution

Assuming the distribution of mass and temperature symmetrical over the plate, the form of symmetrical deflection satisfying boundary conditions of the clamped circular plate with immovable edges can be written as follows:

$$w(r,t) = A\left[1-\left(\frac{r}{R}\right)^2\right]^2 F(t) \tag{2.18}$$

where A is the maximum deflection at the center of the plate i.e. the amplitude of vibrations at the center of the plate and $F(t)$ is a function of time.

Substitution of Equation (2.18) into Equation (2.16) yields

$$r^2 u,_{rr} + ru,_r - u = C_1 r^3 + C_2 r^5 + C_3 r^7 + \varphi(r) \tag{2.19}$$

where

$$C_1 = \frac{8(\nu_c - 3)A^2F^2(t)}{R^4} \tag{2.20a}$$

$$C_2 = -\frac{16(\nu_c - 5)A^2F^2(t)}{R^6} \tag{2.20b}$$

$$C_3 = \frac{8(\nu_c - 7)A^2F^2(t)}{R^8} \tag{2.20c}$$

$$\varphi(r) = \frac{(1-\nu_c^2)}{E_c H} r^2 (N_T),_r \tag{2.21}$$

The complete solution of Equation (2.19) is obtained in the form

$$u(r,t) = C_0 r + \frac{C_1}{8} r^3 + \frac{C_2}{24} r^5 + \frac{C_3}{48} r^7 + \psi(r) \tag{2.22}$$

where ψ (r) is the particular solution for the term φ (r) of Equation (2.19); the parameter C_0 can be determined from boundary condition (iii) of immovable edges and the same is documented below as

$$C_0 = -\left(\frac{R^2 C_1}{8} + \frac{R^4 C_2}{24} + \frac{R^6 C_3}{48}\right) - \frac{\psi(R)}{R} \tag{2.23}$$

Substituting Equations (2.11a), (2.14), (2.18), (2.20), (2.21), (2.22) and (2.23) into Equation (2.17) and applying Galerkin's error minimizing technique one gets well known nonlinear time differential equation in the following form

$$\ddot{F}(t) + \lambda_1 F(t) + \lambda_3 F^3(t) = 0 \tag{2.24}$$

where, $\ddot{F}(t)$ represents double derivative of $F(t)$ with respect to time; the parameters λ_1 and λ_3 are known and the same are presented as follows

$$\lambda_1 = 11.0 \frac{E_c H^2}{(1-\nu_c^2)\rho R^4} - \frac{5.5}{(1-\nu_c)} \frac{E_c \alpha_c (T_u + T_b)}{\rho R^2} \tag{2.25a}$$

$$\lambda_3 = \frac{E_c A^2}{(1-\nu_c^2)\rho R^4} (2.2653 + 1.8334\nu_c - 0.436\nu_c^2) \tag{2.25b}$$

The solution of the Equation (2.24) subjected to initial conditions $F(0) = 0$ and $\dot{F}(0) = 0$ has been given by Nash, W. A. and Modeer, J. R. (1959) as

$$F(t) = cn(\omega_{NL} t, K) \tag{2.26}$$

where

$$\omega_{NL} = (\lambda_1 + \lambda_3)^{1/2} \tag{2.27}$$

$$K = \left[\frac{\lambda_3}{2(\gamma_1 + \gamma_3)} \right]^{1/2} \tag{2.28}$$

The nonlinear time period is given by

$$T_{NL} = \frac{4K}{\omega_{NL}} \tag{2.29}$$

Here, ω_{NL} and K are positive constants, *cn* is Jacobi's elliptic function. ω_{NL} is the nonlinear frequency and T_{NL} is the nonlinear time period.

By dropping the nonlinear terms in Equation (2.24) on obtains the linear frequency (ω_L) of vibration as

$$\omega_L = [\lambda_1]^{1/2} \tag{2.30}$$

Hence, the relative nonlinear frequency i.e. the ratio of nonlinear frequency to linear frequency becomes

$$\frac{\omega_{NL}}{\omega_L} = \left[1 + \frac{\lambda_3}{\lambda_1} \right]^{1/2} \tag{2.31}$$

Substituting Equations (2.25a) and (2.25b) into Equation (2.31) one obtains

$$\frac{\omega_{NL}}{\omega_L} = \left[1 + \frac{\left(0.20624 + 0.1667 v_c - 0.03965 v_c^2\right)\dfrac{A^2}{H^2}}{1 - \dfrac{\alpha_c (1 + v_c)(T_u + T_b)}{4} \cdot \dfrac{R^2}{H^2}} \right]^{1/2} \tag{2.32}$$

The Equation (2.32) represents the relative nonlinear frequency i.e. the ratio of nonlinear frequency to linear frequency and the same can be used to study the influence of different parameters, which appear in the analysis, on nonlinear dynamic behavior of isotropic circular plates with clamped immovable edges. This equation reveals that in absence of thermal loading the relative nonlinear frequency depends only on relative amplitude $\left(\frac{A}{H}\right)$ of vibrations. In absence of thermal loading besides amplitude (A) and plate thickness (H) natural frequencies (Equations (2.25a), (2.25b), 2.27) and (2.30)), both linear and nonlinear, depend on the radius of the plate by the same proportion and increase of slenderness parameter $\left(\frac{R}{H}\right)$ may be viewed as increase of the radius at constant plate thickness (H). So, with increase of slenderness parameter natural frequencies, both linear and nonlinear, decrease in the same proportion at constant amplitude (A) of vibrations and constant plate thickness (H) and hence relative nonlinear frequency becomes independent of slenderness parameter $\left(\frac{R}{H}\right)$. In presence of thermal loading it also depends on slenderness parameter $\left(\frac{R}{H}\right)$ besides thermal parameters and nondimensional amplitude $\left(\frac{A}{H}\right)$. If the thermal loading is considered to be composed of two parts i.e. (i) the average surface temperature $\left(\frac{T_u + T_b}{2}\right)$ and (ii) the constant linear thermal gradient part $\left(\frac{T_u - T_b}{H} z\right)$ then one observes that the thermal gradient part has no effects on nonlinear dynamic behavior of such structures for the assumed thermal loading. This is due to the fact that the constant linear thermal gradient part does not contribute to the thermal stress resultants since the material properties are assumed independent of temperature (Equations (2.6) and (2.14a)). Further, the constant linear thermal gradient along the thickness direction through out the plate induce constant pure bending moment uniformly distributed along the edges of the plate and no shear will be induced in the thickness direction. Hence, the term $\nabla^2 M_T$ of Equation (2.16) vanishes. As a result the linear thermal gradient part does not influence relative nonlinear frequency of vibrations in the transverse direction of such structures.

2.6 Numerical Results, Observations and Discussions

This section presents numerical results in the graphical form to understand the effects of different parameters viz. (i) the nondimensional amplitude $\left(\frac{A}{H}\right)$, (ii) the thermal loading (the average surface temperature $\frac{(T_u + T_b)}{2.0}$) and (iii) the slenderness parameter $\frac{R}{H}$ ratio etc. on the relative nonlinear frequency of clamped thin isotropic circular plates with immovable edges. Assuming the plates to be made of steel, the coefficient of linear thermal deformations (α) and Poisson's ratio (ν) are assumed as $12\times10^{-6}\, per\,{}^\circ C$ and 0.3 respectively.

2.6.1 The Effects of Nondimensional Amplitude $\left(\frac{A}{H}\right)$

Figures 2.1(a) and 2.1(b) show the effects of nondimensional amplitude on relative nonlinear frequency. It is observed that $\frac{\omega_{NL}}{\omega_L}$ increases with increase of $\frac{A}{H}$ i.e. circular plates with clamped immovable edges exhibit strain-hardening type of nonlinearity. Similar behavior is obtained by Biswas, P. and Kapoor, P. (1984a, 1984b and 1986a) for the when thermal loading is absent; though results are presented in different form. The additional stiffness due to large deflection i.e. stretching of the middle surface increases the nonlinear stiffness and hence, the nonlinear frequency of the plate with increase of amplitude. On the other hand, the linear frequency is independent of amplitude of vibration. This phenomenon explains the strain-hardening behavior of thin circular plate structures and this behavior is valid within the proportional limit of the structure material. Such effects become more pronounced with increase of average surface temperature $\left(\frac{T_u + T_b}{2}\right)$ and/or slenderness parameter $\left(\frac{R}{H}\right)$ in presence of thermal loading. The dynamic behavior under thermal loading is explained in the subsequent sections.

2.6.2 The Effects of Thermal Loading

Figure 2.2 shows the effects of average surface temperature $\frac{(T_u + T_b)}{2}$ on relative nonlinear frequency at $\frac{A}{H} = 1.0$ for different values of slenderness parameter $\left(\frac{R}{H}\right)$. The choice of nondimensional amplitude $\frac{A}{H} = 1.0$ should not be considered to be taken arbitrarily. Numerical values for other values of $\frac{A}{H}$ can be plotted to yield similar curves. It is observed that the relative nonlinear frequency increases with increase $\frac{(T_u + T_b)}{2}$ and after certain stage it approaches infinity. The average surface temperature part of thermal loading induces compressive stress in the circular plate with clamped immovable edges and stiffness i.e. frequency, both linear as well as nonlinear, decreases with increase of average surface temperature $\left(\frac{T_u + T_b}{2}\right)$. Due to the presence of the additional stiffness associated with large deflection the linear frequency decreases more than the nonlinear one proportionately and as a result $\frac{\omega_{NL}}{\omega_L}$ increases. At certain stage linear frequency becomes zero i.e. thermal instability of the structure occurs based on linear theory; but nonlinear frequency is still non-zero which makes $\frac{\omega_{NL}}{\omega_L}$ infinity. So, thermal instability is delayed due to additional stiffness associated with large deflection.

2.6.3 The Effects of Slenderness Parameter $\left(\frac{R}{H}\right)$

In absence of thermal loading, besides amplitude (A) of vibrations and plate thickness (H) natural frequencies, both linear and nonlinear, also depend on the radius of the plate by the same proportion [Equations (2.25), (2.27) and (2.30)] and increase of slenderness ratio may be achieved by increase of the radius at constant plate thickness. So, with increase of slenderness ratio natural frequencies, both linear and nonlinear, decrease in the same proportion and become asymptotic around zero; but the relative nonlinear frequency remains constant in absence of thermal loading.

In presence of thermal loading, the variation of relative nonlinear frequency with the slenderness parameter $\left(\frac{R}{H}\right)$ at $\frac{A}{H} = 1.0$ for some particular values of average surface temperature $\left(\frac{T_u + T_b}{2}\right)$ is shown in Figure 2.3. In presence of thermal loading, increase of slenderness parameter $\left(\frac{R}{H}\right)$ reduces stiffness and hence frequency of vibration; both linear as well as nonlinear; but the linear frequency decreases more than the nonlinear one proportionately due to the presence of the additional stiffness associated with large deflection and as a result $\frac{\omega_{NL}}{\omega_L}$ increases. At certain stage linear frequency becomes zero due to presence of thermal loading i.e. thermal instability occurs based on linear theory; but nonlinear frequency is still non-zero which makes $\frac{\omega_{NL}}{\omega_L}$ infinity.

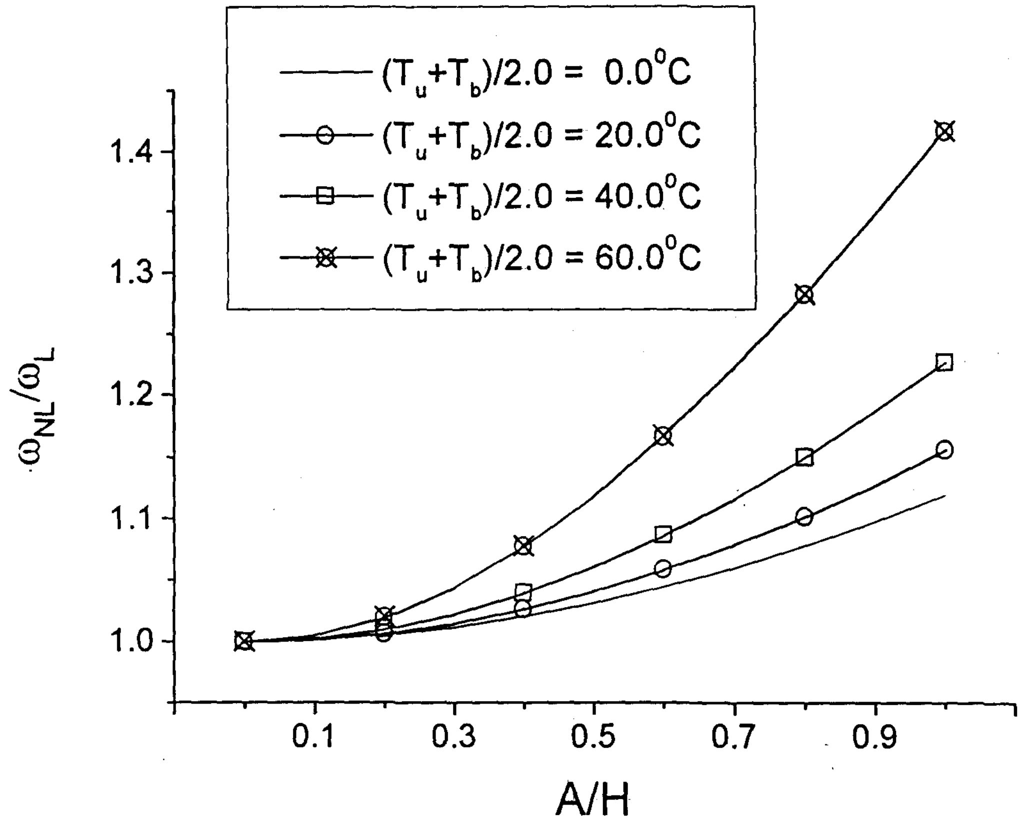

Fig. 2.1(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{A}{H}$ **for different values of** $\frac{(T_u + T_b)}{2.0}$ **at** $\frac{R}{H}$ **= 50.0.**

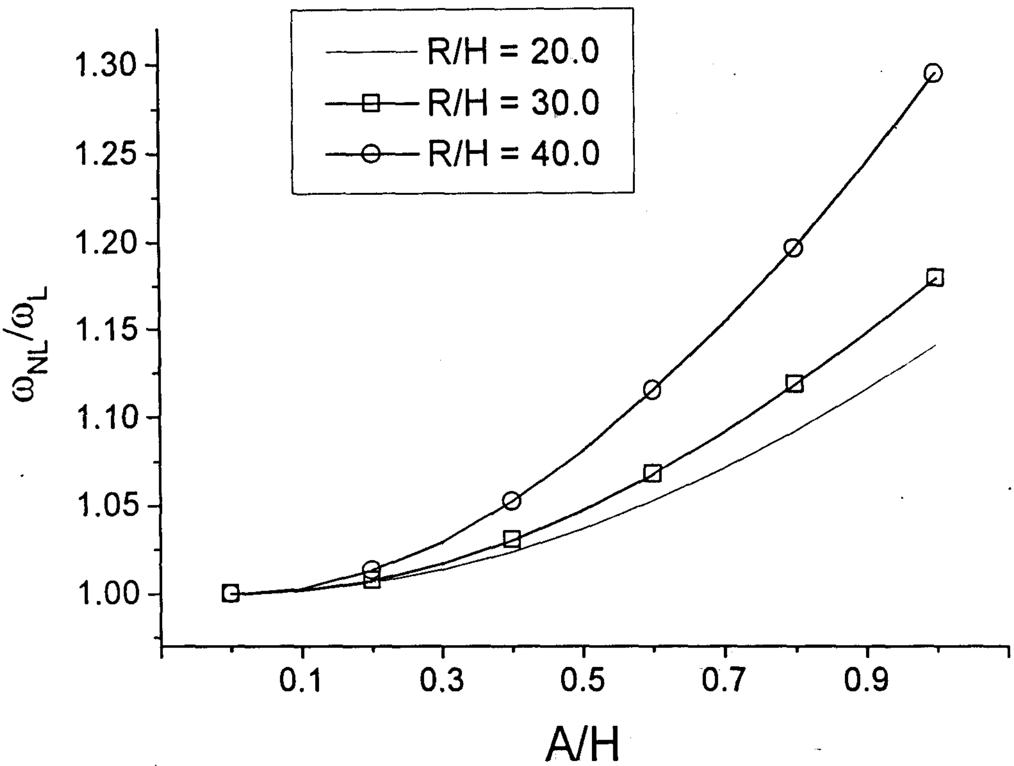

Fig. 2.1(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{A}{H}$ **for different values of** $\frac{R}{H}$ **at** $\frac{(T_u + T_b)}{2.0} = 50.0^0 C$ **.**

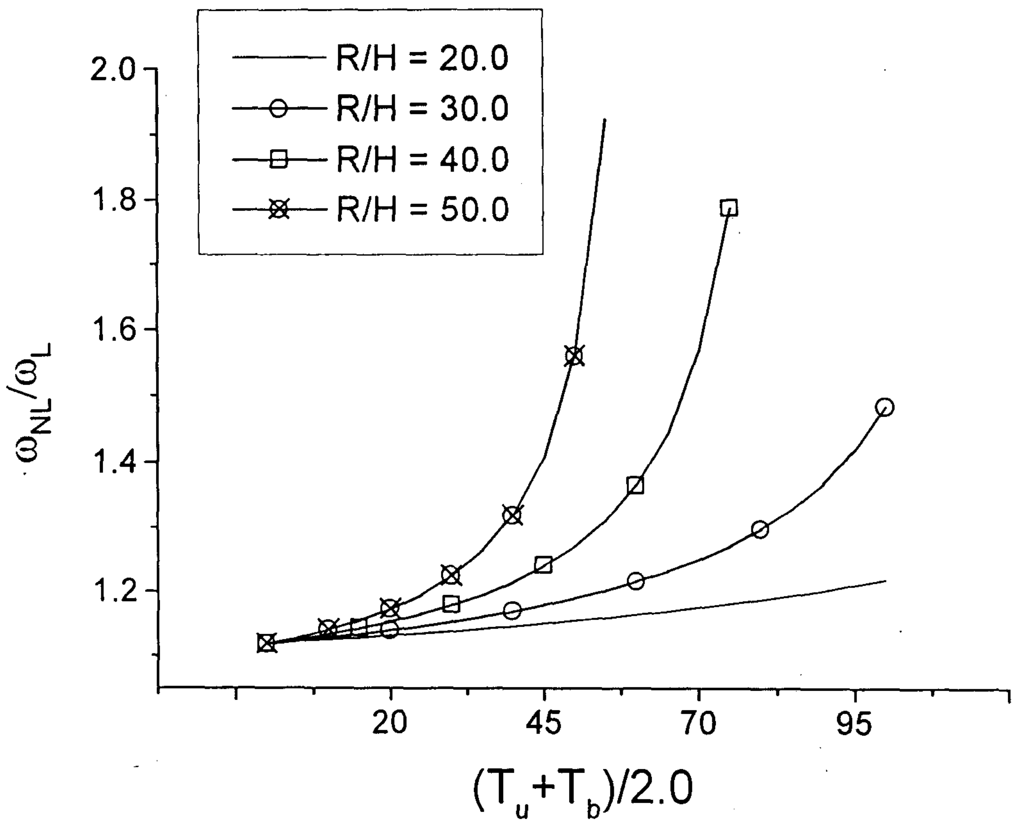

Fig. 2.2 $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{(T_u + T_b)}{2.0}$ **for different values of** $\frac{R}{H}$ **at** $\frac{A}{H} = 1.0$

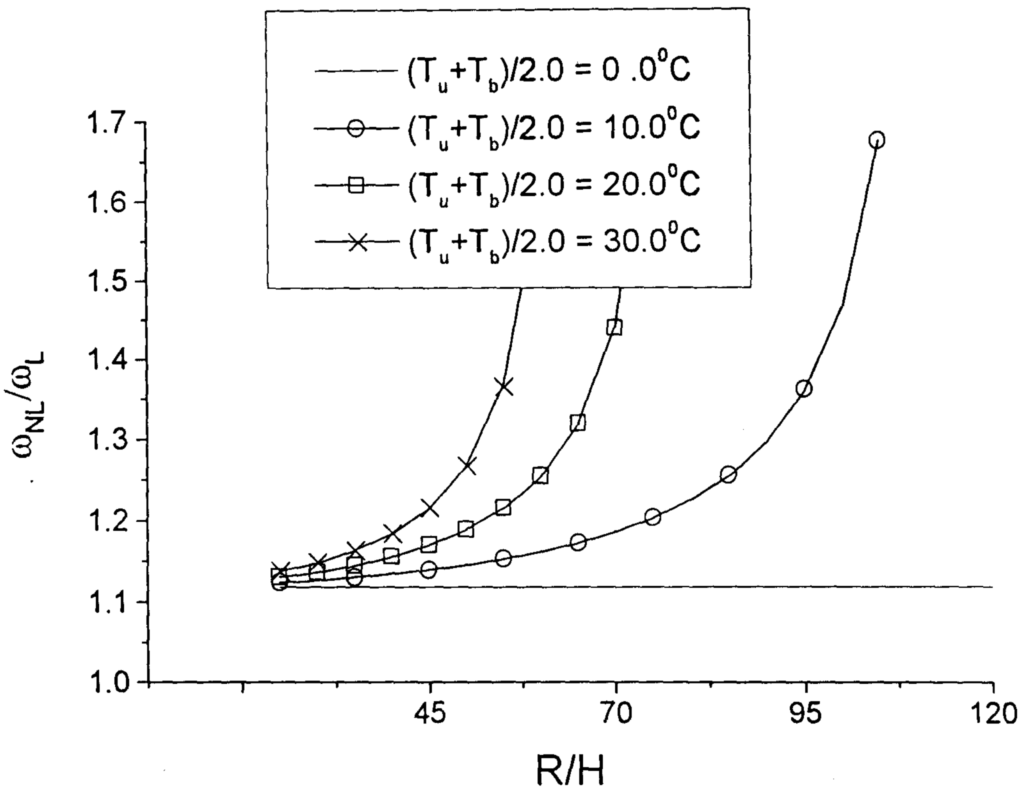

Fig. 2.3 $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{R}{H}$ **for different values of** $\frac{(T_u + T_b)}{2.0}$ **at** $\frac{A}{H} = 1.0$

CHAPTER III

NONLINEAR VIBRATIONS OF THIN ELASTIC PLATES OF VARIOUS SHAPES UNDER THERMAL LOADING

3.1 Introduction

Thin plates of various shapes are frequently used as structural components in many structures viz. chemical plants, nuclear plants, offshore and ship-structures, spacecrafts, storage structures etc. and the study of their nonlinear dynamic behavior is of interest to a designer. This Chapter presents nonlinear free vibration analysis of isotropic triangular and parabolic elastic plates including thermal loading. Literature survey reveals that many researchers have studied nonlinear vibrations of isosceles right angled and equilateral triangles with different boundary conditions viz. Vendhan, C. P., and Dhoopar, B. L. (1973), Biswas, P. (1976c,1979), Choudhury, S. K. (1981), Biswas, P. and Kapoor, P. (1984c). Thermal loading has been considered in some of the research works mentioned above [Biswas, P. (1976c, 1979), Biswas, P. and Kapoor, P. (1984c)].

Biswas, P. and Kapoor, P. (1986c) also presented large amplitude free vibration of heated parabolic elastic plate with clamped immovable edges. They have considered a parabolic plate of particular geometric configuration having the base dimension equal to the length of the axis and the nature of temperature field is not clearly defined.

In this Chapter a triangular plate of an arbitrary geometric configuration and a parabolic plate of more general shape, having arbitrary base dimension and length of the axis, have been considered. Clamped immovable boundary conditions have been used for both the cases. A steady-state temperature field, characterized by constant surface temperatures T_u and T_b, measured from stress free temperature, for the upper and lower surfaces respectively is considered for this study.

It is well-established that large amplitude vibration analysis using decoupled nonlinear governing differential equations derived on the basis of Berger's approximation i.e. ignoring the energy contribution due to second strain invariant (e_2) of the middle surface yields sufficiently accurate results for practical purposes and the same have been used. These equations have been solved by Galerkin's method. Parametric studies of such structures have been presented to understand the effects of different parameters related to plate geometry, nondimensional amplitude and thermal loading parameter on the relative nonlinear frequency. This study reveals some interesting nonlinear dynamic features of such structures.

3.2. Temperature Distribution

A steady-state thermal field is considered. The upper surface and the lower surface are subjected to constant temperatures, measured from stress free temperature, denoted by T_u and T_b respectively. Heat loss through the edges is assumed negligible. So, heat flow will takes place along the thickness direction i.e. z direction only. The temperature field $T(x, y, z)$, also denoted by T, may be represented by three dimensional heat conduction equation in the Cartesian coordinate system for isotropic material with constant thermal conductivity [Sachdeva, R. C., 1988] as

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q}{k_c}=\frac{1}{\eta'}\frac{\partial T}{\partial t} \tag{3.1}$$

with usual notations.

For one dimensional steady-state of heat conduction along z-direction without heat generation within the plate

$$\frac{\partial T}{\partial t}=0;\ \frac{q}{k_c}=0;\ \frac{\partial^2 T}{\partial x^2}=0 \text{ and } \frac{\partial^2 T}{\partial y^2}=0 \tag{3.2}$$

By virtue of Equation (3.2) Equation (3.1) reduces to

$$\frac{\partial^2 T}{\partial z^2}=0 \tag{3.3}$$

The solution of Equation of (3.3) is

$$T=C_1^T z+C_2^T \tag{3.4}$$

The constants C_1^T and C_2^T are determined from the thermal boundary conditions of the plate (i) $T=T_u$ at $z=\frac{H}{2}$ and (ii) $T=T_b$ at $z=-\frac{H}{2}$ as

$$C_1^T=\frac{T_u-T_b}{H} \tag{3.5a}$$

$$C_2^T=\frac{T_u+T_b}{2} \tag{3.5b}$$

The Equation (3.4) becomes

$$T(x,y,z)=\frac{T_u+T_b}{2}+\frac{T_u-T_b}{H}z \tag{3.6}$$

The plate is subjected to the thermal field $T(x, y, z)$ defined by the Equation (3.6) that is independent of x and y coordinates. It has linear thermal gradient in the thickness i.e. z direction only.

3.3 Governing Differential Equations

The plate is assumed to be made of elastic and homogeneous material. The quasi-linear uncoupled partial differential equations in terms of Cartesian coordinate system, based on Berger's approximation i.e. ignoring the energy contribution due to second strain invariant (e_2) of the middle surface in the total potential energy expression, for nonlinear free vibrations of elastic isotropic plates are given by [Jones, R. et al., 1980]

$$D_c \nabla^4 w + C_b \nabla^2 w + \rho H \ddot{w} + \frac{\nabla^2 M_T}{(1 - \nu_c)} = 0 \tag{3.7}$$

$$\frac{N_T}{(1 - \nu_c)} - \frac{12 D_c e_1}{H^2} = C_b \tag{3.8}$$

where,

$$e_1 = \varepsilon_x + \varepsilon_y = u_{,x} + v_{,y} + \frac{w_{,x}^2}{2} + \frac{w_{,y}^2}{2}; \tag{3.9}$$

$$e_2 = \varepsilon_x \varepsilon_y - \frac{1}{4}\gamma_{xy}^2; \tag{3.10a}$$

$$\varepsilon_x = u_{,x} + \frac{1}{2} w_{,x}^2; \tag{3.10b}$$

$$\varepsilon_y = v_{,y} + \frac{1}{2} w_{,y}^2; \tag{3.10c}$$

$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x} w_{,y}; \tag{3.10d}$$

C_b = Berger's constant which is independent of x and y; but involves time 't'; e_1 and e_2 are first and second strain invariants respectively of middle surface of the plate; ε_x and ε_y are strains of middle surface of the plate in x and y- directions. The thermal stress resultant N_T and the resultant thermal stress couple (bending moment) M_T due to temperature field T(x, y, z) are given by

$$N_T = \frac{\alpha_c E_c}{(1-\nu_c)} \int_{-H/2}^{H/2} T(x, y, z) dz \tag{3.11}$$

$$M_T = \frac{\alpha_{ct} E_c}{(1-\nu_c)} \int_{-H/2}^{H/2} zT(x, y, z) dz \tag{3.12}$$

In the present analysis, temperature distribution has linear thermal gradient along the z-direction only and is independent of in-plane coordinates and hence, by virtue of Equation (3.12), Equation (3.7) reduces to

$$D_c \nabla^4 w + C_b \nabla^2 w + \rho H \ddot{w} = 0 \tag{3.13}$$

3.4 Triangular Plates

A thin isotropic triangular plate having uniform thickness is considered. The geometry of the plate and the coordinate system are shown in Fig. 3.1. The Symbols ς and η represent the oblique coordinates and skew angle is denoted by ϕ. The edges of the plate are assumed clamped immovable. The relations between oblique coordinates and Cartesian coordinates are given by

$$x = \varsigma + \eta \sin\phi \tag{3.14a}$$

$$y = \eta \cos\phi \tag{3.14b}$$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \varsigma} \tag{3.15a}$$

$$\frac{\partial}{\partial y}=\frac{1}{\cos\phi}\left(\frac{\partial}{\partial \eta}-\sin\phi\frac{\partial}{\partial \varsigma}\right) \tag{3.15b}$$

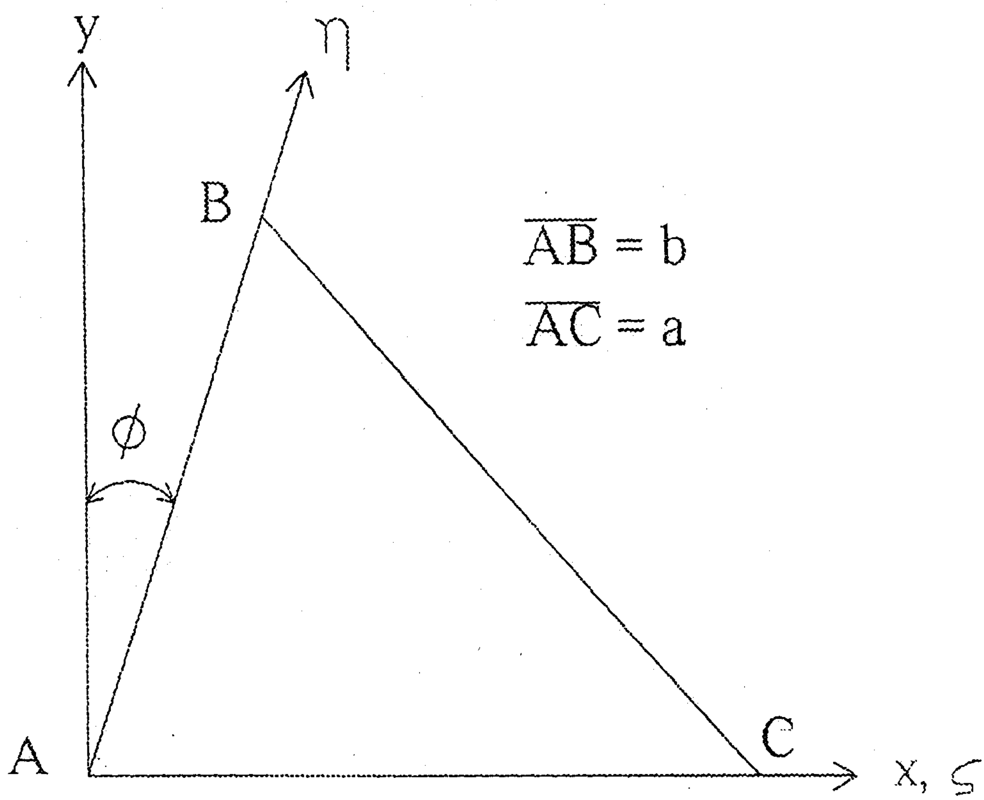

Fig. 3.1 Geometry and coordinate system of a triangular plate of arbitrary shape.

$$\nabla^2 = \frac{1}{\cos^2\phi}\left(\frac{\partial^2}{\partial\varsigma^2} - 2\sin\phi\frac{\partial^2}{\partial\varsigma\partial\eta} + \frac{\partial^2}{\partial\eta^2}\right) \tag{3.16a}$$

$$\nabla^4 = \frac{1}{\cos^4\phi}\left\{\frac{\partial^4}{\partial\varsigma^4} - 4\sin\phi\frac{\partial^4}{\partial\varsigma^3\partial\eta} + \left(2+4\sin^2\phi\right)\frac{\partial^4}{\partial\varsigma^2\partial\eta^2} - 4\sin\phi\frac{\partial^4}{\partial\varsigma\partial\eta^3} + \frac{\partial^4}{\partial\eta^4}\right\} \tag{3.16b}$$

3.4.1 Solution

For fundamental mode of vibrations, satisfying clamped immovable boundary conditions, the deflection $w(\varsigma,\eta,t)$ is assumed in the form.

$$w(\varsigma,\eta,t) = A\frac{\varsigma^2\eta^2}{a^4b^4}(\varsigma b+\eta a-ab)^2F(t) \tag{3.17}$$

Combining Equations (3.13), (3.16) and (3.17) and applying Galerkin's procedure one gets, after necessary integrations, the following equation

$$\frac{3.1746D_cF(t)}{\cos^4\phi}\left\{2\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+\left(2+4\sin^2\phi\right)-4\sin\phi\left(\frac{a}{b}+\frac{b}{a}\right)\right\}-$$

$$-\frac{0.0481abC_bF(t)}{\cos^2\phi}\left\{\left(\frac{a}{b}+\frac{b}{a}\right)-\sin\phi\right\}+0.0015857\rho Ha^2b^2\ddot{F}(t)=0 \tag{3.18}$$

The in-plane displacements u and v may be assumed suitably so that the terms $\int_A u_{,\varsigma}\, dA$ and $\int_A v_{,\eta}\, dA$ vanish. Now, substituting Equations (3.9), (3.11), (3.15) and (3.17) into Equation (3.8) and integrating the latter over the area of the plate one obtains

$$abC_b = \frac{abN_T^*D_c}{H^2} - \frac{D_c}{H^2}\left\{\frac{b}{a}+\frac{1}{\cos^2\phi}\left(\frac{a}{b}+\frac{b}{a}\sin^2\phi-\sin\phi\right)\right\}A^2F^2(t) \tag{3.19}$$

where,

$$N_T^* = \frac{H^3}{D_c}\iint\left[\frac{\alpha_c E_c}{ab(1-\nu_c)}\left\{\int_{-H/2}^{H/2}\frac{T(x,y,z)}{H}dz\right\}\right]dxdy = 12\times 0.5(1+\nu_c)\alpha_c\frac{(T_u+T_b)}{2.0} \tag{3.20}$$

Eliminating C_b from Equations (3.18) and (3.19), one gets the same cubic time differential Equation (2.24) as in case of circular plate of Chapter-II:

$$\ddot{F}(t)+\lambda_1 F(t)+\lambda_3 F^3(t)=0 \tag{2.24}$$

where λ_1 and λ_3 are known parameters.

The solution of the Equation (2.24) subjected to initial conditions $F(0)=0$ and $\dot{F}(0)=0$ yields the relative nonlinear frequency i.e. the ratio of nonlinear frequency to linear frequency [Nash and Modeer (1959)] as

$$\frac{\omega_{NL}}{\omega_L}=\left[1+\frac{\lambda_3}{\lambda_1}\right]^{1/2} \tag{2.31}$$

where

$$\lambda_1=\frac{630.9148D_c}{\rho Ha^2b^2}\left[\begin{array}{l}\dfrac{3.1746}{\cos^4\phi}\left\{2\left(\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}\right)+\left(2+4\sin^2\phi\right)-4\sin\phi\left(\dfrac{a}{b}+\dfrac{b}{a}\right)\right\}-\\[2ex] -\dfrac{0.5772(1+\nu_c)\alpha_c}{\cos^2\phi}\dfrac{(T_u+T_b)}{2.0}\dfrac{ab}{H^2}\left\{\left(\dfrac{a}{b}+\dfrac{b}{a}\right)-\sin\phi\right\}\end{array}\right] \tag{3.21}$$

$$\lambda_3=\frac{930.8875D_c}{\rho Ha^2b^2\cos^2\phi}\frac{A^2}{H^2}\left\{\left(\frac{a}{b}+\frac{b}{a}\right)-\sin\phi\right\}\left\{\frac{b}{a}+\frac{1}{\cos^2\phi}\left(\frac{a}{b}+\frac{b}{a}\sin^2\phi-\sin\phi\right)\right\} \tag{3.22}$$

By dropping the nonlinear terms in Equation (2.24) one obtains the linear frequency (ω_L) of vibration as

$$\omega_L=(\lambda_1)^{1/2} \tag{2.30}$$

3.4.2 Numerical Results, Observations and Discussions

Numerical results are presented in this section to understand the effects of different parameters viz. (i) the nondimensional amplitude $\left(\frac{A}{H}\right)$, (ii) the aspect ratio $\left(\frac{a}{b}\right)$, (iii) the skew angle (ϕ), (iv) the average surface temperature $\frac{(T_u + T_b)}{2}$ and (v) the slenderness parameter $\left(\frac{ab}{H^2}\right)$ etc. on nonlinear free vibrations of triangular plates with clamped immovable edges. Assuming the plates to be made of steel, the coefficient of linear thermal deformations (α) and Poisson's ratio (v) are assumed as $12\times10^{-6}\,per^{o}C$ and 0.3 respectively.

3.4.2.1 The Effects of Nondimensional Amplitude $\left(\frac{A}{H}\right)$

The effects of nondimensional amplitude on relative nonlinear frequency are documented graphically in Figures 3.2(a) to 3.2(d) for various combinations of other parameters. It is observed that triangular plates with clamped immovable edges exhibit strain-hardening type of non-linearity i.e. $\frac{\omega_{NL}}{\omega_L}$ increases with increase of $\frac{A}{H}$. This can be explained in the same way as discussed in case of the circular plates [Section 2.6.1 of Chapter II]. The additional stiffness due to large deflection i.e. stretching of the middle surface increases the nonlinear stiffness and hence, the nonlinear frequency of the plate with increase of amplitude. On the other hand, the linear frequency is independent of amplitude of vibration. This phenomenon explains the strain-hardening behavior of such structures and this behavior is valid within the proportional limit of the structure material. Vendhan, C. P., and Dhoopar, B. L. (1973) obtained similar behavior for particular cases of triangular plates viz. right angled and isosceles triangular plates with clamped edges; results being presented in different form. Other characteristics are discussed in the following sections.

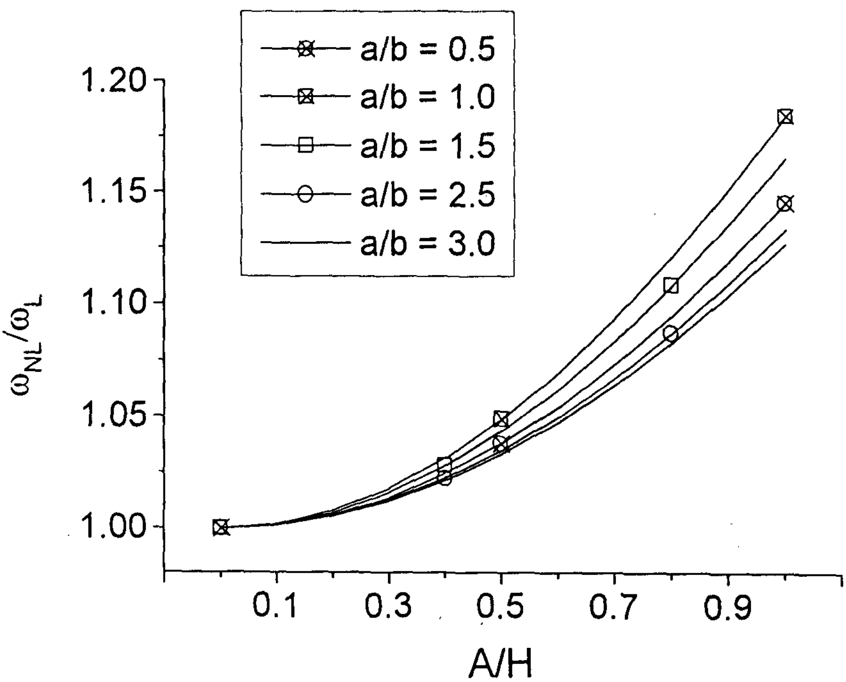

Fig. 3.2(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{A}{H}$ **for different values of** $\frac{a}{b}$ **ratio at** $\frac{ab}{H^2} = 2500.0$,

$\phi = 30^0$ **and** $\frac{T_u + T_b}{2.0} = 50.0^0 C$.

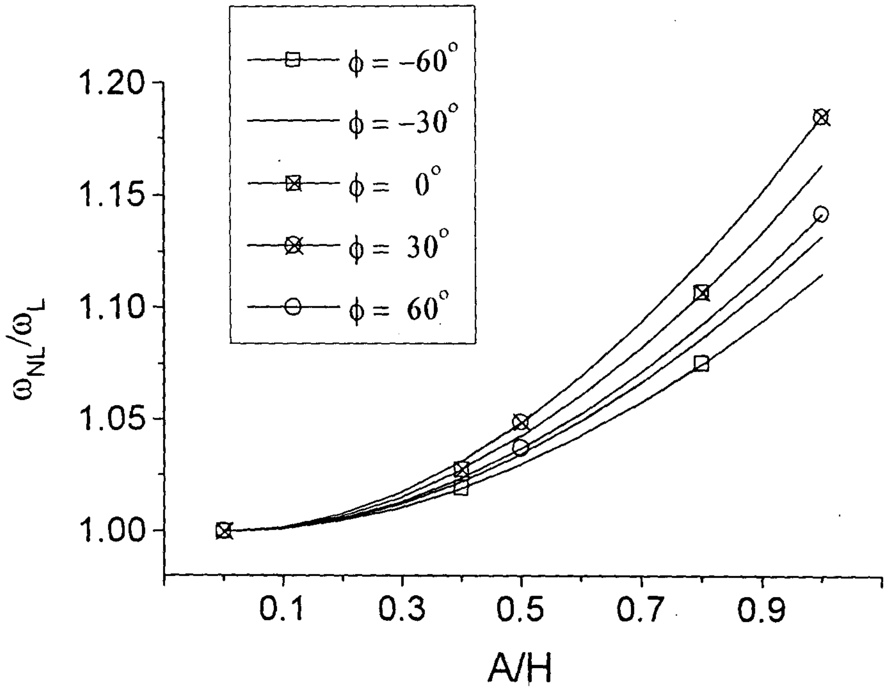

Fig. 3.2(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{A}{H}$ **for different values of** ϕ **at** $\frac{a}{b} = 1.0$,

$\frac{ab}{H^2} = 2500.0$ **and** $\frac{T_u + T_b}{2.0} = 50.0^{\circ}C$.

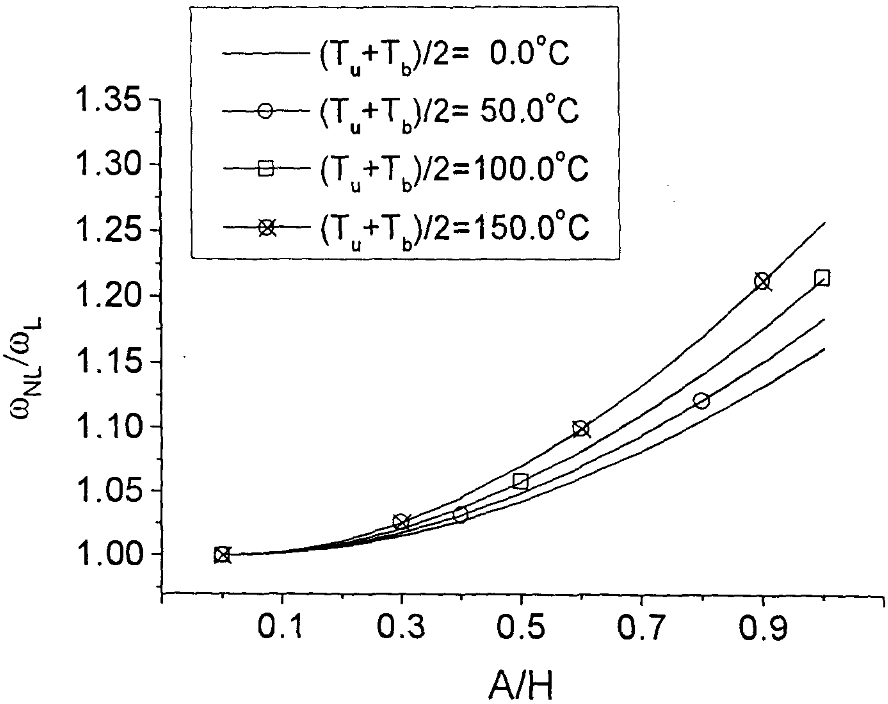

Fig. 3.2(c) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{A}{H}$ **for different values of** $\frac{T_u+T_b}{2.0}$ **at** $\frac{a}{b}=1.0$,

$\frac{ab}{H^2}=2500.0$ **and** $\phi=30^o$.

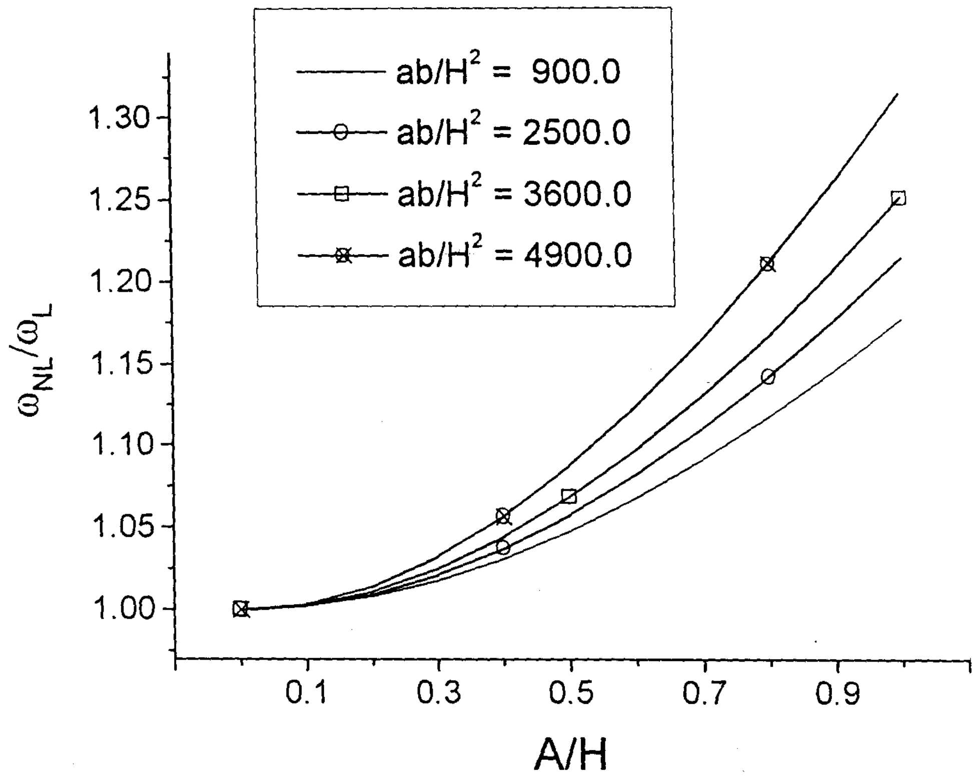

Fig. 3.2(d) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{A}{H}$ **for different values of** $\frac{ab}{H^2}$ **at** $\phi = 30^o$,

$\frac{a}{b} = 1.0$ **and** $\frac{T_u + T_b}{2.0} = 100.0^0 C$.

3.4.2.2 The Effects of Aspect Ratio $\left(\frac{a}{b}\right)$ and Skew Angle (ϕ)

The Figures 3.3(a) to 3.3(c) show the plot of aspect ratio against relative nonlinear frequency and the Figures 3.4(a) to 3.4(c) present graphically skew angle verses relative nonlinear frequency for different combinations of other parameters. Study of these Figures reveals that with increase of aspect ratio/skew angle relative nonlinear frequency increases up to a maximum value and then decreases. This is due to the fact that as the deviation from the geometric configuration of triangular plates having aspect ratio around 1.0 and skew angle around 30^0 increases, any two supports relatively come closer or one of the perpendicular from an apex to the opposite side becomes shorter and as a result the stiffness of the plate increases. In other words, as the deviation from the geometric configuration of triangular plates having aspect ratio around 1.0 and skew angle around 30^0 decreases the stiffness of the plate and hence, the frequency decreases. However, the rate of decrease of linear frequency is more than nonlinear frequency because of the additional stiffness of the plate due to large deflection and the relative nonlinear frequency increases. This phenomenon explains the occurrence of the peak in all these Figures. Such effects are more pronounced at higher values of average surface temperature $\frac{(T_u + T_b)}{2}$ (Figure 3.3(b) and Figure 3.4(b)) and/or slenderness parameter $\left(\frac{ab}{H^2}\right)$ in presence of thermal loading (Figure 3.3(c) and Figure 3.4(c)); which are explained in the subsequent sections. It is important to notice that triangular plates having aspect ratio around 1.0 and skew angle around 30^0 are more susceptible to dynamic instability with or without thermal loading.

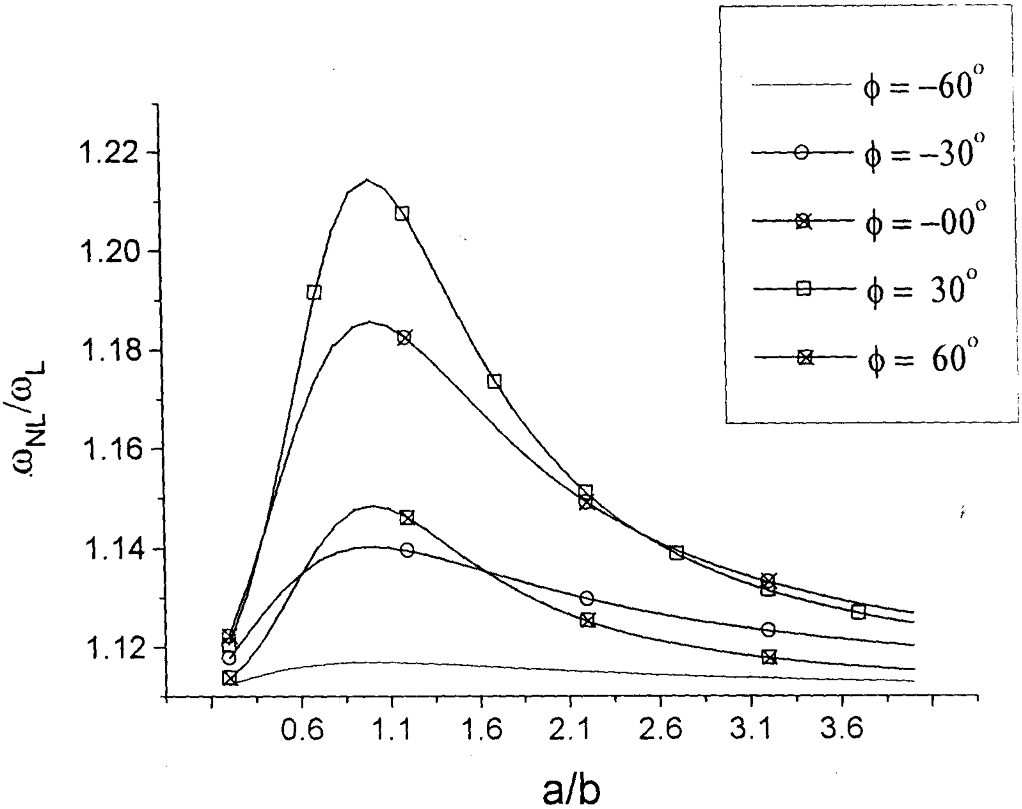

Fig. 3.3(a) $\dfrac{\omega_{NL}}{\omega_L}$ **vs.** $\dfrac{a}{b}$ **for different values of** ϕ **at** $\dfrac{ab}{H^2} = 2500.0$

$\dfrac{A}{H} = 1.0$ **and** $\dfrac{T_u + T_b}{2.0} = 100.0^0 C$.

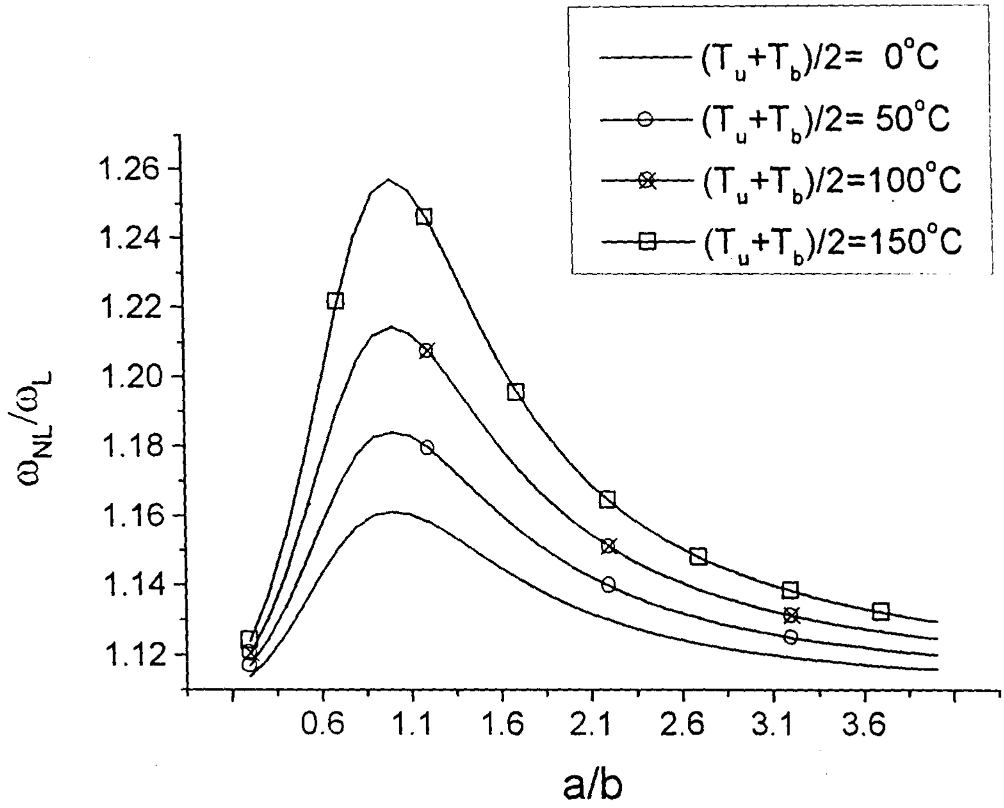

Fig. 3.3(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{a}{b}$ **for different values of** $\frac{T_u + T_b}{2.0}$ **at** $\frac{A}{H} = 1.0$,

$\frac{ab}{H^2} = 2500.0$ **and** $\phi = 30^0$.

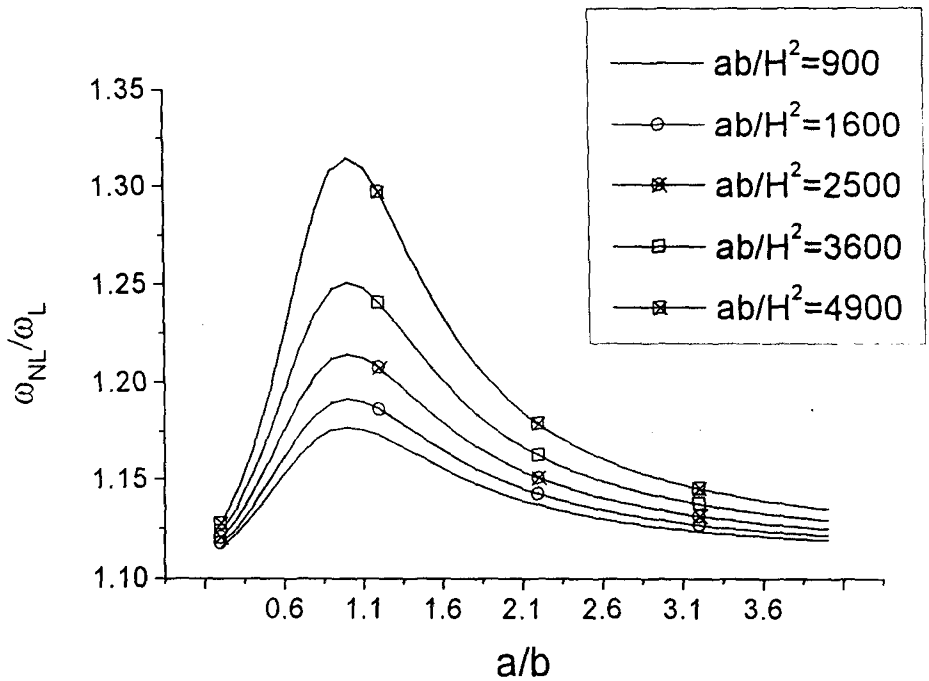

Fig. 3.3(c) $\dfrac{\omega_{NL}}{\omega_L}$ **vs.** $\dfrac{a}{b}$ **for different values of** $\dfrac{ab}{H^2}$ **at** $\dfrac{A}{H} = 1.0$,

$\phi = 30^0$ **and** $\dfrac{T_u + T_b}{2.0} = 100.0^0 C$.

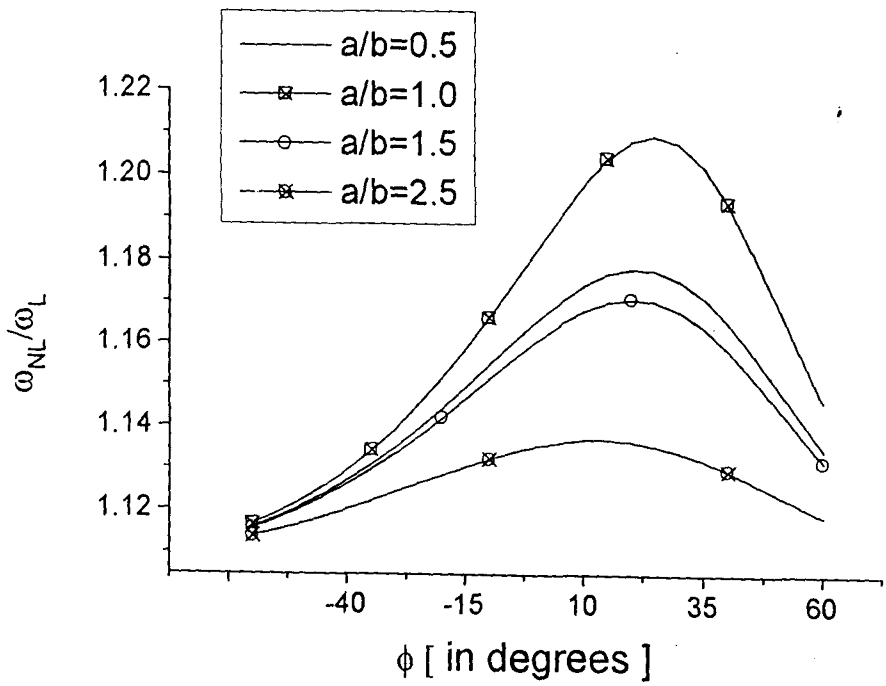

Fig. 3.4(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** ϕ **for different values of** $\frac{a}{b}$ **ratio at** $\frac{A}{H} = 1.0$,

$\frac{ab}{H^2} = 2500.0$ **and** $\frac{T_u + T_b}{2.0} = 100.0^0 C$.

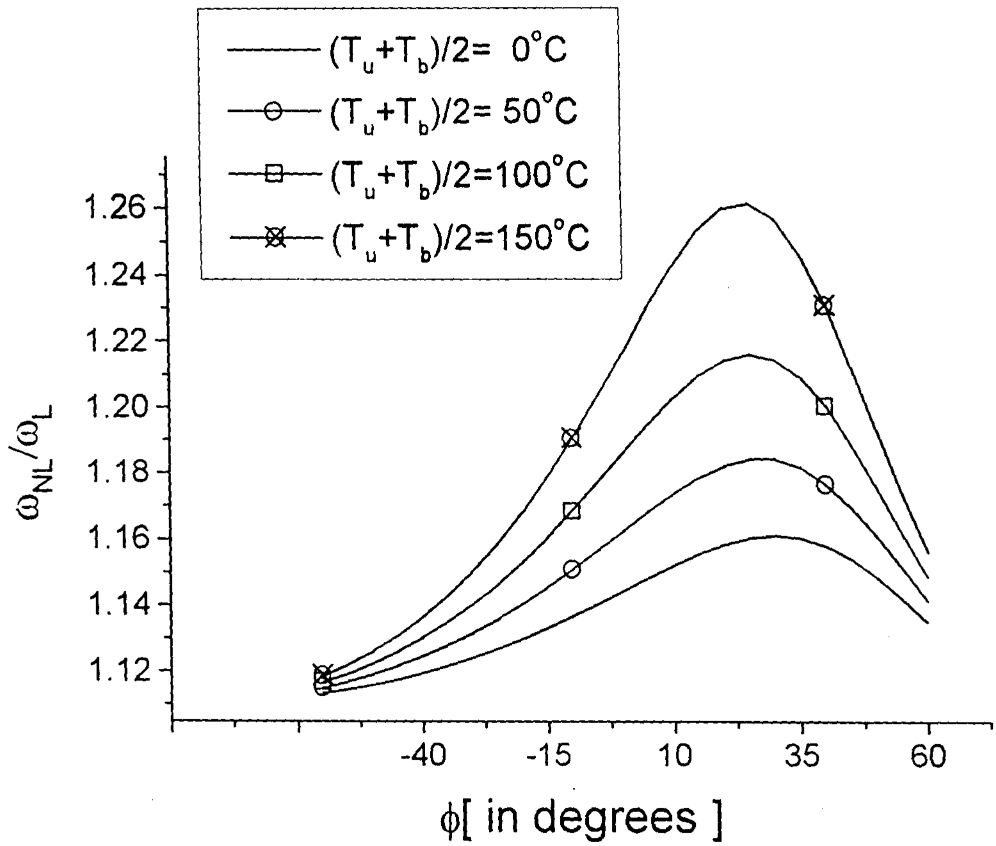

Fig. 3.4(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** ϕ **for different values of** $\frac{T_u + T_b}{2.0}$ **at** $\frac{A}{H} = 1.0$

$\frac{a}{b} = 1.0$ **and** $\frac{ab}{H^2} = 2500.0$.

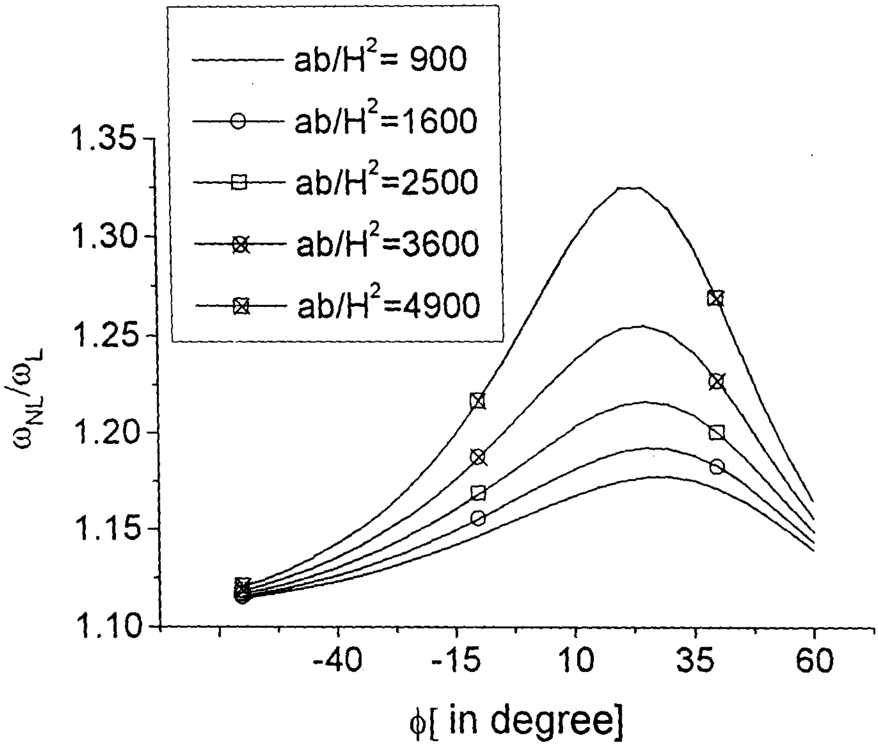

Fig. 3.4(c) $\frac{\omega_{NL}}{\omega_L}$ **vs.** ϕ **for different values of** $\frac{ab}{H^2}$ **at** $\frac{A}{H}=1.0$,

$\frac{a}{b}=1.0$ **and** $\frac{T_u+T_b}{2.0}=100.0^0C$.

3.4.2.4 The Effects of the Thermal Loading

Figures 3.5(a) to 3.5(c) describe pictorially the effects of average surface temperature $\frac{(T_u+T_b)}{2}$ on relative nonlinear frequency. For particular values of $\frac{A}{H}$, $\frac{a}{b}$, ϕ, and $\frac{ab}{H^2}$ it is observed that relative nonlinear frequency increases with increase of $\frac{(T_u+T_b)}{2}$ and finally it becomes infinite. As discussed in case of circular plates of Chapter II the stiffness and hence, the natural frequency, both linear as well as nonlinear, diminishes with increase of average surface temperature $\frac{(T_u+T_b)}{2}$; but the linear stiffness i.e. the linear frequency decreases faster compared to the nonlinear stiffness due to presence of additional stiffness associated with large deflection. After certain stage, the linear stiffness becomes zero leading to thermal instability of the plate as per linear theory and the linear frequency becomes zero and on the other hand the nonlinear stiffness i.e. the nonlinear frequency still remains non-zero due to additional stiffness associated with large deflection. This phenomenon makes the relative nonlinear frequency infinite. So, thermal instability is delayed due to additional stiffness associated with large deflection. Further, the thermal gradient part $\left(\frac{T_u-T_b}{H}z\right)$ does not influence the free vibrations in the transverse direction of the plates as explained in case of circular plate of Chapter II.

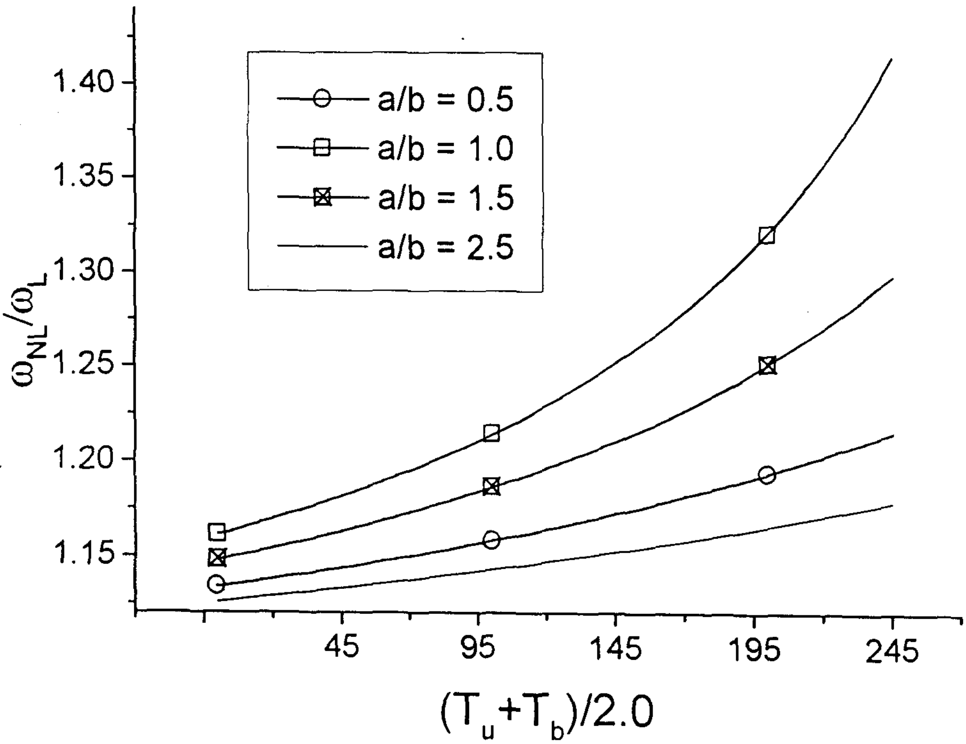

Fig. 3.5(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{T_u + T_b}{2.0}$ **for different values of** $\frac{a}{b}$ **ratio at** $\frac{A}{H} = 1.0$,

$\phi = 30^0$ **and** $\frac{ab}{H^2} = 2500.0$.

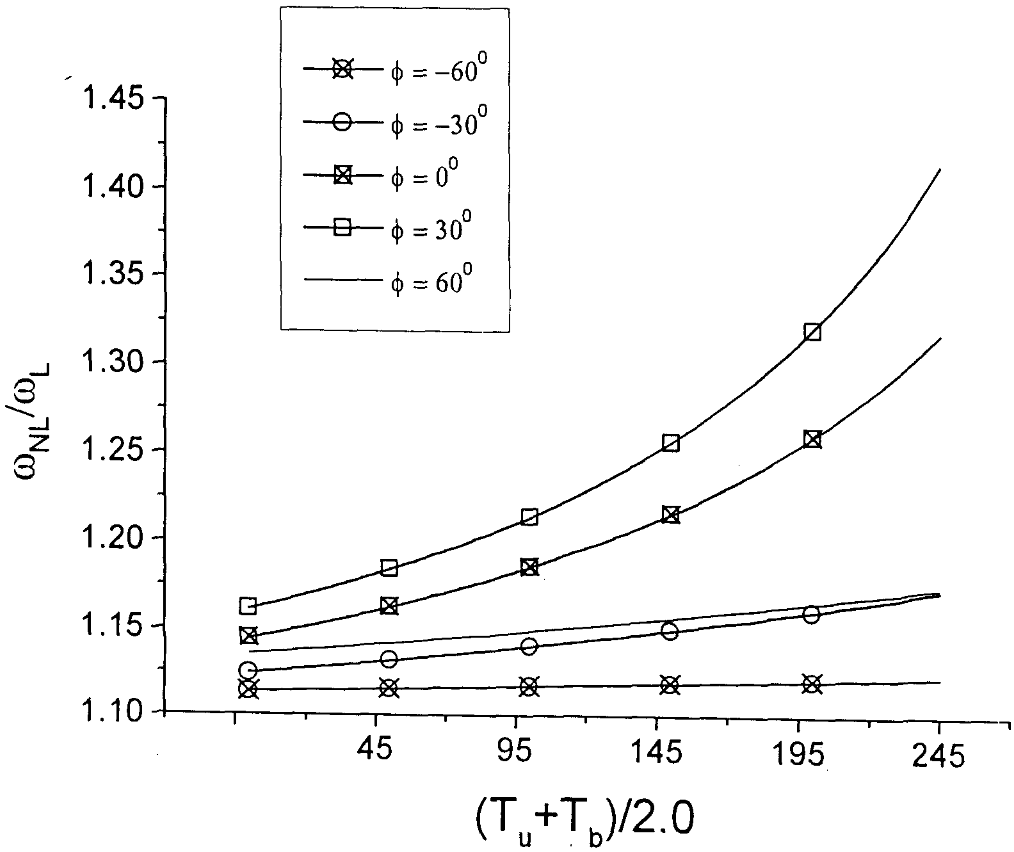

Fig. 3.5(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{T_u + T_b}{2.0}$ **for different values of** ϕ **at** $\frac{A}{H} = 1.0$,

$\frac{a}{b} = 1.0$ **and** $\frac{ab}{H^2} = 2500.0$.

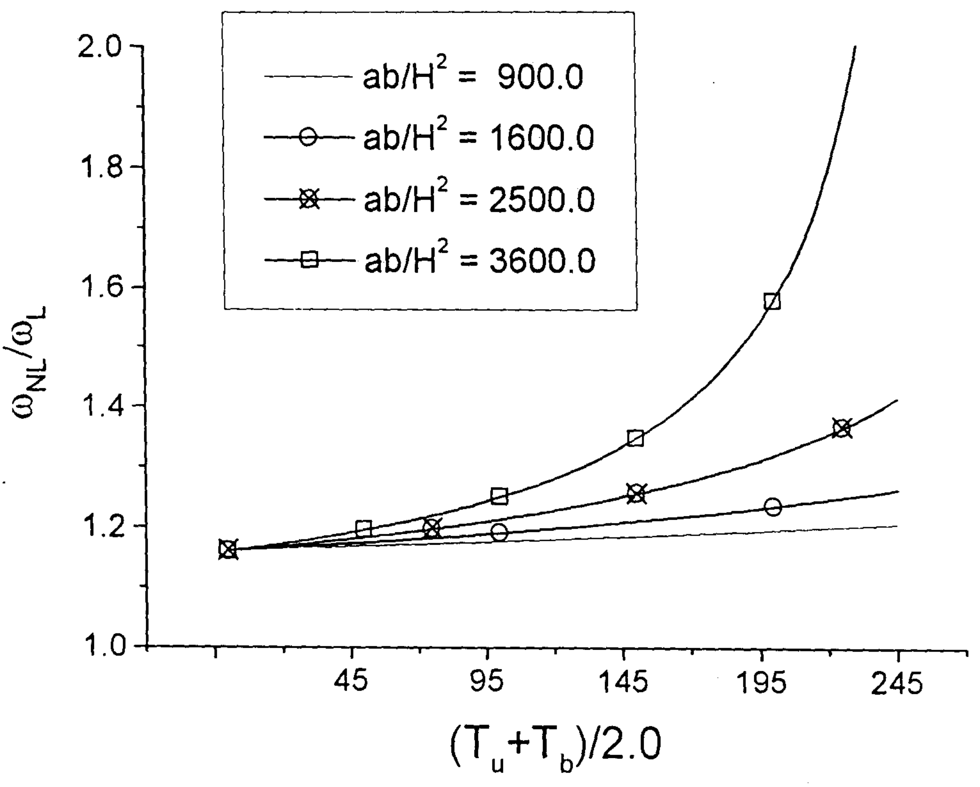

Fig. 3.5(c) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{T_u + T_b}{2.0}$ **for different values of** $\frac{ab}{H^2}$ **at** $\frac{A}{H} = 1.0$,

$\frac{a}{b} = 1.0$ **and** $\phi = 30^o$.

3.4.2.5 The Effects of Slenderness Parameter $\left[\frac{ab}{H^2}\right]$

Fig. 3.6(a) to 3.6(c) furnish the effects of the slenderness parameter $\left(\frac{ab}{H^2}\right)$ on relative nonlinear frequency in presence of thermal loading. For particular values of $\frac{A}{H}$, $\frac{a}{b}$, ϕ and $\frac{(T_u+T_b)}{2}$, the relative nonlinear frequency increases with increase of the slenderness parameter and after certain stage it becomes infinite. This is due to the fact that the linear stiffness i.e. the linear frequency diminishes faster than the nonlinear frequency with increase of the slenderness parameter $\left(\frac{ab}{H^2}\right)$; because additional stiffness due to large deflection as per nonlinear theory. After certain stage, the linear stiffness becomes zero, due to presence of thermal loading, leading to thermal instability of the plate as per linear theory and the linear frequency becomes zero and on the other hand the nonlinear stiffness i.e. the nonlinear frequency still remains non-zero which makes the relative nonlinear frequency infinite.

In absence of thermal loading, besides amplitude(A) of vibrations, plate thickness(H), aspect ratio$\left(\frac{a}{b}\right)$ and skew angle(ϕ) natural frequencies, both linear and nonlinear, also depend on the plate dimensions in the same proportion [Equations (2.27), (2.30), (3.21) and (3.22)] and increase of slenderness ratio may be achieved by increasing plate dimensions at constant plate thickness(H). So, with increase of slenderness ratio natural frequencies, both linear and nonlinear, decrease in the same proportion and become asymptotic around zero; but the relative nonlinear frequency remains constant in absence of thermal loading.

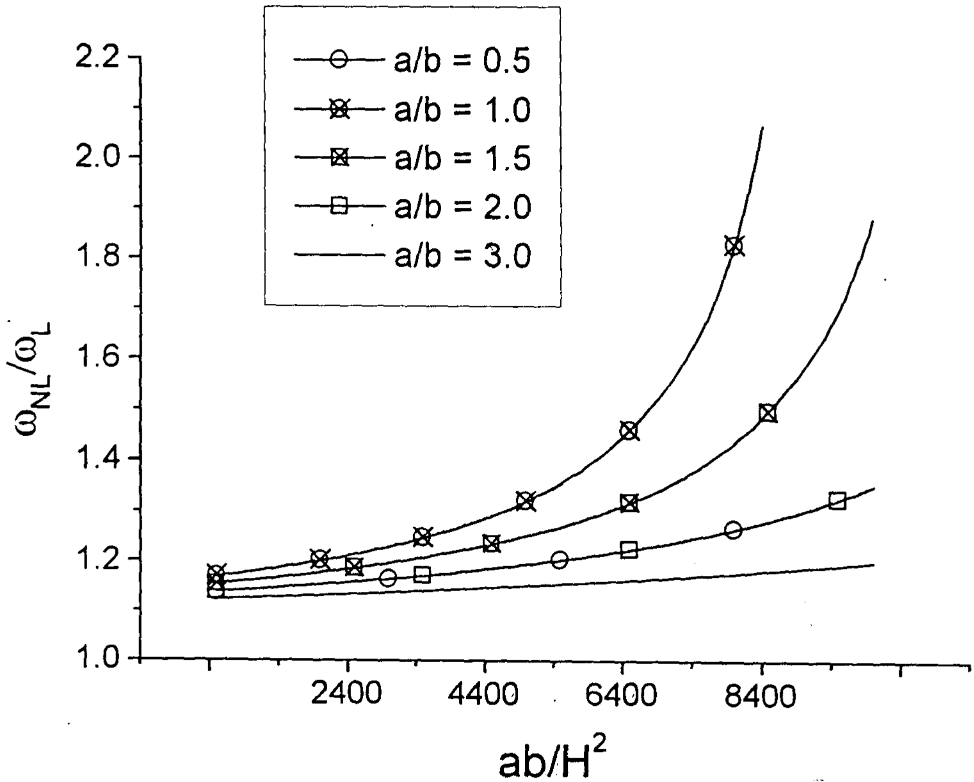

Fig. 3.6(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{ab}{H^2}$ **for different values of** $\frac{a}{b}$ **at** $\frac{A}{H} = 1.0$

$\phi = 30^o$ **and** $\frac{T_u + T_b}{2.0} = 100.0^0 C$.

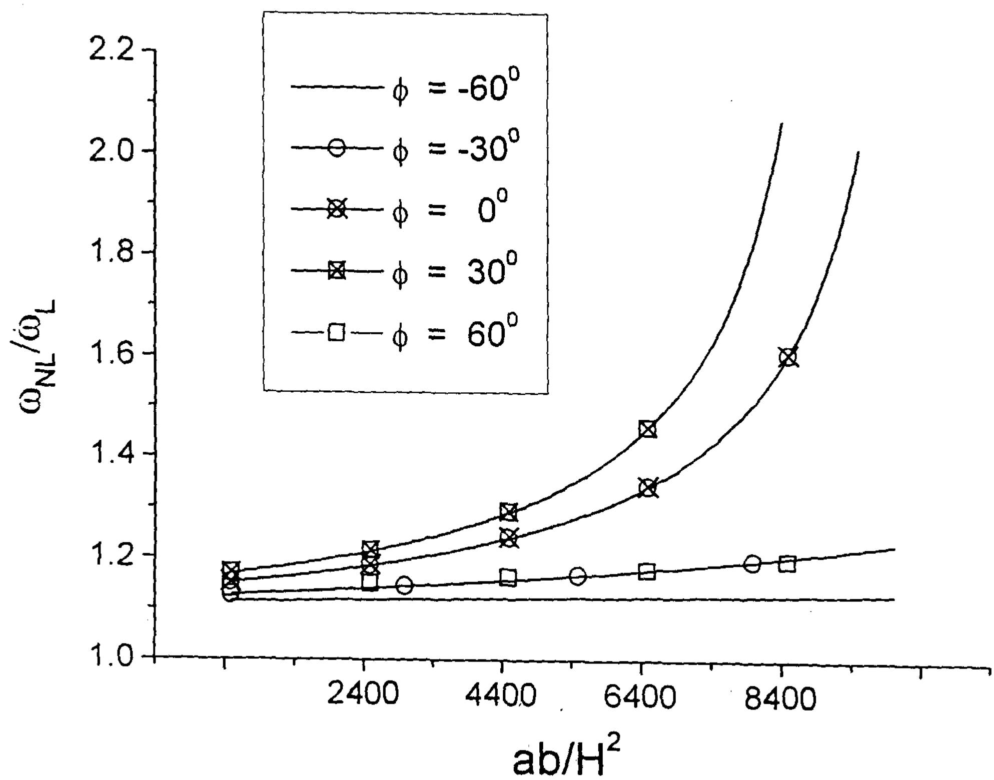

Fig. 3.6(b) $\dfrac{\omega_{NL}}{\omega_L}$ **vs.** $\dfrac{ab}{H^2}$ **for different values of** ϕ **at** $\dfrac{A}{H} = 1.0$

$\dfrac{a}{b} = 1.0$ **and** $\dfrac{T_u + T_b}{2.0} = 100.0^0 C$

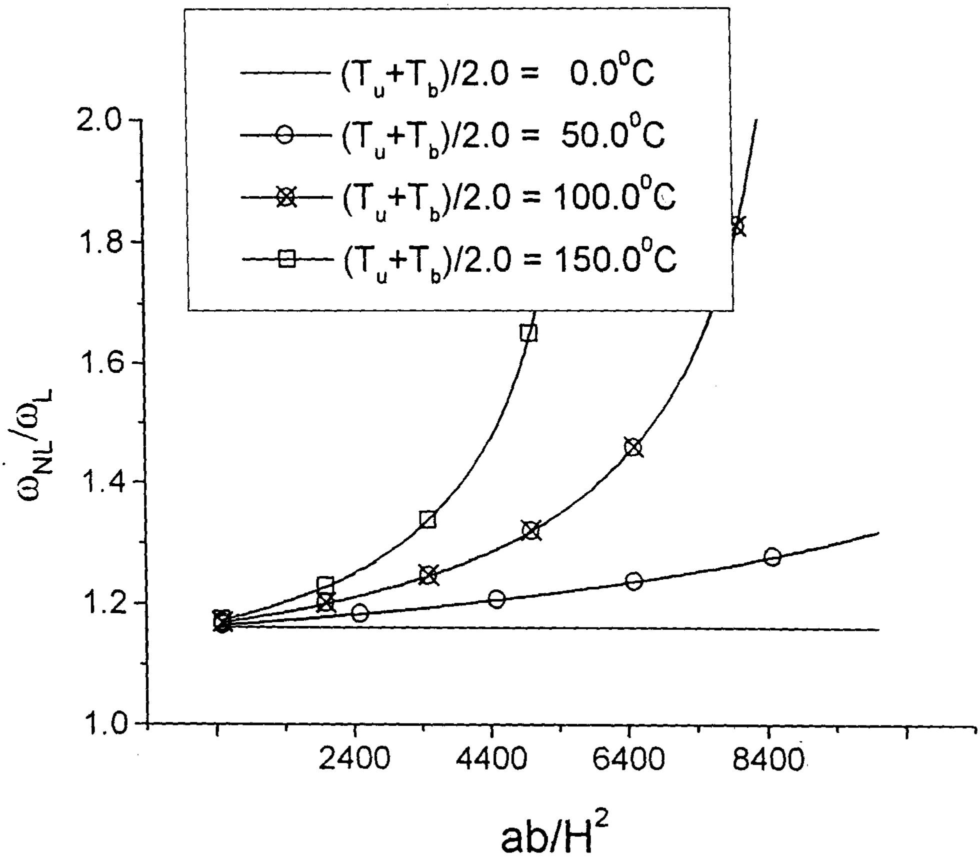

Fig. 3.6(c) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{ab}{H^2}$ **for different values of** $\frac{T_u + T_b}{2.0}$ **at** $\frac{A}{H} = 1.0$

$\frac{a}{b} = 1.0$ **and** $\phi = 30^o$**.**

3.5 Parabolic Plates

A thin isotropic parabolic plate, made of elastic and homogeneous material, is considered as shown in Fig. 3.7 with clamped immovable boundary defined by

$$x^2 = \frac{a^2}{2b}(2b - y), \quad y = 0 \tag{3.23}$$

where 2a and 2b are the base and the axis of the parabola respectively.

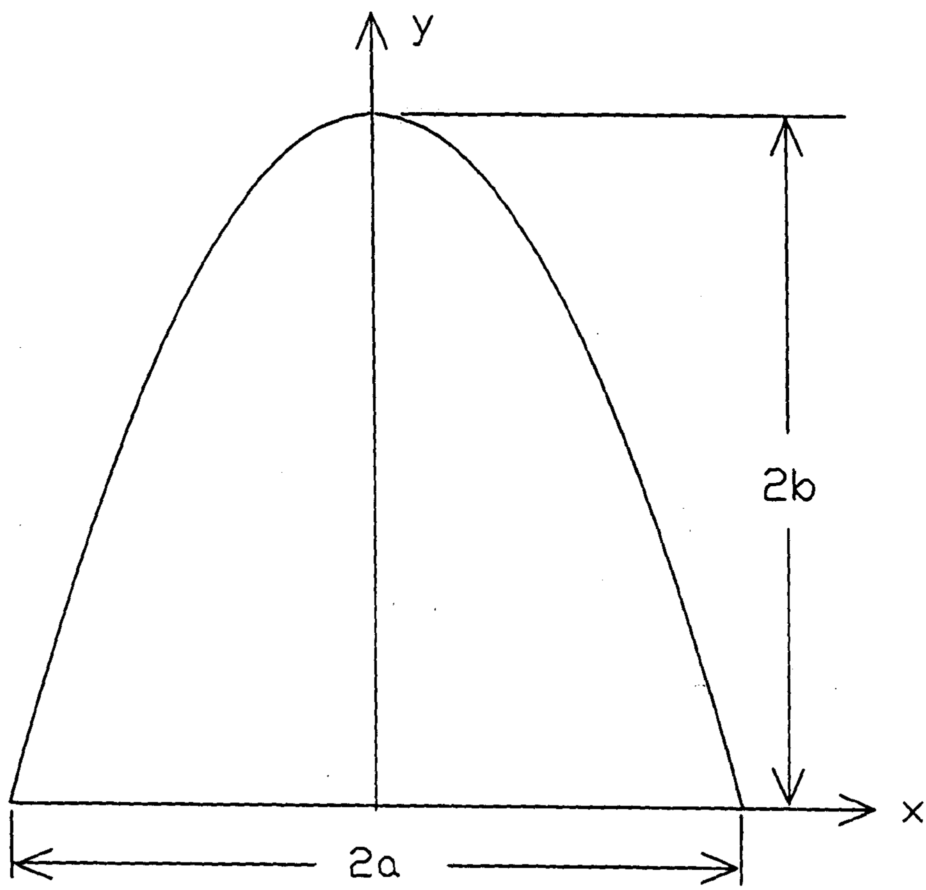

Fig. 3.7 Geometry and coordinate system of the parabolic plate.

3.5.1 Solution

For fundamental mode of vibrations, satisfying clamped immovable boundary conditions, the deflection $w(x, y, t)$ is assumed in the form.

$$w(x,y,t) = A\frac{4y^2}{a^4b^2}\left\{\frac{a^2}{2b}(2b-y)-x^2\right\}^2 F(t) \tag{3.24}$$

Combining Equations (3.13), 3.16) and (3.24) and applying Galerkin's procedure one gets, after necessary integrations, the following equation

$$\begin{aligned} &0.028828a^2b^2\rho H\ddot{F}(t) + \left(4.655687\frac{b^2}{a^2}+1.182107\frac{a^2}{b^2}+1.939867\right)D_cF(t) - \\ &-C_b\left(0.182576\frac{b}{a}+0.0969934\frac{a}{b}\right)abF(t)=0 \end{aligned} \tag{3.25}$$

The in-plane displacements u(x, y, t) and v(x, y, t) may be assumed suitably so that the terms $\int_A u,_x dA$ and $\int_A v,_y dA$ vanish. Now, substituting Equations (3.9), (3.11), (3.15) and (3.24) into Equation (3.8) and integrating the latter over the area of the plate one obtains

$$2.6667abC_b = \left\{\frac{abHD_c}{H^3}N_T^* - \frac{96D_cA^2F^2(t)}{H^2}\left(0.18257576\frac{b}{a}+0.0969934\frac{a}{b}\right)\right\} \tag{3.26}$$

where,

$$N_T^* = \frac{H^3}{D_c}\iint\left[\frac{\alpha_t E_c}{ab(1-v_c)}\left\{\int_{-H/2}^{H/2}\frac{T(x,y,z)}{H}dz\right\}\right]dxdy = 12\times 2.6669(1+v_c)\alpha_t\frac{(T_u+T_b)}{2.0} \tag{3.27}$$

Eliminating C_b from the Equations (3.25) and (3.26) one gets well known time differential cubic equation given by the Equation (2.24) as below

$$\ddot{F}(t) + \lambda_1 F(t) + \lambda_3 F^3(t) = 0 \tag{2.24}$$

where λ_1 and λ_3 are known parameters.

The solution of the Equation (2.24) subjected to initial conditions $F(0) = 0$ and $\dot{F}(0) = 0$ gives the relative nonlinear frequency [Nash and Modeer (1959)] as

$$\frac{\omega_{NL}}{\omega_L} = \left[1 + \frac{\lambda_3}{\lambda_1}\right]^{1/2} \tag{2.31}$$

where

$$\lambda_1 = \frac{\left(4.6557\frac{b^2}{a^2} + 1.182107\frac{a^2}{b^2} + 1.939867\right) - \left(0.18257576\frac{b}{a} + 0.0969934\frac{a}{b}\right)\left(\frac{abN_T^*}{2.6669H^2}\right)}{0.028828\frac{\rho H a^2 b^2}{D_c}} \tag{3.28}$$

$$\lambda_3 = \frac{36\left(0.18257576\frac{b}{a} + 0.0969934\frac{a}{b}\right)^2}{0.028828\frac{\rho H a^2 b^2}{D_c}} \frac{A^2}{H^2} \tag{3.29}$$

By dropping the nonlinear terms in Equation (2.24) on obtains the linear frequency (ω_L) of vibration as

$$\omega_L = (\lambda_1)^{1/2} \tag{2.30}$$

3.5.2 Numerical Results, Observations and Discussion

Numerical results are presented in this section to understand the effects of different parameters viz. (i) the nondimensional amplitude $\left(\frac{A}{H}\right)$, (ii) the aspect ratio $\left(\frac{a}{b}\right)$, (iii) the thermal loading and (iv) the slenderness parameter $\left(\frac{ab}{H^2}\right)$ etc. on dynamic behaviors of parabolic plates with clamped immovable edges. Assuming the plates to be made of steel, the coefficient of linear thermal deformations (α) and Poisson's ratio (ν) are assumed as $12\times10^{-6}\, per^{o}C$ and 0.3 respectively.

3.5.2.1 The Effects of Nondimensional Amplitude $\left(\frac{A}{H}\right)$

Figures 3.8(a) to 3.8(c) present the effects of nondimensional amplitude on relative nonlinear frequency for various combinations of other parameters. It is observed that parabolic plates with clamped immovable edges show strain-hardening type of non-linearity. Such behavior can be explained in the same way as discussed in Section 3.4.2.1. of this Chapter. Similar behavior is obtained by Biswas, P. and Kapoor, P.(1986c) for the particular case $\frac{a}{b}=1.0$ only. The effects of other parameters are explained in the subsequent sections.

3.5.2.3 The Effects of aspect ratio $\left(\frac{a}{b}\right)$

Figures 3.9(a) and 3.9(b) describe the influence of aspect ratio on relative nonlinear frequency for different combinations of other parameters. For particular values of $\frac{A}{H}$, $\frac{ab}{H^2}$ and $\frac{(T_u + T_b)}{2}$ it is observed that with increase of aspect ratio relative nonlinear frequency increases up to a maximum around the value of aspect ratio 1.4 and then decreases. This is due to the fact that as the deviation from the geometric configuration of a parabolic plate having aspect ratio around 1.4 increases, opposite supports relatively come closer and as a result the stiffness of the plate in the transverse direction increases. In other words, as the deviation from the geometric configuration of a parabolic plate having aspect ratio around 1.4 decreases, the stiffness of the plate and hence the frequency decreases. However, the rate of decrease of linear frequency is more than nonlinear frequency because of the additional stiffness associated with large deflection. This effect is more pronounced at higher values of average surface temperature $\frac{(T_u + T_b)}{2}$ [Figure 3.9(a)] and/or slenderness parameter $\left(\frac{ab}{H^2}\right)$ [Figure 3.9(b)]; which are discussed in the subsequent sections. A parabolic plate having aspect ratio around 1.4 is more prone to develop dynamic instability with or without thermal loading due to the reasons discussed above.

3.5.2.3 The Effects of average surface temperature $\left[\frac{T_u + T_b}{2}\right]$

Figures 3.10(a) and 3.10(b) show the effects of average surface temperature $\frac{(T_u + T_b)}{2}$ on relative nonlinear frequency for various combinations of other parameters. It is learned that the parabolic plates behave like circular/triangular plates and relative nonlinear frequency increases with increase of $\frac{(T_u + T_b)}{2}$ and finally it becomes infinite. The same can be explained similarly as discussed in case of the triangular plates [Section 3.4.2.3].

3.5.2.4 The Effects of Slenderness Parameter $\left(\frac{ab}{H^2}\right)$

Figures (3.11a) and (3.11b) furnish the effects of slenderness factor $\left(\frac{ab}{H^2}\right)$ on relative nonlinear frequency. The relative nonlinear frequency increases with increase of slenderness factor and after certain stage it becomes infinite. It is observed that in absence of thermal loading slenderness factor does not affect the relative nonlinear frequency. These behaviors are again similar to those of the triangular plates and the reasons for such behavior are explained in the Section 3.4.2.4.

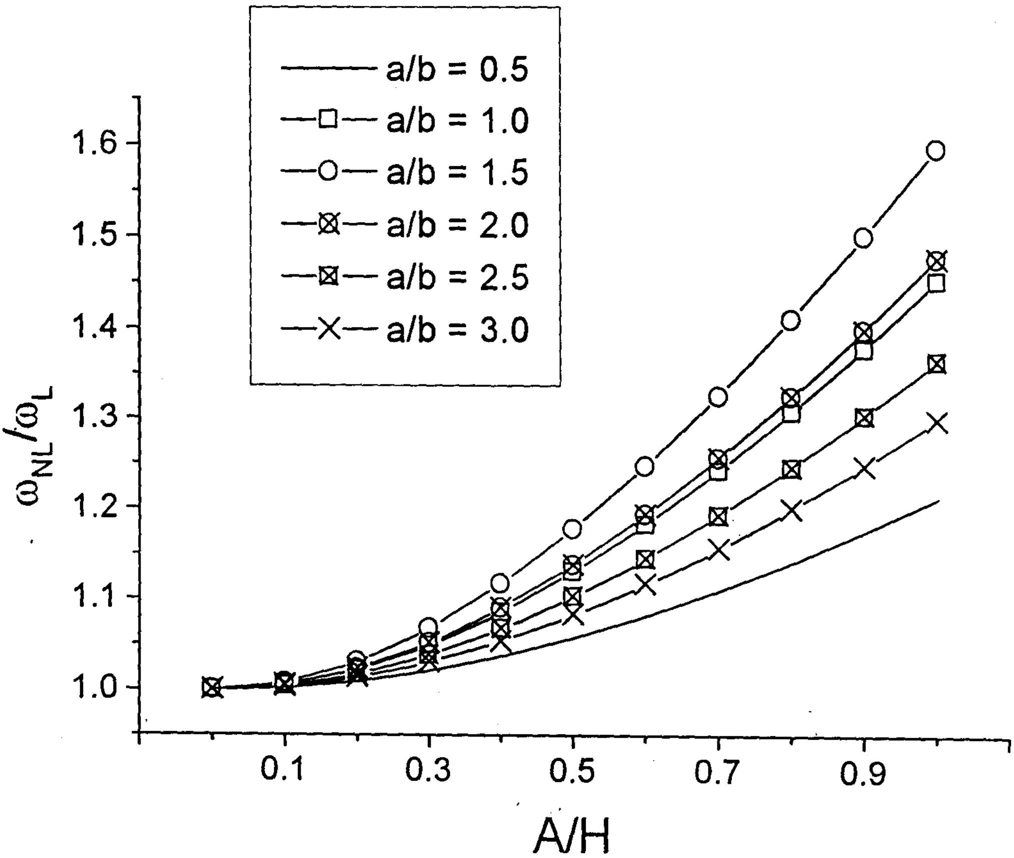

Fig. 3.8(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{A}{H}$ **for different values of** $\frac{a}{b}$ **ratio at** $\frac{ab}{H^2} = 2500.0$

and $\frac{T_u + T_b}{2} = 40.0^{\circ}C$.

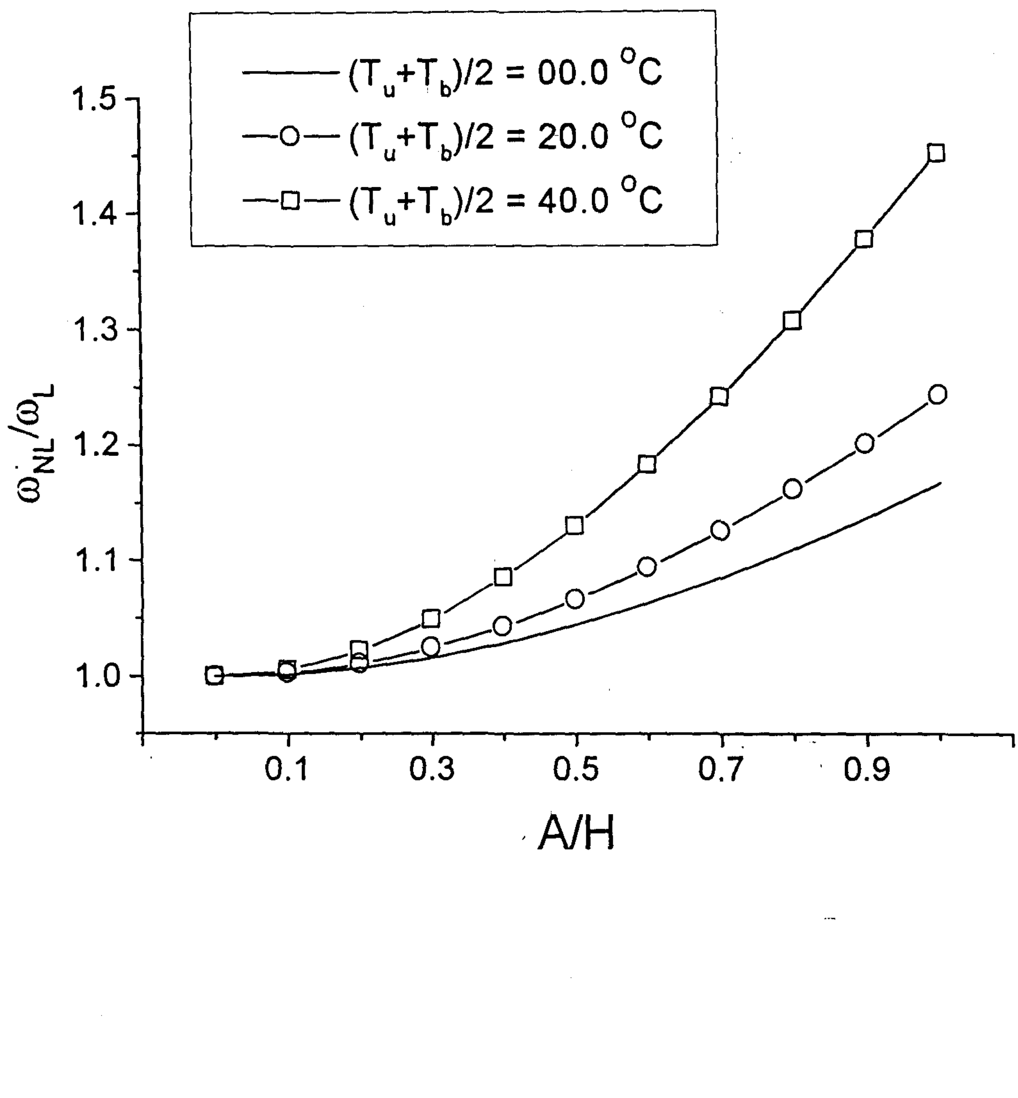

Fig. 3.8(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{A}{H}$ **for different values of** N_T^* **ratio at** $\frac{a}{b}=1$

And $\frac{ab}{H^2}=2500.0$.

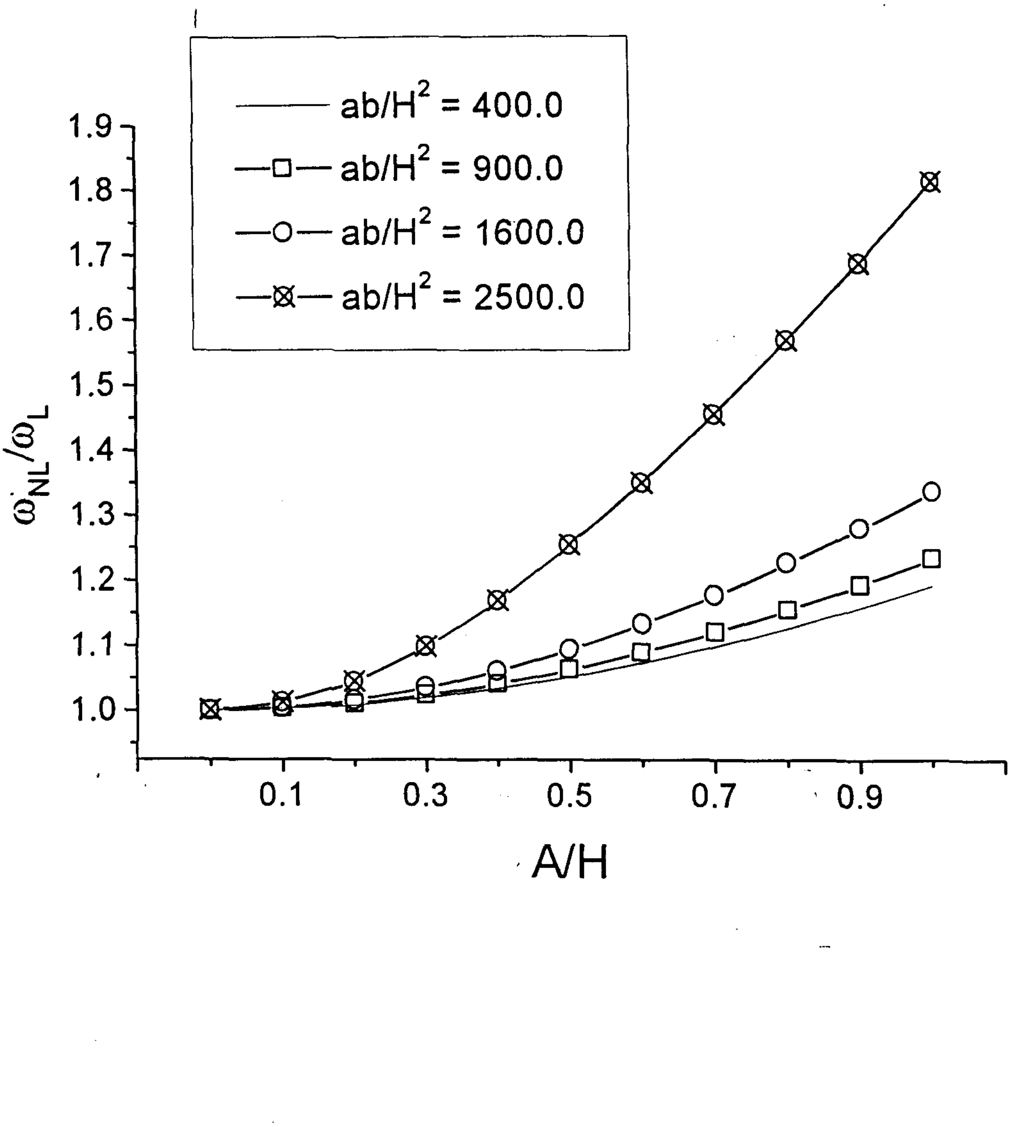

Fig. 3.8 (c) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{A}{H}$ **for different values of** $\frac{ab}{H^2}$ **at** $\frac{a}{b} = 1.0$

and $\frac{T_u + T_b}{2} = 50.0^0 C$.

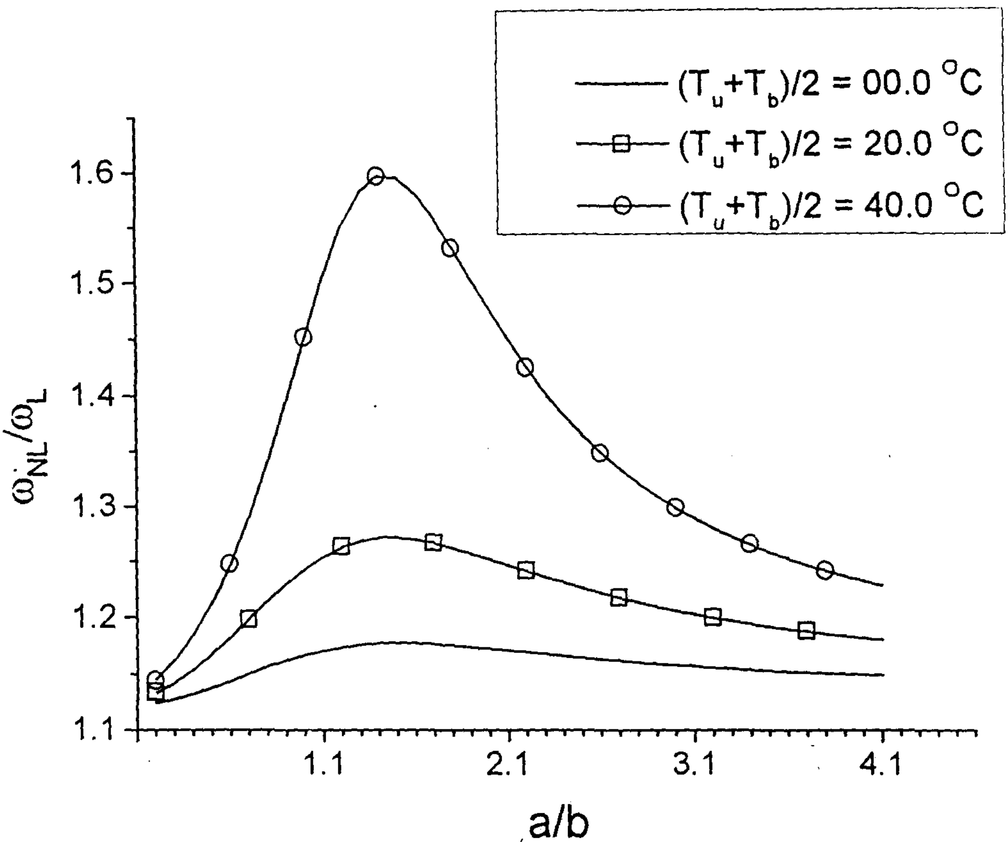

Fig. 3.9(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{a}{b}$ **ratio for different values of** $\frac{T_u+T_b}{2}$ **at** $\frac{A}{H}=1.0$

and $\frac{ab}{H^2}=2500.0$.

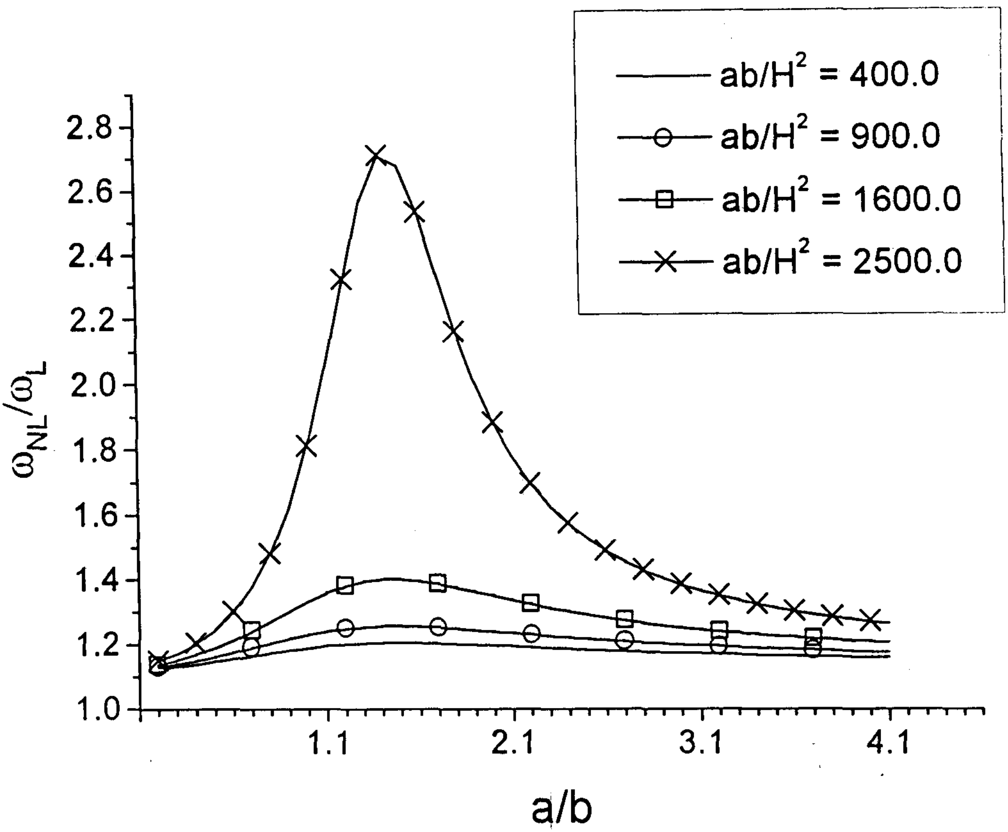

Fig. 3.9(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{a}{b}$ **ratio for different values of** $\frac{ab}{H^2}$ **at** $\frac{A}{H} = 1.0$

and $\frac{T_u + T_b}{2} = 50.0^0 C$.

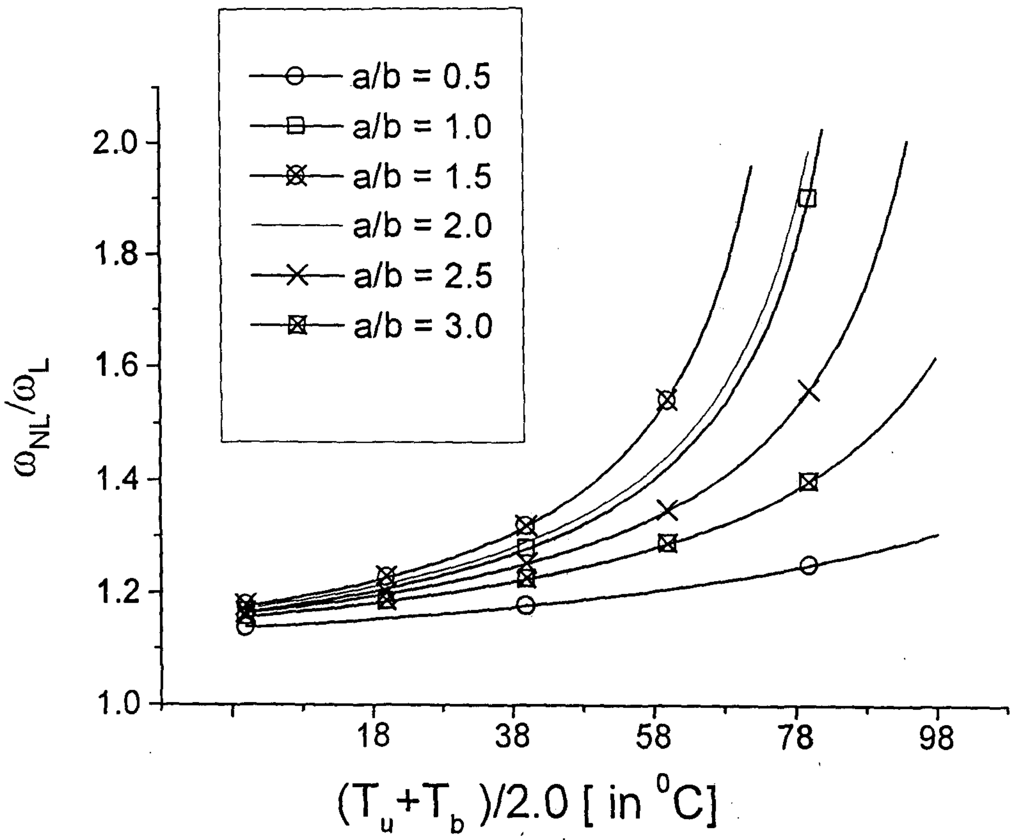

Fig. 3.10(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{(T_u + T_b)}{2.0}$ **for different values of** $\frac{a}{b}$ **ratio at** $\frac{A}{H} = 1.0$ **and** $\frac{ab}{H^2} = 1600.0$.

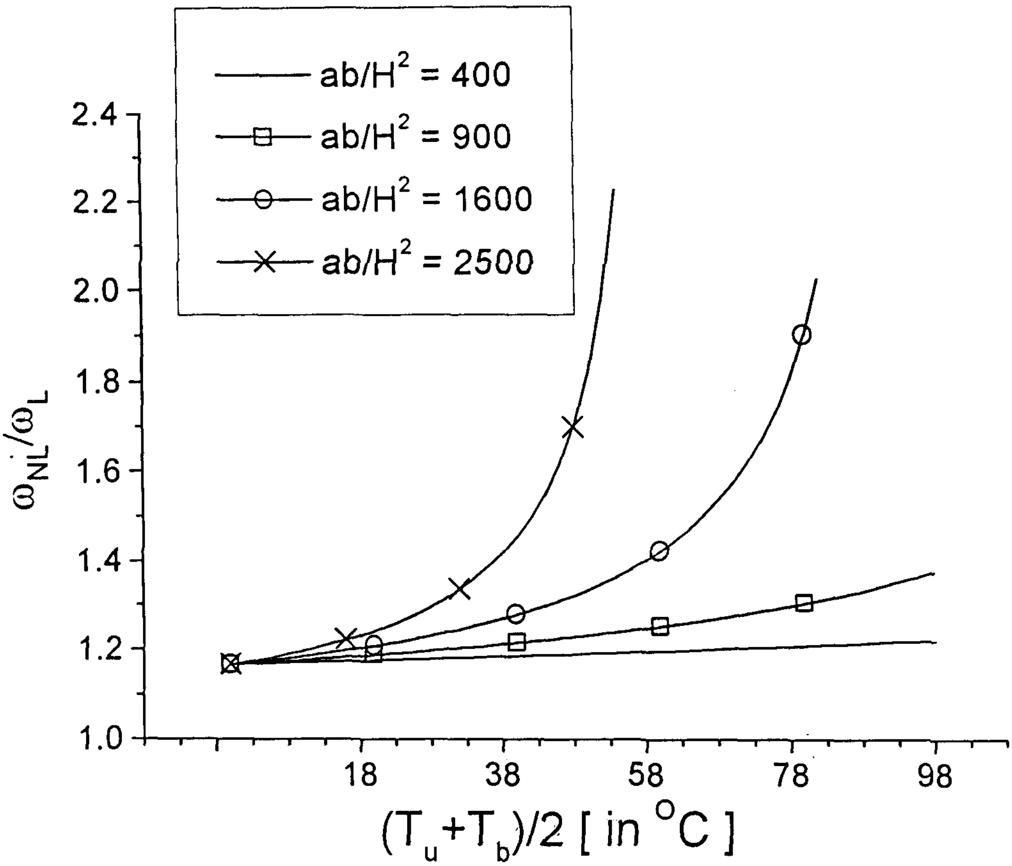

Fig. 3.10(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{(T_u + T_b)}{2}$ **for different values of** $\frac{ab}{H^2}$ **ratio at** $\frac{A}{H} = 1.0$ **and** $\frac{a}{b} = 1.0$.

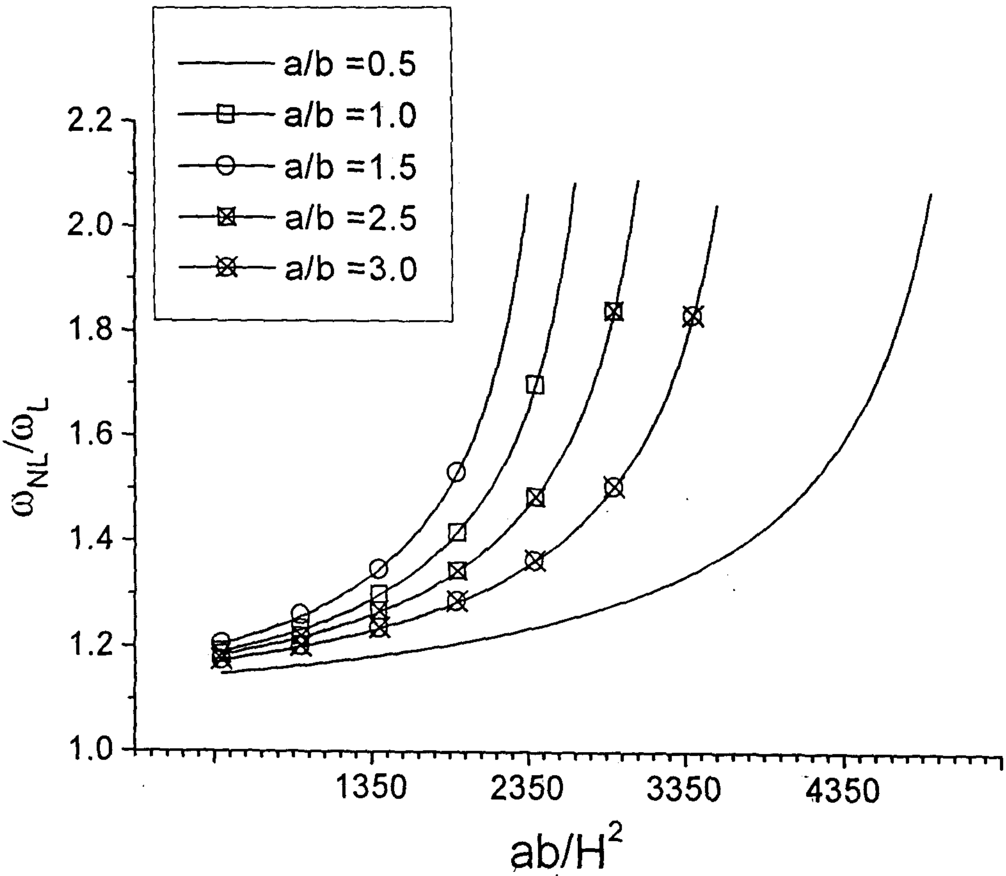

Fig. 3.11(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{ab}{H^2}$ **for different values of** $\frac{a}{b}$ **ratio at** $\frac{A}{H} = 1.0$

and $\frac{T_a + T_b}{2} = 50^0 C$.

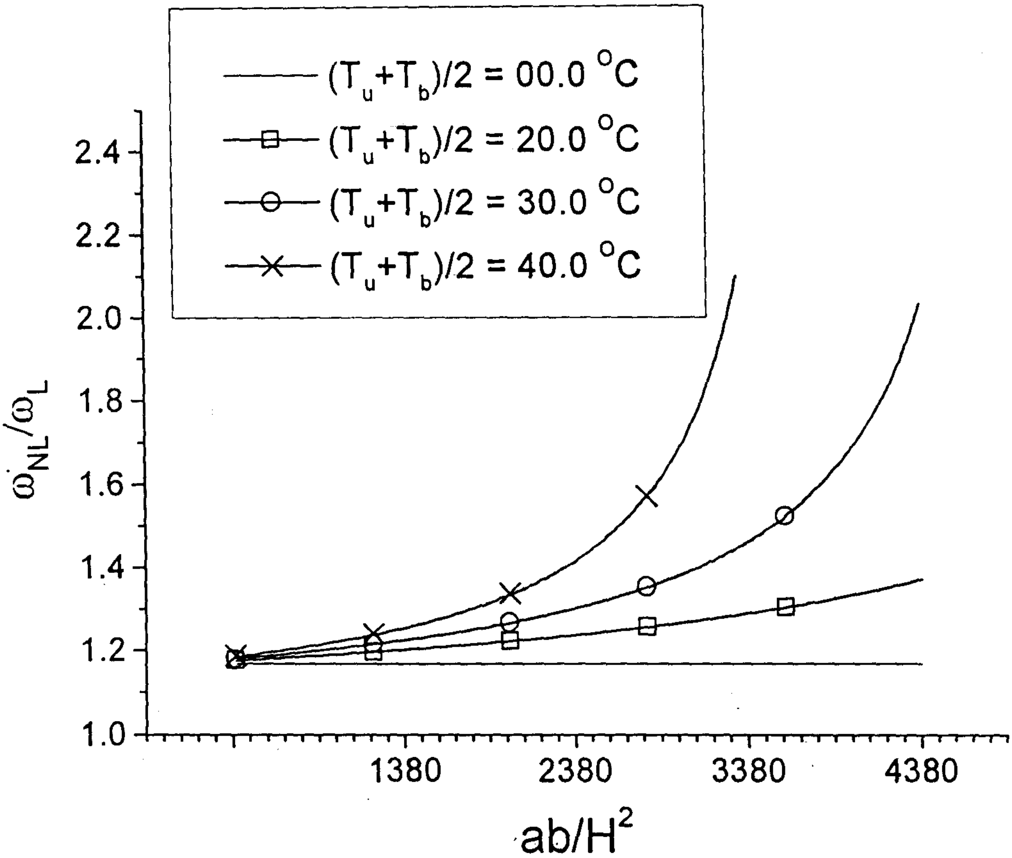

Fig. 3.11(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{ab}{H^2}$ **for different values of** $\frac{(T_u+T_b)}{2}$ **at** $\frac{A}{H} = 1.0$ **and** $\frac{a}{b} = 1.0$.

CHAPTER IV

NONLINEAR ANALYSIS OF NONHOMOGENEOUS THIN CIRCULAR ELASTIC PLATES UNDER THERMAL LOADING

4.1 Introduction

The thermal and mechanical properties of materials viz. specific heat, thermal conductivity, density, coefficient of thermal deformations, modulus of elasticity etc. of most engineering materials are temperature-dependent. Steel, Titanium alloy etc. are very common examples of such materials. The thermal and mechanical properties of materials become functions of space variables in presence temperature field with thermal gradient. A plate made of such engineering materials with temperature-dependent material properties becomes nonhomogeneous in presence temperature field with thermal gradient.

Literature survey shows that many investigations of dynamic behavior of plate and shell structures subjected to thermal loading considering temperature-dependent elastic coefficients of material have been carried out on the basis of linear theory. However, very few investigations in this direction, based on large deflection theory, have been reported in the literature as mentioned in Section 1.2 of Chapter I.

This chapter intends to study the nonlinear dynamics of thin circular elastic plates under thermal loading considering nonhomogeneity arising due to temperature dependency of material properties such as thermal conductivity, modulus of elasticity, Poisson's ratio and coefficient of thermal deformations under thermal loading with temperature gradient along the thickness direction. The basic governing differential equations have been derived in the von Karman sense in terms of displacement components and solved with the help of Galerkin procedure. Further, to solve the steady-state heat conduction problem temperature dependency of thermal conductivity has been considered. Parametric studies have been presented to understand the influences of different parameters.

4.2 Temperature Distribution

A thin circular plate of radius R and uniform thickness H, having clamped immovable edges, is considered. The origin of the polar coordinate system is located at the center of the middle surface of the plate. The z-coordinate is normal to the middle surface of the plate. The plate is made of elastic, isotropic and homogeneous material. The material properties viz. the thermal conductivity, the modulus of elasticity, the Poison's ratio and the coefficient of thermal linear deformation are considered to be dependent of temperature.

A steady-state thermal field is considered. The upper surface and the lower surface of the plate are subjected to constant temperatures denoted by T_u and T_b respectively. Heat loss through the edges of the plate is assumed negligible. So, heat flow will takes place along the thickness direction i.e. z direction only. The modulus of elasticity, the Poison's ratio and the coefficient of thermal linear deformation are represented by $E\{T(z)\}$, $\nu\{T(z)\}$ and $\alpha\{T(z)\}$ respectively.

The steady-state one dimensional heat conduction equation for the plate made of material having temperature-dependent thermal conductivity $k\{T(z)\}$ may be written [Sachdeva, R. C., 1988] as

$$\frac{\partial}{\partial z}\left[k\{T(z)\}\frac{\partial T(z)}{\partial z}\right]=0 \qquad \left[-\frac{H}{2}\leq z\leq\frac{H}{2}\right] \tag{4.1}$$

Normally, for limited ranges of temperature it is sufficiently accurate to use the linear or quadratic expression for $k\{T(z)\}$ and accordingly, represented by

(i) $k\{T(z)\}=k_0+k_1T(z)$ (4.2)

(ii) $k\{T(z)\}=k_0+k_1T(z)+k_2T(z)$ (4.3)

where, k_0 is a constant; k_1 and k_2 are the temperature coefficients of thermal conductivity.

Case – I: Linear variation of thermal conductivity with temperature

Substituting equation (4.2) the steady-state heat conduction equation (4.1) becomes

$$\frac{\partial}{\partial z}\left[(k_0+k_1T)\frac{\partial T}{\partial z}\right]=0 \tag{4.4}$$

The integration of equation (4.4) twice with respect to z coordinate yield

$$\left[k_0T+k_1\frac{T^2}{2}\right]=C_1^Tz+C_2^T \tag{4.5}$$

The constants C_1^T and C_2^T are to be determined from the thermal boundary conditions of the plate (i) $T=T_u$ at $z=\frac{H}{2}$ and (ii) $T=T_b$ at $z=-\frac{H}{2}$ as

$$C_1^T = \frac{T_u - T_b}{H} k_0 + \frac{T_u^2 - T_b^2}{2H} k_1 \tag{4.6}$$

$$C_2^T = \frac{T_u + T_b}{2} k_0 + \frac{T_u^2 + T_b^2}{4} k_1 \tag{4.7}$$

The solution of the quadratic equation (4.5) gives the temperature field as

$$T(z) = -\frac{k_0}{k_1} + \sqrt{\frac{\left(T_u + \frac{k_0}{k_1}\right)^2 - \left(T_b + \frac{k_0}{k_1}\right)^2}{H} z + \frac{\left(T_u + \frac{k_0}{k_1}\right)^2 + \left(T_b + \frac{k_0}{k_1}\right)^2}{2}} \tag{4.8}$$

Case – II: Quadratic variation of thermal conductivity with temperature

Substituting equation (4.3) the steady-state heat conduction equation (4.1) becomes

$$\frac{\partial}{\partial z}\left[\left(k_0 + k_1 T + k_2 T^2\right)\frac{\partial T}{\partial z}\right] = 0 \tag{4.9}$$

The integration of equation (4.9) twice with respect to z coordinate yield

$$k_0 T + k_1 \frac{T^2}{2} + k_2 \frac{T^3}{3} = C_1^T z + C_2^T \tag{4.10}$$

The constants C_1^T and C_2^T are to be determined from the thermal boundary conditions of the plate (i) $T = T_u$ at $z = \frac{H}{2}$ and (ii) $T = T_b$ at $z = -\frac{H}{2}$ as

$$C_1^T = \frac{1}{6H}\left\{6k_0\left(T_u - T_b\right) + 3k_1\left(T_u^2 - T_b^2\right) + 2k_2\left(T_u^3 - T_b^3\right)\right\} \tag{4.11}$$

$$C_2^T = \frac{1}{12}\left\{6k_0\left(T_u + T_b\right) + 3k_1\left(T_u^2 + T_b^2\right) + 2k_2\left(T_u^3 + T_b^3\right)\right\} \tag{4.12}$$

Finally, the equation (4.10) turns to

$$T^3 + s_2 T^2 + s_1 T + s_0 = 0 \tag{4.13}$$

where

$$s_0 = \frac{3\left(C_1^T z + C_2^T\right)}{k_2} \tag{4.14}$$

$$s_1 = \frac{3k_0}{k_2} \tag{4.15}$$

$$s_2 = \frac{3k_1}{2k_2} \tag{4.16}$$

The temperature field is represented by the solution of the cubic equation (4.13). The solution procedure of the equation (4.13) is described in Appendix 4-A at the end of this Chapter.

4.3 Governing Differential Equations

The strain-displacement relationships at any point at a distance z from the middle surface of such plate are given by equations (2.9) of Chapter II and those are reproduced here for convenience.

$$\varepsilon_{rr} = u,_r + \frac{1}{2} w,_r^2 - z w,_{rr} \tag{2.9a}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} - z\frac{w,_r}{r} \tag{2.9b}$$

$$\varepsilon_{r\theta} = 0 \tag{2.9c}$$

The stresses at a distance z from the middle surface of the plate subjected to thermal loading are:

$$\sigma_{rr} = \frac{E\{T(z)\}}{1-[\nu\{T(z)\}]^2}\left[\left(\varepsilon_{rr} - \int_{T_0}^{T} \alpha\{T(z)\}dT\right) + \nu\{T(z)\}\left(\varepsilon_{\theta\theta} - \int_{T_0}^{T} \alpha\{T(z)\}dT\right)\right] \tag{4.17a}$$

$$\sigma_{\theta\theta} = \frac{E\{T(z)\}}{1-[\nu\{T(z)\}]^2}\left[\left(\varepsilon_{\theta\theta} - \int_{T_0}^{T}\alpha\{T(z)\}dT\right) + \nu\{T(z)\}\left(\varepsilon_{rr} - \int_{T_0}^{T}\alpha\{T(z)\}dT\right)\right] \tag{4.17b}$$

$$\sigma_{r\theta} = 0 \tag{4.17c}$$

The equations (2.9) are substituted in equations (4.17) and the resulting equations are integrated with respect to z coordinate to yield the stress resultants and the stress couples:

$$N_{rr} = HE_1\left(u,_r + \frac{1}{2}w,_r^2\right) - H^2E_2w,_{rr} + HE_4\frac{u}{r} - H^2E_5\frac{w,_r}{r} - HN_T \tag{4.18a}$$

$$N_{\theta\theta} = HE_1\frac{u}{r} - H^2E_2\frac{w,_r}{r} + HE_4\left(u,_r + \frac{1}{2}w,_r^2\right) - H^2E_5w,_{rr} - HN_T \tag{4.18b}$$

$$M_{rr} = H^2E_2\left(u,_r + \frac{1}{2}w,_r^2\right) - H^3E_3w,_{rr} + HE_5\frac{u}{r} - H^3E_6\frac{w,_r}{r} - H^2M_T \tag{4.19a}$$

$$M_{\theta\theta} = H^2E_2\frac{u}{r} - H^3E_3\frac{w,_r}{r} + H^2E_5\left(u,_r + \frac{1}{2}w,_r^2\right) - H^3E_6w,_{rr} - H^2M_T \tag{4.19b}$$

where

$$E_1 = \frac{1}{H}\int_{-H/2}^{H/2}\left[\frac{E\{T(z)\}}{1-[\nu\{T(z)\}]^2}\right]dz \tag{4.20a}$$

$$E_2 = \frac{1}{H^2}\int_{-H/2}^{H/2}\left[\frac{E\{T(z)\}}{1-[\nu\{T(z)\}]^2}\right]zdz \tag{4.20b}$$

$$E_3 = \frac{1}{H^3}\int_{-H/2}^{H/2}\left[\frac{E\{T(z)\}}{1-[\nu\{T(z)\}]^2}\right]z^2dz \tag{4.20c}$$

$$E_4 = \frac{1}{H}\int_{-H/2}^{H/2}\left[\frac{E\{T(z)\}\nu\{T(z)\}}{1-[\nu\{T(z)\}]^2}\right]dz \tag{4.20d}$$

$$E_5 = \frac{1}{H^2}\int_{-H/2}^{H/2}\left[\frac{E\{T(z)\}\nu\{T(z)\}}{1-[\nu\{T(z)\}]^2}\right]zdz \tag{4.20e}$$

$$E_6 = \frac{1}{H^3}\int_{-H/2}^{H/2}\left[\frac{E\{T(z)\}\nu\{T(z)\}}{1-[\nu\{T(z)\}]^2}\right]z^2 dz \tag{4.20f}$$

$$N_T = \frac{1}{H}\int_{-H/2}^{H/2}\left[\frac{E\{T(z)\}}{1-\nu\{T(z)\}}\left(\int_{T_0}^{T}\alpha\{T(z)\}dT\right)\right]dz \tag{4.21}$$

$$M_T = \frac{1}{H^2}\int_{-H/2}^{H/2}\left[\frac{E\{T(z)\}}{1-\nu\{T(z)\}}\left(\int_{T_0}^{T}\alpha\{T(z)\}dT\right)\right]zdz; \tag{4.22}$$

The equilibrium equations of nonlinear free vibrations in the transverse direction of axisymmetric thin circular plate including thermal loading in the von Karman sense are given by equations (2.15) and these are:

$$-(rN_{rr}),_r + N_{\theta\theta} = 0; \tag{2.15a}$$

$$-(rN_{rr}w,_r),_r - (rQ),_r = -\rho H r\ddot{w}; \tag{2.15b}$$

and

$$-(rM_{rr}),_r + M_{\theta\theta} + rQ = 0 \tag{2.15c}$$

Combining equations (2.15), (4.18) and (4.19) one obtains the following governing differential equations of nonlinear free vibrations of such circular plates subjected to thermal loading having thermal gradient along the thickness direction.

$$\begin{aligned} E_1H\left(ru,_{rr} + u,_r - \frac{u}{r}\right) = -E_1H\left(\frac{w,_r^2}{2} + rw,_r w,_{rr}\right) + E_4H\frac{w,_r^2}{2} + \\ + E_2H^2\left(w,_{rr} + rw,_{rrr} - \frac{w,_r}{r}\right) \end{aligned} \tag{4.23}$$

$$E_3H^3r\nabla^4 w - E_1H\left(\frac{3}{2}rw,_r^2 w,_{rr} + \frac{1}{2}w,_r^3\right) - (3E_5 - E_2)H^2 w,_r w,_{rr} -$$

$$- E_1H\{(w,_r + rw,_{rr})u,_r + rw,_r u,_{rr}\} + E_4H(w,_{rr} u + w,_r u,_r) - \tag{4.24}$$

$$- E_2H^2\left(\frac{u}{r^2} - \frac{u,_r}{r} + 2u,_{rr} + ru,_{rrr}\right) + HN_T(w,_r + rw,_{rr}) + Hr\nabla^2 M_T + \rho Hr\ddot{w} = 0$$

4.4 Boundary Conditions

Clamped immovable edges are considered. So the boundary conditions (at r = R) are

(i) $w = 0$ (deflection along z-direction is zero)

(ii) $\dot{w},_r = 0$ (slope is zero in r-z plane)

(iii) u = 0 (radial displacement is zero for immovable edges)

4.5 Solution

Assuming the distribution of temperature and mass symmetrical over the plate, the form of symmetrical transverse deflection satisfying boundary conditions (i) and (ii) of the clamped circular plate can be written as follows:

$$w(r,t) = AW(r)F(t) = A\left[1 - \left(\frac{r}{R}\right)^2\right]^2 F(t) \tag{4.25}$$

The in-plane radial displacement satisfying boundary condition (iii) and the criteria of symmetric deflection at the centre i.e. $u(r = 0, t) = 0$ is assumed in the form

$$u(r,t) = BU(r)f(t) = B\frac{r}{R}\left[1 - \left(\frac{r}{R}\right)^2\right]f(t) \tag{4.26}$$

such that $U(r=0)=0$ and $U(r=R)=0$.

Now, applying Galerkin procedure to equation (4.23) with respect to spatial function U(r) one obtains

$$Bf(t)=\frac{F_2}{F_1}\frac{A^2}{R}F^2(t)+\frac{F_3}{F_1}\frac{HA}{R}F(t) \tag{4.27}$$

where

$$F_1=\frac{E_1}{R}\iint_A\left(rU_{,rr}+U_{,r}-\frac{U}{r}\right)Urdrd\theta \tag{4.28a}$$

$$F_2=E_1\iint_A\left(\frac{W_{,r}^2}{2}+rW_{,r}W_{,rr}\right)Urdrd\theta+E_4\iint_A\left(\frac{W_{,r}^2}{2}\right)Urdrd\theta \tag{4.28b}$$

$$F_3=E_2\iint_A\left(W_{,rr}+rW_{,rrr}-\frac{W_{,r}}{r}\right)Urdrd\theta \tag{4.28c}$$

Combining equations (4.21), (4.22), (4.25), (4.26) and (4.27) with equation (4.24) and applying Galerkin procedure one gets well known nonlinear time differential equation in the following form

$$\rho HAR^3\lambda_0\ddot{F}(t)+\frac{H^3A}{R}\left(\lambda_1-\frac{R^2}{H^2}N_T^*\right)F(t)+\frac{H^2A^2}{R}\lambda_2F^2(t)+\frac{HA^3}{R}\lambda_3F^3(t)=0 \tag{4.29}$$

where, $\ddot{F}(t)$ represents double derivative of $F(t)$ with respect to time; the parameters λ_1 λ_2 and λ_3 are known and the same are presented as follows

$$\lambda_0=\frac{1}{R^3}\int_0^R rW.Wdr \tag{4.30}$$

$$\lambda_1=(\lambda_{11}+\lambda_{12}) \tag{4.31}$$

$$\lambda_2=(\lambda_{21}+\lambda_{22}+\lambda_{23}+\lambda_{24}) \tag{4.32}$$

$$\lambda_3 = (\lambda_{31} + \lambda_{32} + \lambda_{33}) \tag{4.33}$$

$$N_T^* = \frac{N_T}{R}\int_0^R (W_{,r} + rW_{,rr})Wrdr \tag{4.34}$$

$$\lambda_{11} = E_3 R\int_0^R \left[rW_{,rrrr} + 2W_{,rrr} - \frac{1}{r}W_{,rr} + \frac{1}{r^2}W_{,r} \right]Wrdr \tag{4.35a}$$

$$\lambda_{12} = -E_2 R\frac{F_3}{F_1}\int_0^R \left(\frac{U}{r^2} + \frac{U_{,r}}{r} + 2U_{,rr} + rU_{,rrr} \right)Wrdr \tag{4.35b}$$

$$\lambda_{21} = (3E_5 - E_2)R\int_0^R [W_{,r} W_{,rr}]Wrdr \tag{4.36a}$$

$$\lambda_{22} = -E_1 R\frac{F_3}{F_1}\int_0^R [(W_{,r} + rW_{,rr})U_{,r} + rW_{,r} U_{,rr}]Wrdr \tag{4.36b}$$

$$\lambda_{23} = -E_4 R\frac{F_3}{F_1}\int_0^R (W_{,r} U_{,r} + W_{,rr} U)Wrdr \tag{4.36c}$$

$$\lambda_{24} = -E_2 R\frac{F_2}{F_1}\int_0^R \left(\frac{U}{r^2} + \frac{U_{,r}}{r} + 2U_{,rr} + rU_{,rrr} \right)Wrdr \tag{4.36d}$$

$$\lambda_{31} = -\frac{E_1 R}{2}\int_0^R [3rW_{,r}^2 W_{,rr} + W_{,r}^3]Wrdr \tag{4.37a}$$

$$\lambda_{32} = -E_1 R\frac{F_2}{F_1}\int_0^R [(W_{,r} + rW_{,rr})U_{,r} + rW_{,r} U_{,rr}]Wrdr \tag{4.37b}$$

$$\lambda_{33} = -E_4 R\frac{F_2}{F_1}\int_0^R (W_{,r} U_{,r} + W_{,rr} U)Wrdr \tag{4.37c}$$

The solution of the equation (4.29) subjected to initial conditions $F(0) = 0$ and $\dot{F}(0) = 0$ has been given by Bhattacharyee, A. P. (1976) and the relative nonlinear frequency (the ratio of nonlinear and linear frequencies) takes the form

$$\frac{\omega_{NL}}{\omega_L}=\left[1+\frac{A^2}{H^2}\left\{\frac{3}{4}\frac{\lambda_3}{\left(\lambda_1-\frac{R^2}{H^2}N_T^*\right)}-\frac{5}{6}\left(\frac{\lambda_2}{\left(\lambda_1-\frac{R^2}{H^2}N_T^*\right)}\right)^2\right\}\right]^{\frac{1}{2}} \tag{4.38}$$

where ω_{NL} and ω_L are nonlinear and linear frequencies of the structure respectively.

By dropping the nonlinear terms in equation (4.29) on obtains the linear frequency (ω_L) of vibration as

$$\omega_L=\left[\frac{H^2\left(\lambda_1-\frac{R^2}{H^2}N_T^*\right)}{\rho R^4\lambda_0}\right]^{\frac{1}{2}} \tag{4.39}$$

4.6 Numerical Results, Observations and Discussions

To illustrate the foregoing analysis, numerical results are presented The plates is assumed to be composed of Titanium Alloy$(Ti-6Al-4V)$. The material constants and their thermal dependency are presented analytically as well as graphically (Figures 4.1, 4.2, 4.3 and 4.4) valid for the temperature range of $27^\circ C\le T\le 1027^\circ C$.

For Titanium Alloy$(Ti-6Al-4V)$ [Tanigawa Y. et. al.(1996)]:

$$k=\{1.1+0.017(T+273)\}\quad [W/(m^0C)]$$

$$E=\{(122.7-0.0565(T+273))\times 10^9\}\quad [N/m^2]$$

$$\alpha=\left\{\begin{array}{l}7.43\times10^{-6}+5.56\times10^{-9}(T+273)-2.69\times10^{-12}(T+273)^2\\ for 27^\circ C\le T\le 827^\circ C\\ 10.291\times10^{-6}\, for 827^\circ C\le T\le 1027^\circ C\end{array}\right\}\quad [1/^\circ C]$$

$$\nu=\{0.2888+32\times10^{-6}(T+27)\}$$

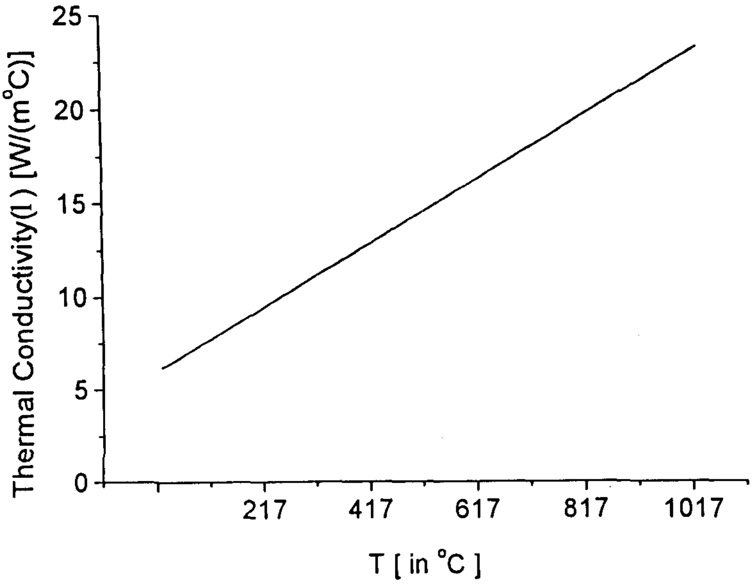

Fig. 4.1. Thermal conductivity vs. temperature for Titanium Alloy $(Ti-6Al-4V)$.

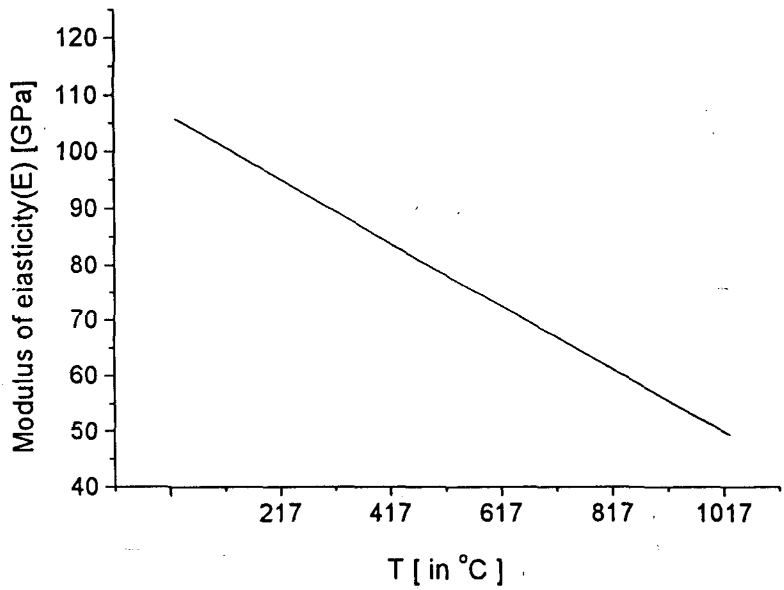

Fig. 4.2. Modulus of elasticity vs. temperature for Titanium Alloy $(Ti-6Al-4V)$.

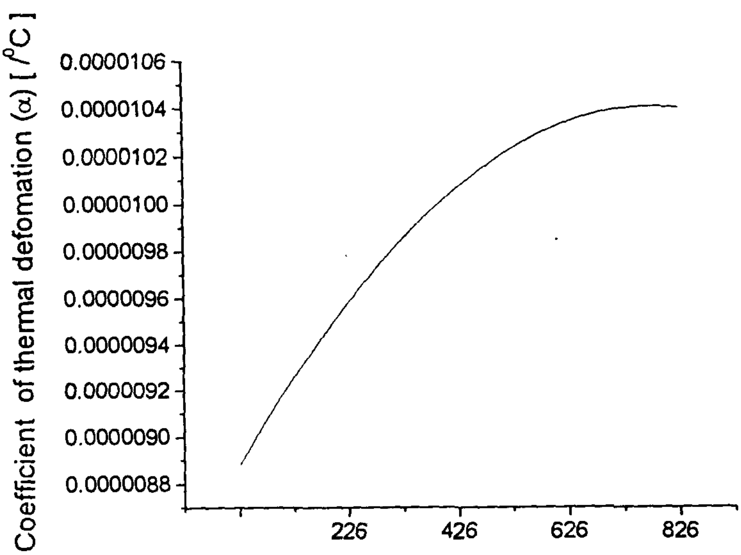

Fig. 4.3. Coefficient of thermal deformation vs. temperature for Titanium Alloy $(Ti-6Al-4V)$.

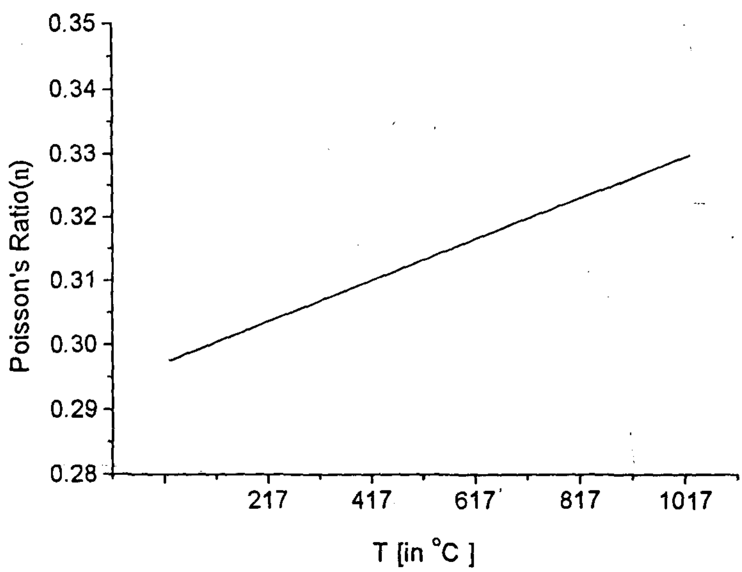

Fig. 4.4. Poisson's ratio vs. temperature for Titanium Alloy $(Ti-6Al-4V)$.

The effects of nondimensional amplitude$\left(\frac{A}{H}\right)$, thermal loading and slenderness parameter, considering temperature-independent material properties, on nonlinear free vibrations of thin isotropic elastic circular have been discussed in Chapter II. Only the effects of temperature dependency of material properties under thermal loading are discussed here.

4.6.1 The Effect of Temperature-Dependent Coefficient of Thermal Deformation, Modulus of Elasticity, and Poison's Ratio

In Chapter II it is observed that increase of average surface temperature $\left(\frac{T_u + T_b}{2}\right)$ reduces stiffness and hence frequency of vibration; both linear as well as nonlinear. Due to presence of additional stiffness associated with large deflection the linear frequency decreases more than the nonlinear one proportionately and as a result $\frac{\omega_{NL}}{\omega_L}$ increases. At certain stage linear frequency becomes zero i.e. thermal instability of the structure occurs based on linear theory and at this stage $\frac{\omega_{NL}}{\omega_L}$ becomes infinity. These behaviors of circular plate are confirmed again through Figure 4.6 to Figure 4.9. In this section, effects of temperature dependency of coefficient of thermal deformation, modulus of elasticity and Poison's ratio are discussed.

The modulus of elasticity decreases with increase of temperature (Figure 4.2) and hence, the stiffness of the plate reduces which results in reduction of nonlinear frequency of vibration (Figure 4.5). However, Figure 4.5 reveals that this reduction is very small compared to difference of nonlinear frequency from linear one due to large deflection as well as decrease of natural frequency due to heating effect. This behavior is easily understood if one analyzes the results of the problem considered. As the average surface temperature increases from 27^0C to 90^0C (within the range of temperature which causes thermal instability), the modulus of elasticity decreases around 3.366% and the rigidity of the plate decreases to the same tune approximately. Since nonlinear frequency is roughly directly proportional to the square-root of the rigidity of the plate, the former reduces by 1.67% approximately for the change of temperature from 27^0C to 90^0C . On the other hand, the effect of large deflection as well as the heating effect on natural frequency is much higher.

Figure 4.3 shows that the coefficient of thermal deformation increases with increase of temperature in the temperature range under consideration. Since the presence of thermal loading reduces plate stiffness, the increase of coefficient of thermal deformation intensifies effects of thermal loading and hence reduces plate stiffness resulting in reduction of nonlinear frequency (Figure 4.6). Again, Figure 4.6 reveals that such reduction is very small compared to the effect of large deflection as well as heating effect.

Figure 4.7 depicts that effect of temperature dependency of Poisson's ratio on nonlinear frequency is negligible.

Finally, figure 4.8 presents the combined effect of temperature dependency of modulus of elasticity, coefficient of thermal deformation and Poisson's ratio which shows that nonlinear frequency does not decrease appreciably for such structures.

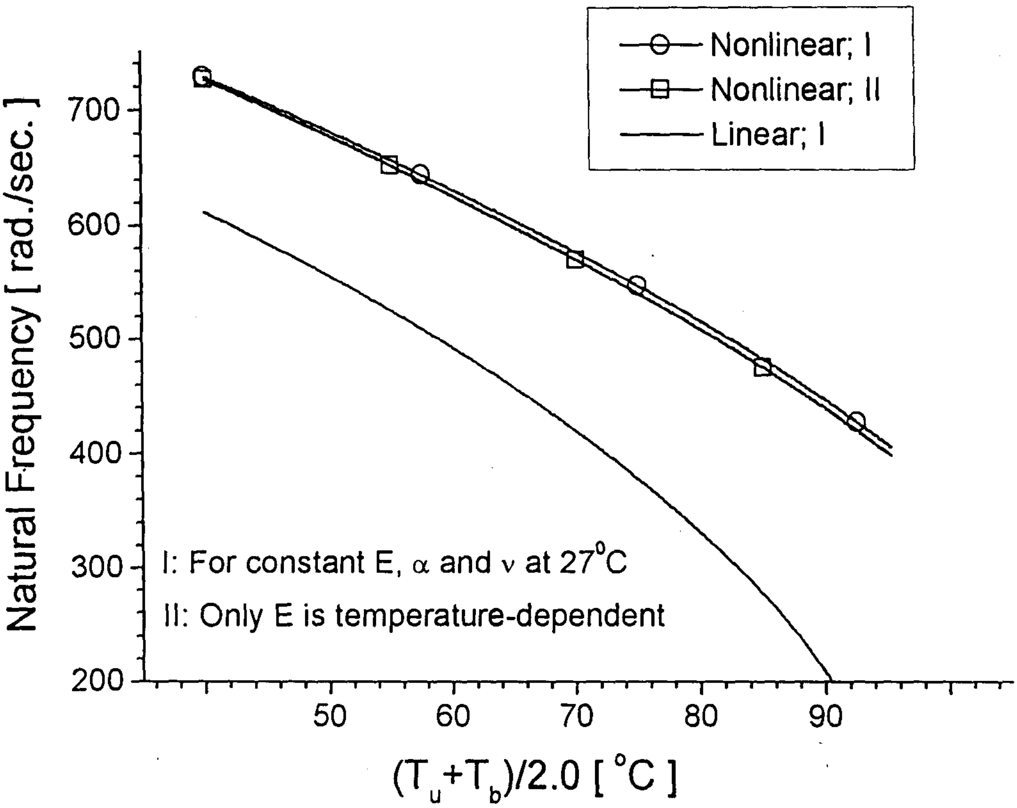

Fig. 4.5 Natural frequency vs. $\frac{T_u + T_b}{2}$ **when** $\frac{R}{H} = 50.0$ **and** $\frac{A}{H} = 1.0$ **showing effect of temperature- dependent modulus of elasticity (E) only.**

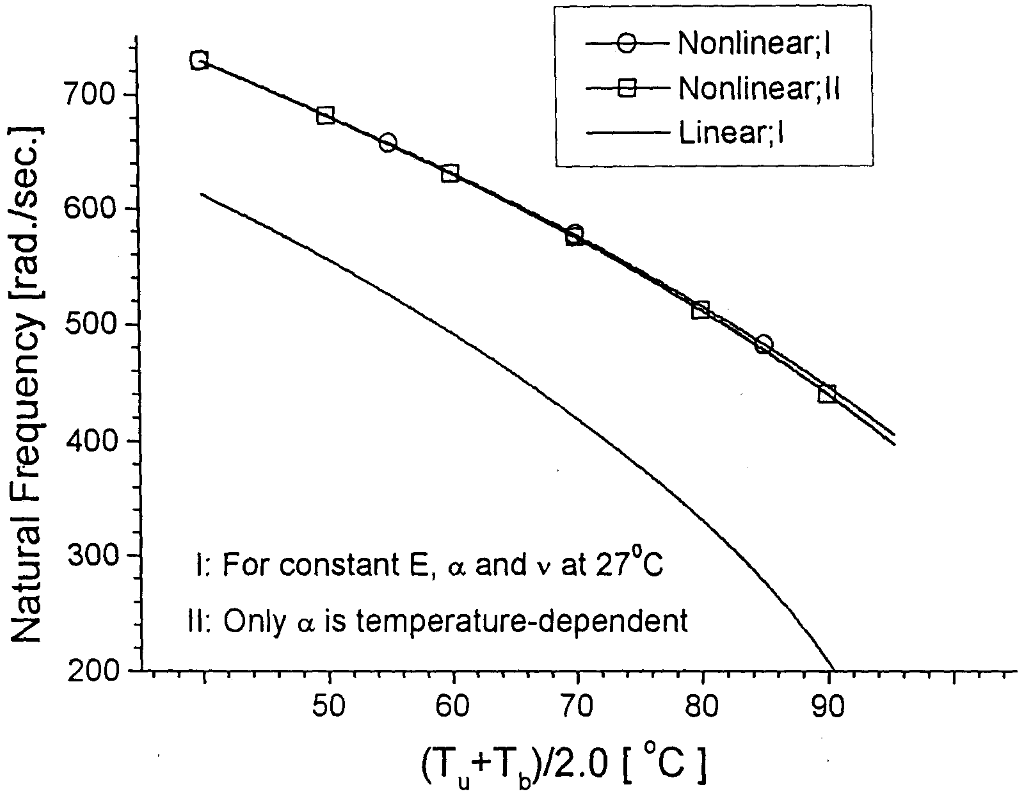

Fig. 4.6 Natural frequency vs. $\frac{T_u + T_b}{2}$ **when** $\frac{R}{H} = 50.0$ **and** $\frac{A}{H} = 1.0$ **showing effect of temperature-dependent coefficient of thermal deformation** (α) **only.**

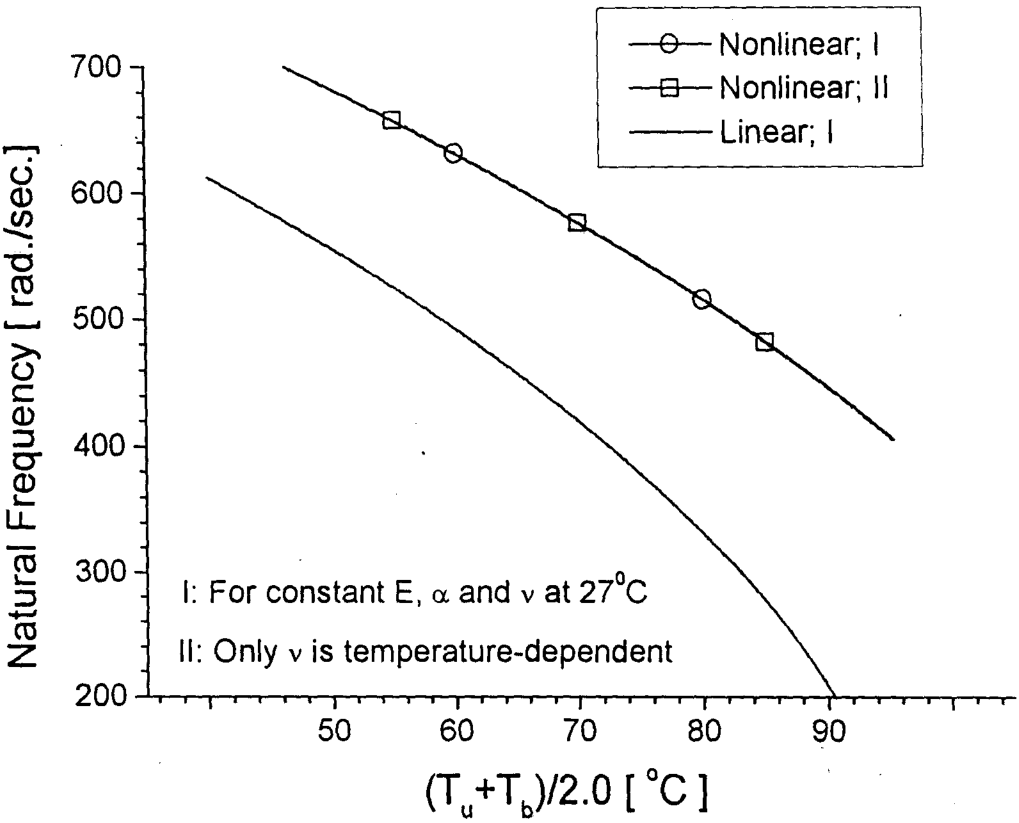

Fig. 4.7 Natural frequency vs. $\frac{T_u + T_b}{2}$ when $\frac{R}{H} = 50.0$ and $\frac{A}{H} = 1.0$ showing

effect of

temperature-dependent Poisson's ratio (ν) only

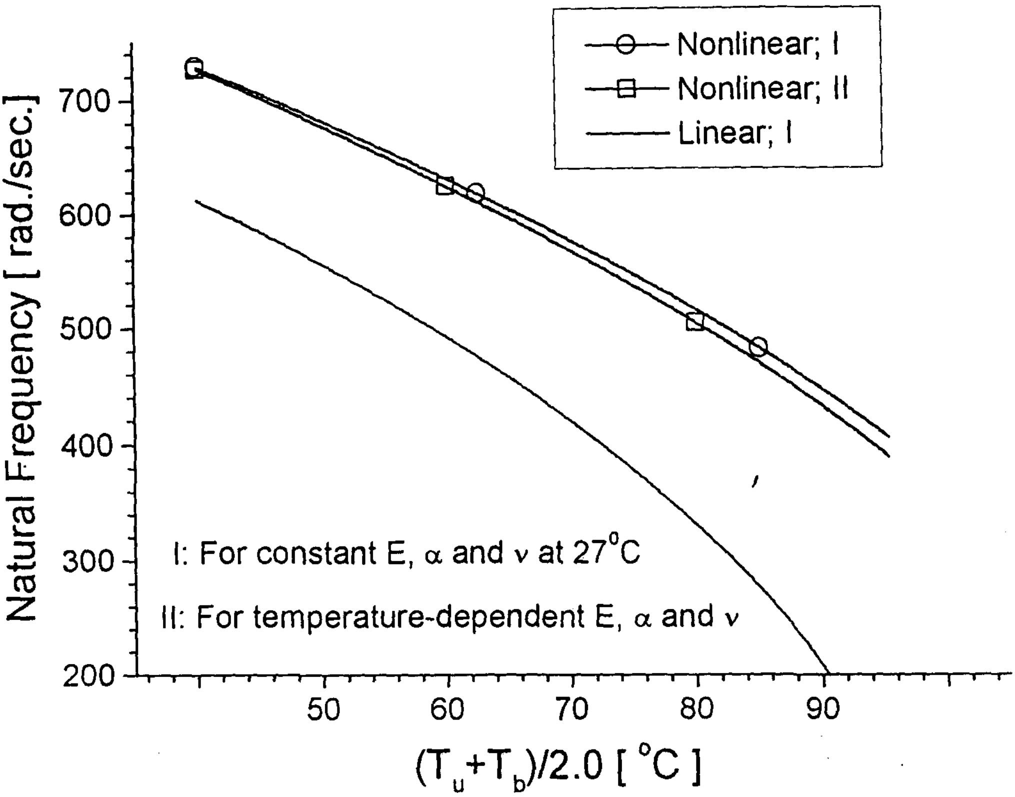

Fig. 4.8 Natural frequency vs. $\frac{T_u + T_b}{2}$ **when** $\frac{R}{H} = 50.0$ **and** $\frac{A}{H} = 1.0$ **showing combined effect of temperature-dependent E,** α **and** ν**.**

4.6.2 The Effect of Temperature-Dependent Thermal Conductivity

Figure 4.9 compares the steady-state temperature distribution across the thickness of the plate due to temperature-dependent thermal conductivity with that due to temperature-independent thermal conductivity(Equation 2.6; Chapter II) for the particular case when $T_u = 150^0 C$ and $T_b = 50^0 C$. It reveals that the temperature dependency of thermal conductivity does not alter temperature distribution significantly within the temperature level considered. However, the temperature level across the thickness increases marginally due to higher value of thermal conductivity at increased level of temperature and as a result temperature-dependent thermal conductivity does not influence the natural frequency significantly for such structures. Though numerical results for natural frequency are not presented, it decreases the natural frequency slightly due to marginal increased level of temperature across the thickness.

4.6.3 The Effect of Temperature Difference between the Surfaces $(T_u - T_b)$

Figures 4.10(a) and 4.10(b) describe the effects of temperature difference between the surfaces $(T_u - T_b)$ on nonlinear frequency. When the material properties are independent of temperature the nonlinear frequency does not change (Figure 4.10(a)). This observation is pointed out in Chapter II and the same is explained there. When the material properties except thermal conductivity (k) are dependent of temperature, also the change of nonlinear frequency is insignificant (Figure 4.10(a)). Figure 4.10(b) reveals clearly that the nonlinear frequency decreases with increase of the temperature difference between the surfaces when the material properties including the thermal conductivity are temperature-dependent; although the magnitude of reduction is very small compared to the effects of large deflection as well as heating. This due to the fact that the thermal conductivity increases at higher temperature level which causes more heat flow under higher temperature difference between the surfaces to increase the level of temperature across the cross-section slightly.

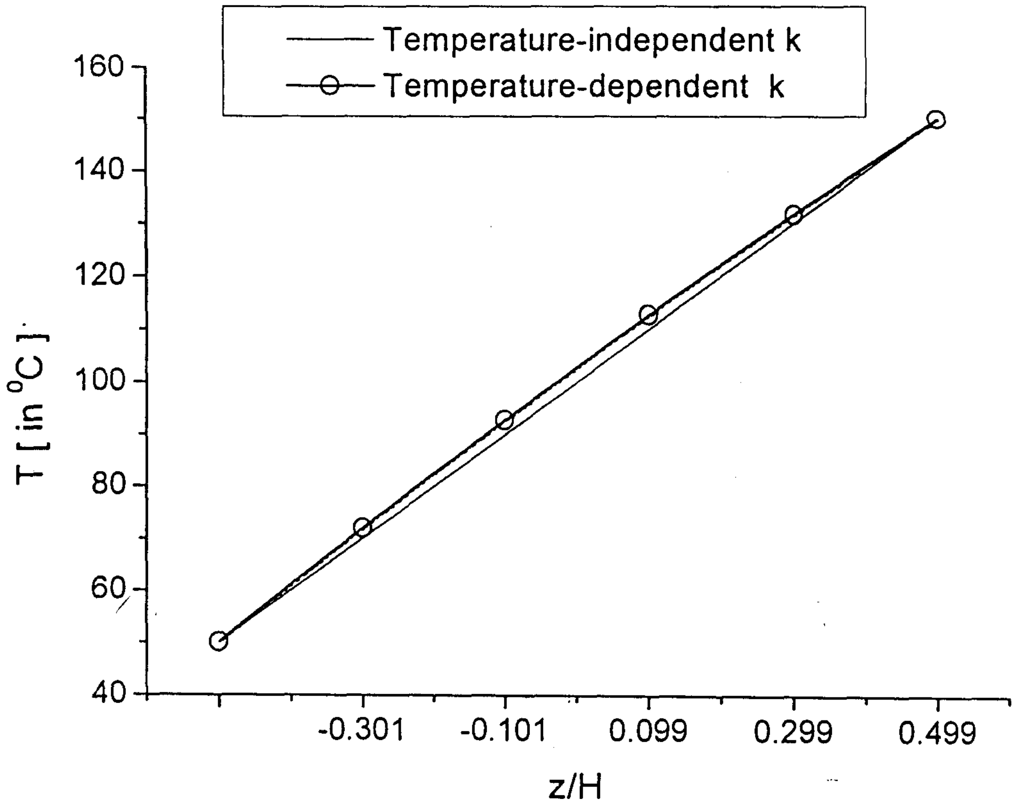

Fig. 4.9 Temperature distribution over the thickness of the plate for

$T_u = 150^0 C$ **and** $T_b = 50^0 C$

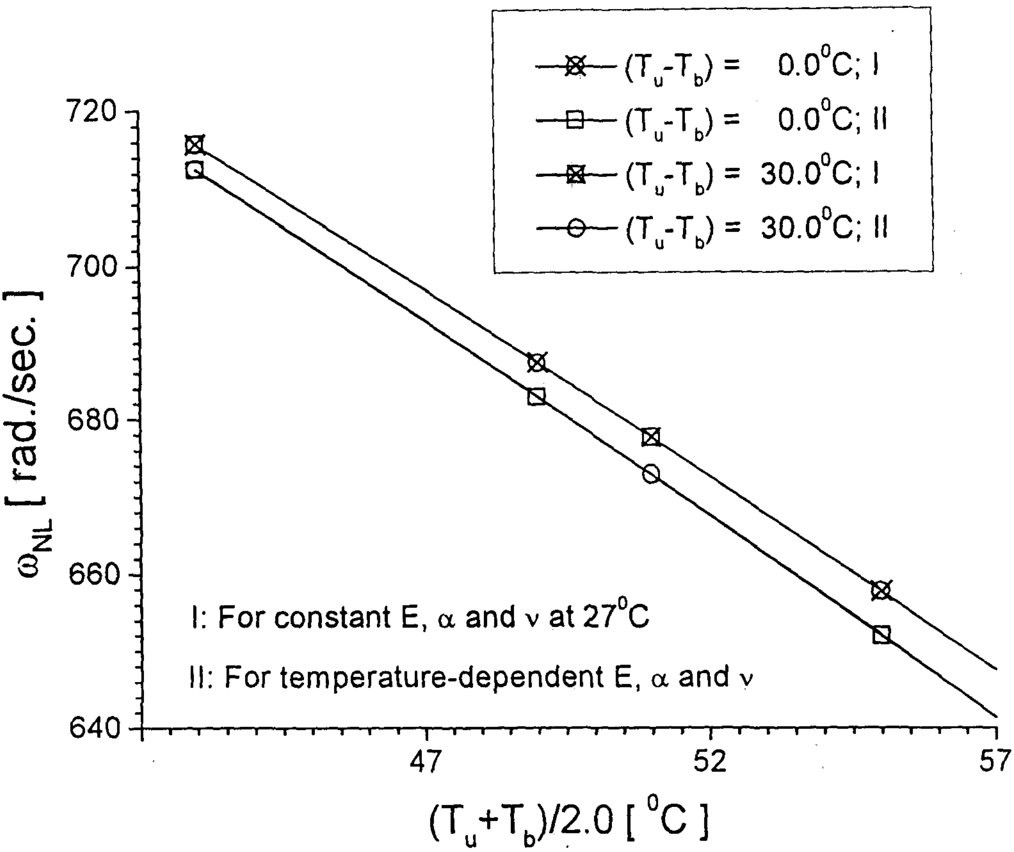

Fig. 4.10(a) Natural frequency vs. $\frac{T_u + T_b}{2}$ when $\frac{R}{H} = 50.0$ and $\frac{A}{H} = 1.0$ for different values of $(T_u - T_b)$ thermal conductivity is independent of temperature.

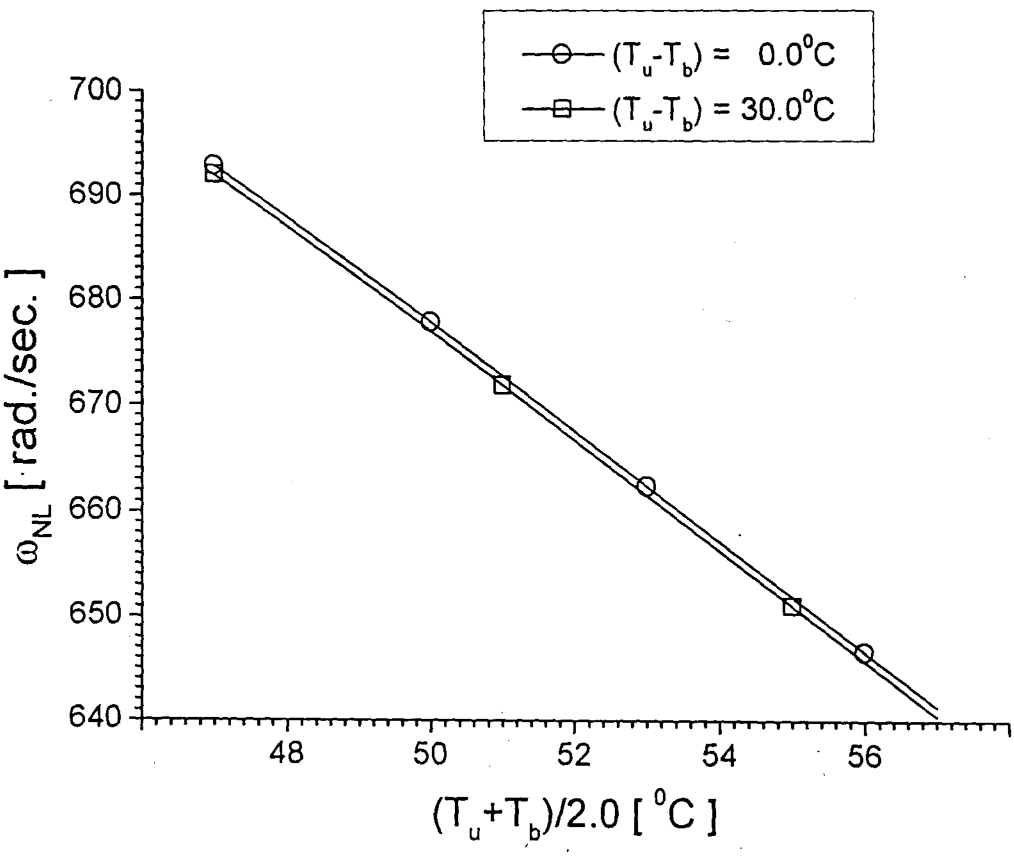

Fig. 4.10(b) Natural frequency vs. $\frac{T_u + T_b}{2}$ **when** $\frac{R}{H} = 50.0$ **and** $\frac{A}{H} = 1.0$ **for different values of** $(T_u - T_b)$ **and temperature-dependent** k **, E,** α **and** ν **.**

APPENDIX – 4-A

Solution of Cubic Equation

To explain the solution procedure of cubic equation the following equation is considered.

$$T^3 + s_2 T^2 + s_1 T + s_0 = 0 \tag{4.13}$$

If $s_0 \neq 0$ and $s_2 \neq 0$, it is assumed that

$$T = y - \frac{s_2}{3} \tag{A.1}$$

The equation (4.13) reduces to the form

$$y^3 + py + q = 0 \tag{A.2}$$

where

$$p = s_1 - \frac{s_2^2}{3} \tag{A.3}$$

$$q = s_0 + \frac{2s_2^3}{27} - \frac{s_1 s_2}{3} \tag{A.4}$$

If $p = 0$, the equation (A.2) results

$$T = \sqrt[3]{-q} - \frac{s_2}{3}$$

If $q = 0$, the equation (A.2) gives

$$T = -\frac{s_2}{3} \text{ or } T = \pm\sqrt{-p} - \frac{s_2}{3}$$

If $pq \neq 0$, it is again assumed that

$$y = z - \frac{p}{3z}$$

The equation (A.2) changes to

$$z^6 + qz^3 - \frac{p^3}{27} = 0$$

This equation yields

$$z^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^2}{27}}}{2}$$

The temperature T is obtained as

$$T = \sqrt[3]{z^3} - \frac{p}{\sqrt[3]{z^3}} - \frac{s_2}{3}$$

CHAPTER V

NONLINEAR VIBRATIONS OF THIN SHALLOW SPHERICAL ELASTIC SHELLS OF VARIABLE THICKNESS

5.1 Introduction

Large amplitude (nonlinear) dynamic behavior of thin shallow shell structures is of some technical importance to the designers due to its wide application in many fields of engineering. Containers, tanks, domes etc. are common examples of practical application of such structures. Although many research works have been carried out on nonlinear vibration of thin shallow spherical elastic shell of uniform thickness, few research works are available on the nonlinear vibration of thin shallow spherical elastic shells of variable thickness [Biswas, P. (1991)]. There he presented only analytical formulation for large amplitude free vibration of a thin shallow spherical shell of variable thickness in the von Karman sense in terms of displacement components without presenting any numerical computations.

In this chapter nonlinear free vibration analysis of thin shallow spherical elastic shells of variable thickness with tangentially clamped immovable edges has been performed by using both (i) coupled governing differential equations derived in the von Karman sense in terms of displacement components as well as (ii) decoupled nonlinear governing differential equations on the basis of Berger approximation (i.e. neglecting second strain invariant e_2) derived from energy expression applying Hamilton's principle and Euler's variational equations. The governing differential equations are solved by Galerkin error minimizing technique incorporating clamped immovable edge conditions. A parametric study is presented to understand the effects of various parameters on nonlinear dynamic behavior of such structures and the same reveals some interesting features. It should be noted that nonlinear vibrations of thin shallow spherical elastic shells of variable thickness under thermal loading may be considered as a separate problem in future investigation by researchers.

5.2 Geometry, Coordinate System and Deflection Components

A thin axisymmetric shallow spherical shell of variable thickness with tangentially clamped immovable edges is considered. Fig. 5.1 shows the geometry, coordinate system and deflection components. The shell is assumed to be made of elastic and homogeneous material. The vertical component of displacement of a point on the middle surface of the shell is denoted by $w(r,t)$ and the radial component is denoted by $u(r,t)$ measured horizontally. The elevation of the middle surface of the shell above the horizontal base plane is represented by

$$Z(r)=\left(\frac{R^2}{2R_0}\right)\left\{1-\left(\frac{r}{R}\right)^2\right\} \tag{5.1}$$

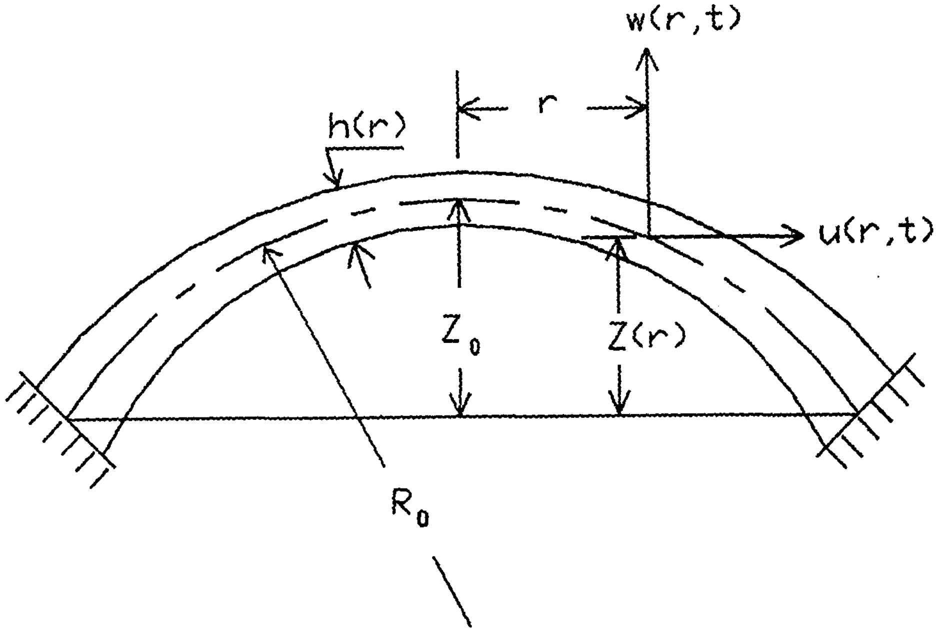

Fig. 5.1 Geometry, coordinate system and deflection components of a thin spherical shell of variable thickness.

The variation of the shell thickness is assumed as the same as taken by Timoshenko, S. and Woinowskty-Krieger, S.(1959) for circular plate of variable thickness. Biswas, P. (1991) also adopted the same variation and it is given by.

$$h(r) = H_0 e^{-\frac{\tau r^2}{6R^2}} \qquad (5.2)$$

where H_0 = the thickness of the shell at the center; R = the radius of the base circle of the shell, R_0 = the radius of curvature of the shell, h(r) = the thickness of the shell at radial distance r from the center and τ = a constant, called thickness parameter. So, the flexural rigidity of the shell at a distance r from the center of the shell and the same at the center of the shell are represented by

$$D(r) = \frac{E_c h^3}{12(1-\nu_c^2)} \tag{5.3a}$$

and

$$D_0 = \frac{E_c H_0^{\ 3}}{12(1-\nu_c^2)} \tag{5.3b}$$

respectively.

5.3 Classical Large Deflection Theory (in the von Karman Sense)

5.3.1 Governing Differential Equations

The force-displacement relationships [Budiansky, B., (1959) & Nash, W. A. and Mdeer, J. R., (1959)] for shallow spherical shell in the von Karman sense are

$$N_{rr} = \left\{\frac{E_c h}{(1-\nu_c^2)}\right\}\left\{u,_r + \frac{1}{2}w,_r^2 + Z,_r\, w,_r + \nu_c\left(\frac{u}{r}\right)\right\} \tag{5.4a}$$

$$N_{\theta\theta} = \left\{\frac{E_c h}{(1-\nu^2)}\right\}\left\{\frac{u}{r} + \nu\left(u,_r + \frac{1}{2}w,_r^2 + Z,_r\, w,_r\right)\right\} \tag{5.4b}$$

$$M_{rr} = -D\left\{w,_{rr} + \left(\frac{\nu w,_r}{r}\right)\right\} \tag{5.5a}$$

$$M_{\theta\theta} = -D\left\{\nu w,_{rr} + \frac{w,_r}{r}\right\} \tag{5.5b}$$

Neglecting the inplane inertia, the equilibrium equations of nonlinear free vibrations for axisymmetric thin spherical shell are obtained by incorporating the inertia term in the static equilibrium equations [Budiansky, B., 1959 & Nash, W. A. and Mdeer, J. R., (1959)] as

$$-(rN_{rr}),_r + N_{\theta\theta} = 0; \tag{2.15a}$$

$$-(rM_{rr}),_r + M_{\theta\theta} + rQ = 0 \tag{2.15b}$$

and

$$-\{r(w,_r + Z,_r)N_{rr}\},_r - (rQ),_r = -\rho hr\ddot{w} \tag{5.6}$$

with usual notations.

Considering variable shell thickness and combining equations(2.15a), (2.15b), (5.4a), (5.4b), (5.5a), (5.5b) and (5.6) one obtains

$$\begin{aligned} hru,_{rr} + (h + h,_r r)u,_r + \left(\nu h,_r - \frac{h}{r}\right)u &= \frac{(\nu_c - 1)}{2}hw,_r^2 - \frac{1}{2}rh,_r w,_r^2 - rhw,_r w,_{rr} \\ &- hZ,_r w,_r - rh,_r Z,_r w,_r - rhZ,_{rr} w,_r - rhZ,_r w,_{rr} + \nu hZ,_r w,_r \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} rD\nabla^4 w + rD,_r\left(2w,_{rrr} + \frac{2}{r}w,_{rr} + \frac{\nu}{r}w,_{rr} - \frac{1}{r^2}w,_r\right) + rD,_{rr}\left(w,_{rr} + \frac{\nu}{r}w,_r\right) \\ + r\rho h\ddot{w} - (rN_{rr}w,_r),_r + (rN_{rr}Z,_r),_{rr} = 0 \end{aligned} \tag{5.8}$$

These two equations are to be used to study the nonlinear free vibrations of thin shallow spherical shell structures.

5.3.2 Boundary Conditions

A thin shallow spherical shell with tangentially clamped immovable edges is considered. So the boundary conditions at r = R are

(i) $w = 0$ (deflection along z-direction is zero)

(ii) $w_{,r} = 0$ (slope is zero in r-z plane)

(iii) u = 0 (radial displacement is zero for immovable edges)

5.3.3 Solution

Assuming symmetrical distribution of mass over the shell, the form of the deflection components $w(r,t)$ of the thin shallow spherical shell of variable thickness with clamped immovable edges can be written approximately [Sinharay, G. C., and Banerjee, B., (1985a and 1985b), Biswas, P. (1991)] as

$$w(r,t) = AW(r)F(t) = A\left[1-\left(\frac{r}{R}\right)^2\right]^2 F(t) \tag{4.25}$$

which satisfies the boundary conditions (i) and (ii) mentioned above.

The in-plane radial displacement $u(r,t)$ satisfying boundary condition (iii) and the criteria of symmetric deflection at the centre i.e. $u(r=0,t)=0$ is assumed in the form

$$u(r,t) = BU(r)f(t) = B\frac{r}{R}\left[1-\left(\frac{r}{R}\right)^2\right]f(t) \tag{4.26}$$

such that $U(r=0)=0$ and $U(r=R)=0$.

Substituting equations (5.1), (5.2), (4.25) and (4.26) in equation (5.7) and applying Galerkin procedure with respect to the spatial function U(r) one obtains

$$Bf(t)=\frac{F_2}{F_1}\frac{A^2}{R}F^2(t)+\frac{F_3}{F_1}\frac{RA}{R_0}F(t) \tag{5.9}$$

where

$$F_1=\frac{1}{RH_0}\iint_A\left(rhU_{,rr}+(h+h_{,r}r)U_{,r}+\left(\nu_c h_{,r}-\frac{h}{r}\right)U\right)Urdrd\theta \tag{5.10a}$$

$$F_2=\frac{1}{H_0}\iint_A\left(-\frac{rh_{,r}W_{,r}^2}{2}+\frac{(\nu_c-1)hW_{,r}^2}{2}-rhW_{,r}W_{,rr}\right)Urdrd\theta \tag{5.10b}$$

$$F_3=\frac{R_0}{H_0R^2}\iint_A\left((rh_{,r}Z_{,r}+hZ_{,r}+rhZ_{,rr}-\nu hZ_{,r})W_{,r}+rhZ_{,r}W_{,rr}\right)Urdrd\theta \tag{5.10c}$$

Combining equations (5.1), (5.2), (5.3), (5.4a), (4.25) and (4.26) with equation (5.8) and applying Galerkin procedure one gets well known nonlinear time differential equation in the following form

$$\rho H_0AR^3\lambda_0\ddot{F}(t)+\frac{H_0^3A}{R}\left(\lambda_{11}+\frac{R^4}{R_0^2H_0^2}\lambda_{12}\right)F(t)+\frac{RH_0A^2}{R_0}\lambda_2F^2(t)$$
$$+\frac{H_0A^3}{R}\lambda_3F^3(t)=0 \tag{5.11}$$

where, $\ddot{F}(t)$ represents double derivative of $F(t)$ with respect to time; the parameters λ_0, λ_{11}, λ_{12}, λ_2 and λ_3 are known and the same are presented as follows

$$\lambda_0=\frac{1}{R^3}\int_0^R rW.Wrdr \tag{4.30}$$

$$\lambda_{11}=(\lambda_{111}+\lambda_{112}+\lambda_{113}) \tag{5.12}$$

$$\lambda_{12}=(\lambda_{121}+\lambda_{122}) \tag{5.13}$$

$$\lambda_2=(\lambda_{21}+\lambda_{22}+\lambda_{23}+\lambda_{24}) \tag{5.14}$$

$$\lambda_3=(\lambda_{31}+\lambda_{32}) \tag{5.15}$$

$$\lambda_{111} = \frac{R}{H_0^3}\int_0^R D\left[rW_{,rrrr} + 2W_{,rrr} - \frac{1}{r}W_{,rr} + \frac{1}{r^2}W_{,r}\right]Wrdr \tag{5.16a}$$

$$\lambda_{112} = \frac{R}{H_0^3}\int_0^R D_{,r}\left[2rW_{,rrr} + 2W_{,rr} + \nu W_{,rr} - \frac{1}{r}W_{,r}\right]Wrdr \tag{5.16b}$$

$$\lambda_{113} = \frac{R}{H_0^3}\int_0^R D_{,rr}\left[rW_{,rr} + \nu W_{,r}\right]Wrdr \tag{5.16b}$$

$$\lambda_{121} = \frac{R_0^2}{R^3 H_0}\frac{E_c}{\left(1-\nu_c^2\right)}\int_0^R \left\{\left(hZ_{,r}^2 + 2rhZ_{,r}Z_{,rr} + rh_{,r}Z_{,r}^2\right)W_{,r} + rhZ_{,r}^2 W_{,rr}\right\}Wrdr \tag{5.17a}$$

$$\lambda_{122} = \frac{R_0^2}{R^3 H_0}\frac{E_c}{\left(1-\nu_c^2\right)}\frac{F_3}{F_1}\int_0^R \left[\nu_c\begin{pmatrix} h_{,r}Z_{,r} \\ +hZ_{,rr}\end{pmatrix}U + \begin{Bmatrix}\left(h + rh_{,r} + \nu_c h\right)Z_{,r} \\ + rhZ_{,rr}\end{Bmatrix}U_{,r} + rhZ_{,r}U_{,rr}\right]Wrdr$$

(5.17b)

$$\lambda_{21} = \frac{R_0}{RH_0}\frac{E_c}{\left(1-\nu_c^2\right)}\int_0^R \left[\left\{\left(h + rh_{,r}\right)Z_{,r} + rhZ_{,rr}\right\}W_{,r}^2 + 2rhZ_{,r}W_{,r}W_{,rr}\right]Wrdr \tag{5.18a}$$

$$\lambda_{22} = \frac{R_0}{RH_0}\frac{E_c}{\left(1-\nu_c^2\right)}\int_0^R \left[\frac{1}{2}\left\{\left(h + rh_{,r}\right)Z_{,r} + rhZ_{,rr}\right\}W_{,r}^2 + rhZ_{,r}W_{,r}W_{,rr}\right]Wrdr \tag{5.18b}$$

$$\lambda_{23} = \frac{R_0}{RH_0}\frac{E_c}{\left(1-\nu_c^2\right)}\frac{F_2}{F_1}\int_0^R \left\{\nu_c\begin{pmatrix} h_{,r}Z_{,r} \\ +hZ_{,rr}\end{pmatrix}U + \begin{Bmatrix}\left(h + rh_{,r} + \nu_c h\right)Z_{,r} \\ + rhZ_{,rr}\end{Bmatrix}U_{,r} + rhZ_{,r}U_{,rr}\right\}Wrdr$$

(5.18c)

$$\lambda_{24} = \frac{R_0}{RH_0}\frac{E_c}{\left(1-\nu_c^2\right)}\frac{F_3}{F_1}\int_0^R \left[\nu_c\begin{pmatrix} h_{,r}W_{,r} \\ +hW_{,rr}\end{pmatrix}U + \begin{Bmatrix}\left(h + rh_{,r} + \nu_c h\right)W_{,r} \\ + rhW_{,rr}\end{Bmatrix}U_{,r} + rhW_{,r}U_{,rr}\right]Wrdr$$

(5.18d)

$$\lambda_{31} = -\frac{R}{H_0}\frac{E_c}{\left(1-\nu_c^2\right)}\int_0^R \left\{\frac{1}{2}\left(h + rh_{,r}\right)W_{,r}^3 + \frac{3}{2}rhW_{,r}^2 W_{,rr}\right\}Wrdr \tag{5.19a}$$

$$\lambda_{32} = \frac{R}{H_0}\frac{E_c}{\left(1-\nu_c^2\right)}\frac{F_2}{F_1}\int_0^R \left[\nu_c\begin{pmatrix} h_{,r}W_{,r} \\ +hW_{,rr}\end{pmatrix}U + \begin{Bmatrix}\left(h + rh_{,r} + \nu_c h\right)W_{,r} \\ + rhW_{,rr}\end{Bmatrix}U_{,r} + rhW_{,r}U_{,rr}\right]Wrdr$$

(5.19b)

The solution of the equation (5.11) subjected to initial conditions $F(0)=0$ and $\dot{F}(0)=0$ yields the relative nonlinear frequency i.e. the ratio of nonlinear and linear frequencies [Bhattacharyee, A. P., (1976)] as

$$\frac{\omega_{NL}}{\omega_L}=\left[1+\frac{A^2}{H_0^2}\left\{\frac{3}{4}\frac{\lambda_3}{\left(\lambda_{11}+\dfrac{R^4}{R_0^2H_0^2}\lambda_{12}\right)}-\frac{5}{6}\left(\frac{\dfrac{R^2}{R_0H_0}\lambda_2}{\left(\lambda_{11}+\dfrac{R^4}{R_0^2H_0^2}\lambda_{12}\right)}\right)^2\right\}\right]^{1/2} \tag{5.20}$$

where ω_{NL} and ω_L nonlinear and linear frequencies of the structure respectively.

By dropping the nonlinear terms in equation (5.11) one obtains the linear frequency (ω_L) of vibration as

$$\omega_L=\left[\frac{H_0^2\left(\lambda_{11}+\dfrac{R^4}{R_0H_0^2}\lambda_{12}\right)}{\rho R^4\lambda_0}\right]^{1/2} \tag{5.21}$$

5.4 Analysis based on Berger's Approximation

5.4.1 Governing Differential Equations

The total potential energy (P_E), due to bending and axial deformations of the middle surface of the deflected shell, may be written in the polar coordinates [Sinharay, G. C., and Banerjee, B., (1985b)] as

$$P_E=\frac{1}{2}\int_0^{2\pi}\int_0^{R}D(r)\left[\left(\nabla^2 w\right)^2+12\frac{e_1^2}{h^2}-\frac{2(1-\nu)}{r}\left\{\frac{12e_2}{h^2}+w_{,r}w_{,rr}\right\}\right]rdrd\theta \tag{5.22}$$

where the first (e_1) and second (e_2)strain invariants can be expressed as

$$e_1 = u_{,r} + \frac{u}{r} + 0.5w_{,r}^2 + w_{,r}z_{,r} \tag{5.23}$$

$$e_2 = u_{,r}\left(\frac{u}{r} + 0.5w_{,r}^2 + w_{,r}z_{,r}\right) \tag{5.24}$$

The kinetic energy(K_E) of the vibrating shell is given by

$$K_E = \frac{1}{2}\rho\int_0^{2\pi}\int_0^{R}\left(\dot{u}^2 + \dot{w}^2\right)r\,dr\,d\theta \tag{5.25}$$

The Hamilton's principle yields

$$\delta\Omega = \delta\int_{t_1}^{t_2}(K_E - P_E)dt = 0 \tag{5.26}$$

The Hamilton's integral may be represented by the functional form

$$\Omega = \int_{t_1}^{t_2}\int_0^{2\pi}\int_0^{R} F'(t, r, u, w, u_{,r}, w_{,r}, \dot{u}, \dot{w}, w_{,rr})dr\,d\theta\,dt \tag{5.27}$$

According to Hamilton's principle Equation (5.27) will yield an extremum, i.e. $\delta\Omega = 0$,

if the following Euler's variational equations are satisfied

$$\frac{\partial F'}{\partial u} - \frac{\partial}{\partial r}\left(\frac{\partial F'}{\partial u_{,r}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F'}{\partial \dot{u}}\right) = 0 \tag{5.28}$$

$$\frac{\partial F'}{\partial w} - \frac{\partial}{\partial r}\left(\frac{\partial F'}{\partial w_{,r}}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial F'}{\partial w_{,rr}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F'}{\partial \dot{w}}\right) = 0 \tag{5.29}$$

Neglecting second strain invariant (e_2) [Berger, 1955] as well as the inertia effects due to $u(r,t)$) and using Euler's variational equations (5.28) and (5.29), the following set of differential equations for the free vibrations of the shell are obtained

$$\nabla^2\left(D\nabla^2 w\right)-\frac{(1-\nu)}{r}\left(D_{,rr}w_{,r}+D_{,r}w_{,rr}\right)-C_b\left(\nabla^2 w+\nabla^2 z\right)+\rho h\ddot{w}=0 \tag{5.30}$$

$$e_1=u_{,r}+\frac{u}{r}+0.5w_{,r}^2+w_{,r}z_{,r}=\frac{C_b h^2}{12D} \tag{5.31}$$

with usual notations.

5.4.2 Solution

The deflection components $w(r,t)$ and $u(r,t)$ are assumed as

$$w(r,t)=A\left[1-\left(\frac{r}{R}\right)^2\right]^2 F(t) \tag{2.10}$$

and

$$u(r,t)=U(r)f(t) \tag{5.32}$$

respectively.

Substituting equations (2.10), (5.1), (5.2), (5.3a) and (5.32) into equation (5.31) and integrating over the area of the plate considering U = 0 at r = 0 and r = R one obtains

$$C_b=\frac{D_0 AF(t)}{RH_0^2}\cdot\frac{1}{\sum\limits_{n=0}^{\infty}\frac{\tau^n}{6^n n!(2n+2)}}\cdot\left\{\frac{8}{3}\frac{A^2F^2(t)}{R^2H_0^2}+\frac{4AF(t)}{RH_0^2}+\frac{4}{3}\left(\frac{R^2}{R_0H_0}\right)^2\right\} \tag{5.33}$$

Substitution of equations (2.10), (5.1), (5.2), (5.3a) and (5.33) into equation (5.30) and application of Galerkin's procedure yields the following nonlinear time differential equation

$$\lambda_0 \ddot{F}(t) + \lambda_1 F(t) + \lambda_2 F^2(t) + \lambda_3 F^3(t) = 0 \qquad (5.34)$$

The solution of equation (5.34) subjected to initial conditions $F(0) = 0$ and $\dot{F}(t) = 0$ an be obtained after Bhattacharyee, A. P. (1976) as

$$\frac{\omega_{NL}}{\omega_L} = \left[1 + \left\{\frac{3}{4}\frac{\lambda_3}{\lambda_1} - \frac{5}{6}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right\}\right]^{1/2} \qquad (2.26)$$

where,

$$\lambda_0 = \rho H_0 A R^2 \sum_{n=0}^{\infty} \frac{(-\tau)^n}{6^n n!}\left(\frac{1}{2n+2} - \frac{4}{2n+4} + \frac{6}{2n+6} - \frac{4}{2n+8} + \frac{1}{2n+10}\right) \qquad (5.35)$$

$$\lambda_1 = \frac{D_o A}{R^2}\left(\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15}\right) \qquad (5.36)$$

$$\lambda_{11} = 64 \sum_{n=0}^{\infty} \frac{(-\tau)^n}{2^n n!}\left(\frac{1}{2n+2} - \frac{2}{2n+4} + \frac{1}{2n+6}\right) \qquad (5.37a)$$

$$\lambda_{12} = -8\tau \sum_{n=0}^{\infty} \frac{(-\tau)^n}{2^n n!}\left(-\frac{1}{2n+2} + \frac{12}{2n+4} - \frac{21}{2n+6} + \frac{10}{2n+8}\right) \qquad (5.37b)$$

$$\lambda_{13} = 8\tau \sum_{n=0}^{\infty} \frac{(-\tau)^n}{2^n n!}\left\{\begin{aligned}&\left(\frac{1}{2n+2} - \frac{4}{2n+4} + \frac{5}{2n+6} - \frac{2}{2n+8}\right) - \\ &-\tau\left(\frac{1}{2n+4} - \frac{4}{2n+6} + \frac{5}{2n+8} - \frac{2}{2n+10}\right)\end{aligned}\right\} \qquad (5.37c)$$

$$\lambda_{14} = -4(1-\nu)\tau \sum_{n=0}^{\infty} \frac{(-\tau)^n}{2^n n!}\left\{\begin{aligned}&\left(\frac{1}{2n+2} - \frac{4}{2n+4} + \frac{5}{2n+6} - \frac{2}{2n+8}\right) - \\ &-\tau\left(\frac{1}{2n+4} - \frac{3}{2n+6} + \frac{3}{2n+8} - \frac{1}{2n+10}\right)\end{aligned}\right\} \qquad (5.37d)$$

$$\lambda_{15} = \frac{4}{3\sum_{n=0}^{\infty}\frac{\tau^n}{6^n n!(2n+2)}}\left(\frac{R^2}{R_o H_o}\right)^2 \tag{5.38}$$

$$\lambda_2 = \frac{4D_o A^2}{R_o H_o^2 \sum_{n=0}^{\infty}\frac{\tau^n}{6^n n!(2n+2)}} \tag{5.39}$$

$$\lambda_3 = \frac{8D_o A^3}{3R^2 H_o^2 \sum_{n=0}^{\infty}\frac{\tau^n}{6^n n!(2n+2)}} \tag{5.40}$$

5.5 Numerical Results, Observations and Discussions

Numerical results are presented in this section to understand the effects of different parameters viz. (i) nondimensional geometrical parameter $\left(\frac{R^2}{2R_0 H_0}\right)$, (ii) thickness parameter (τ), and (iii) nondimensional amplitude $\left(\frac{A}{H_0}\right)$ etc. on nonlinear vibrations of thin shallow spherical shell. For illustrative examples, the shell material is assumed as steel and the value of Poisson's ratio is considered as $\nu_c = 0.3$.

5.5.1 The Effect of nondimensional geometrical parameter $\left(\frac{R^2}{2R_0 H_0}\right)$

The nondimensional geometrical parameter $\left(\frac{R^2}{2R_0 H_0}\right)$ is nothing but ratio of the rise $\left(\frac{R^2}{2R_0}\right)$ of the shell from the base plane to the shell thickness (H_0) at the center of the shell. It is reasonable to assume that very low value of $\frac{R^2}{2R_0H_0}$ can be attained by decreasing the rise instead of increasing the thickness otherwise a thin shell tends to become a thick shell. Again, the requirement of a shallow shell is $\frac{rise}{R} \leq \frac{1}{5} to \frac{1}{8}$ approximately [Bandyopadhyay, J. N. (1998), Budiansky, B. (1959) & Nash, W. A. and Mdeer, J. R. (1959)] and if one assumes $\frac{R}{H_0} = 40.0$ then $\frac{R^2}{2R_0H_0}$ should be less than or equal to 8.0 to 5.0. Satisfying this requirement of shallow shell higher value of $\frac{R^2}{2R_0H_0}$ can be achieved by lowering the shell thickness.

The Figures 5.2(a) and 5.2(b) portray the effects of nondimensional geometrical parameter on relative nonlinear frequency for different values of τ at $\frac{A}{H_0} = 1.0$. As the value of $\frac{R^2}{2R_0H_0}$ approaches very low value, a thin shallow spherical shell tends to become a circular plate and the shell stiffness, both linear as well as nonlinear and hence relative nonlinear frequency $\left(\frac{\omega_{NL}}{\omega_L}\right)$ tends to become independent of $\frac{R^2}{2R_0H_0}$ which is reflected in the initial part of the plot in Figure 5.2(a). For very low values of $\frac{R^2}{2R_0H_0}$, bending deformations causes stretching of the middle surface of the shell as per large deflection theory, which contribute additional stiffness. As a result, $\left(\frac{\omega_{NL}}{\omega_L}\right)$ is greater than 1.0 i.e. the behavior of very shallow shell is strain-hardening one like circular plate.

With increase of $\frac{R^2}{2R_0H_0}$ the behavior of shallow spherical shell changes from strain-hardening to strain-softening one and the relative nonlinear frequency starts decreasing to a minimum point and then increases monotonically to approach unity. The value of $\frac{R^2}{2R_0H_0}$ for a particular configuration of a shallow shell at which $\frac{\omega_{NL}}{\omega_L}$ becomes unity i.e. the behavior changes from strain-hardening to strain-softening one is called the transition value. During compression phase of vibrations, shallow spherical shell primarily contracts in size without appreciable bending; bending deformations being localized near the built-in edges and try to buckle in a direction opposite to the initial curvature, called snap buckling. Large deflection facilitates snap buckling of shallow shell and the force required to maintain equilibrium reduces i.e. nonlinear stiffness decreases. On the other hand, linear stiffness, which is independent of amplitude of vibrations, increases with increase of $\frac{R^2}{2R_0H_0}$. As a result, relative nonlinear frequency decreases to yield the lowest peak and this peak corresponds to the geometrical configuration of thin shallow shell which is most liable to undergo snap buckling. With further increase of $\frac{R^2}{2R_0H_0}$ the resistance of shallow spherical shell to snap buckling increases and the shell stiffness, both linear as well as nonlinear, increases; the rate of increase of the nonlinear stiffness being slightly greater than the linear one due to additional stiffness associated with large deflection and the relative nonlinear frequency increases and finally it approaches unity. As the geometric parameter $\frac{R^2}{2R_0H_0}$ for thin shallow shell increases the shallow spherical shell primarily contracts in size and the bending deformations, which can be neglected, remain localized near the edges while the remaining part of the shallow spherical shell remains spherical. As a result the contribution of additional stiffness associated with large deflection i.e. stretching of the middle surface tends to become insignificant and the nonlinear stiffness approaches the linear stiffness; hence $\frac{\omega_{NL}}{\omega_L}$ approaches unity.

Figures 5.3(a) and 5.3(b) display the plot of $\frac{\omega_{NL}}{\omega_L}$ vs. $\frac{R^2}{2R_0H_0}$ based on Berger approximation (i.e. neglecting second strain invariant (e_2)). The overall behavior predicted by the Berger method is in good agreement with that obtained in the present study based on the classical large deflection theory in the von Karman sense. The difference of the numerical results obtained by the two methods may be attributed to the different approaches.

Sinharay, G., and Banerjee, B. (1985b), based on modified Berger's method, presented variation of relative nonlinear frequency with nondimensional amplitude graphically of thin shallow spherical elastic shell having clamped immovable edges and uniform thickness for $\frac{R^2}{2R_0H_0} = 0.5$ and 1.0 and they observed both strain-hardening type of behavior for $\frac{R^2}{2R_0H_0} = 0.5$ as well as strain-softening type of behavior for $\frac{R^2}{2R_0H_0} = 1.0$. Again, the difference of numerical results obtained by different method may be attributed to the approximations involved in the method of analysis.

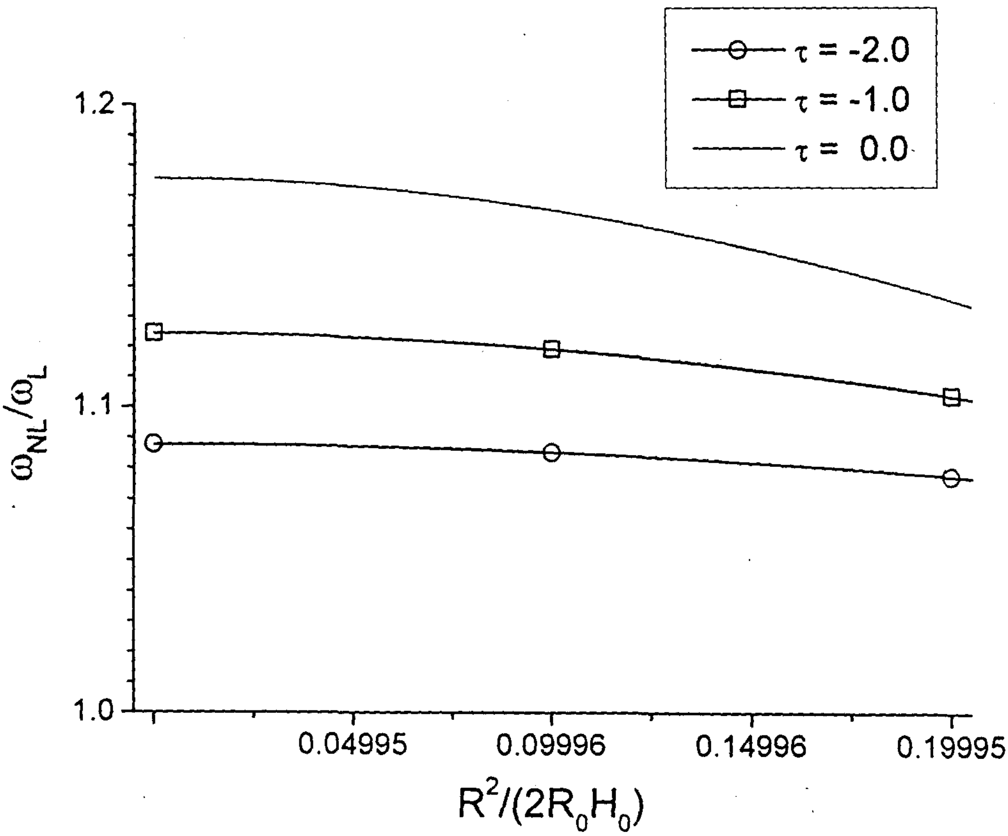

Fig. 5.2(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{R^2}{2R_0H_0}$ **for different values of** τ **at** $\frac{A}{H_0} = 1.0$.

[For very low values of $\frac{R^2}{2R_0H_0}$ **]**

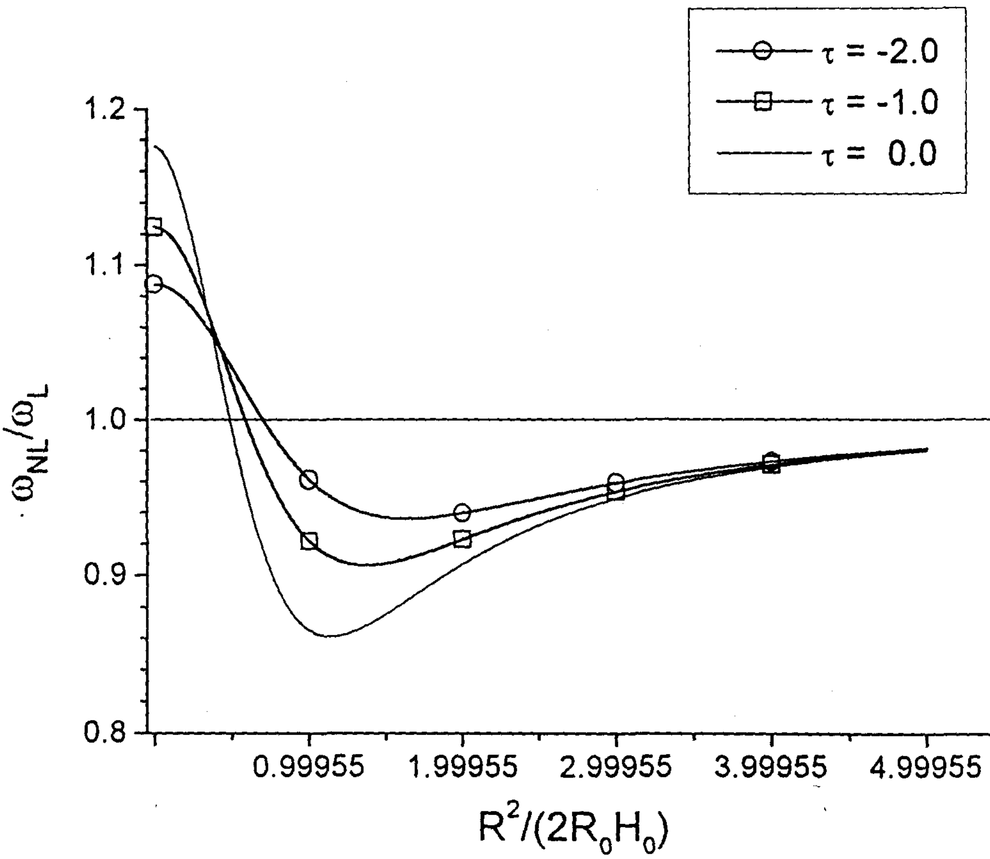

Fig. 5.2(b) $\dfrac{\omega_{NL}}{\omega_L}$ **vs.** $\dfrac{R^2}{2R_0H_0}$ **for different values of** τ **at** $\dfrac{A}{H_0}=1.0$ **and.**

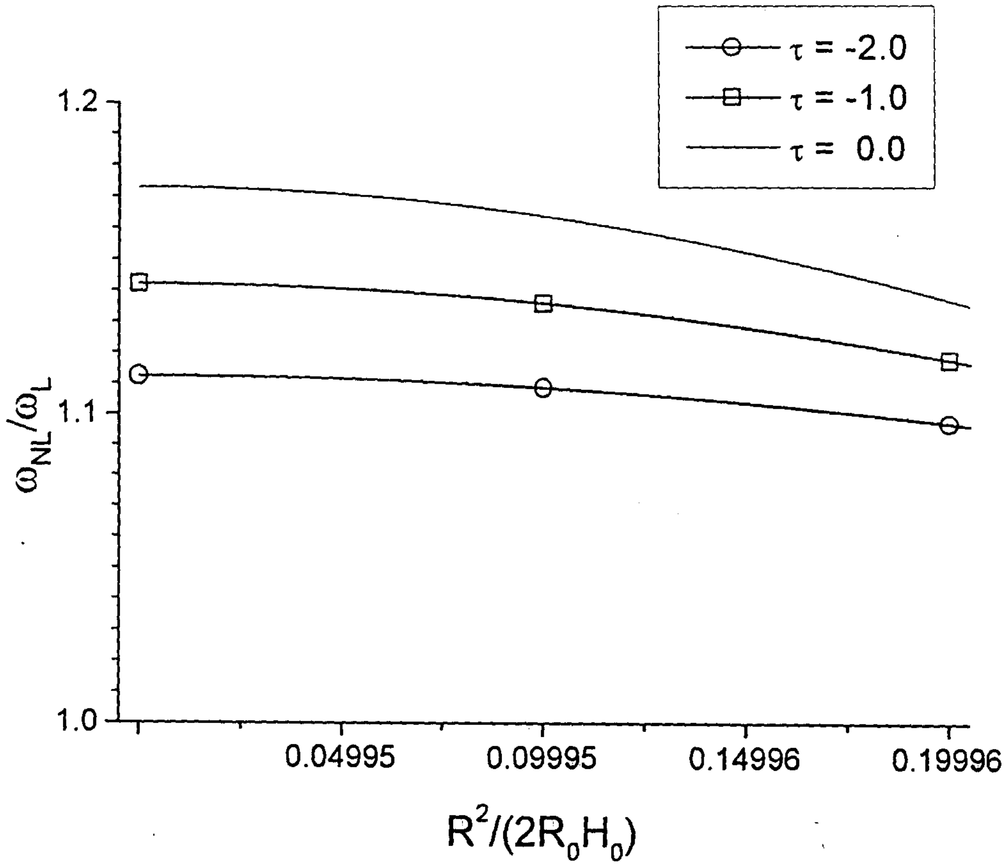

Fig. 5.3(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{R^2}{2R_0H_0}$ **for different values of** τ **at** $\frac{A}{H_0} = 1.0$.

[For very low values of $\frac{R^2}{2R_0H_0}$ **]**

[Berger's Method]

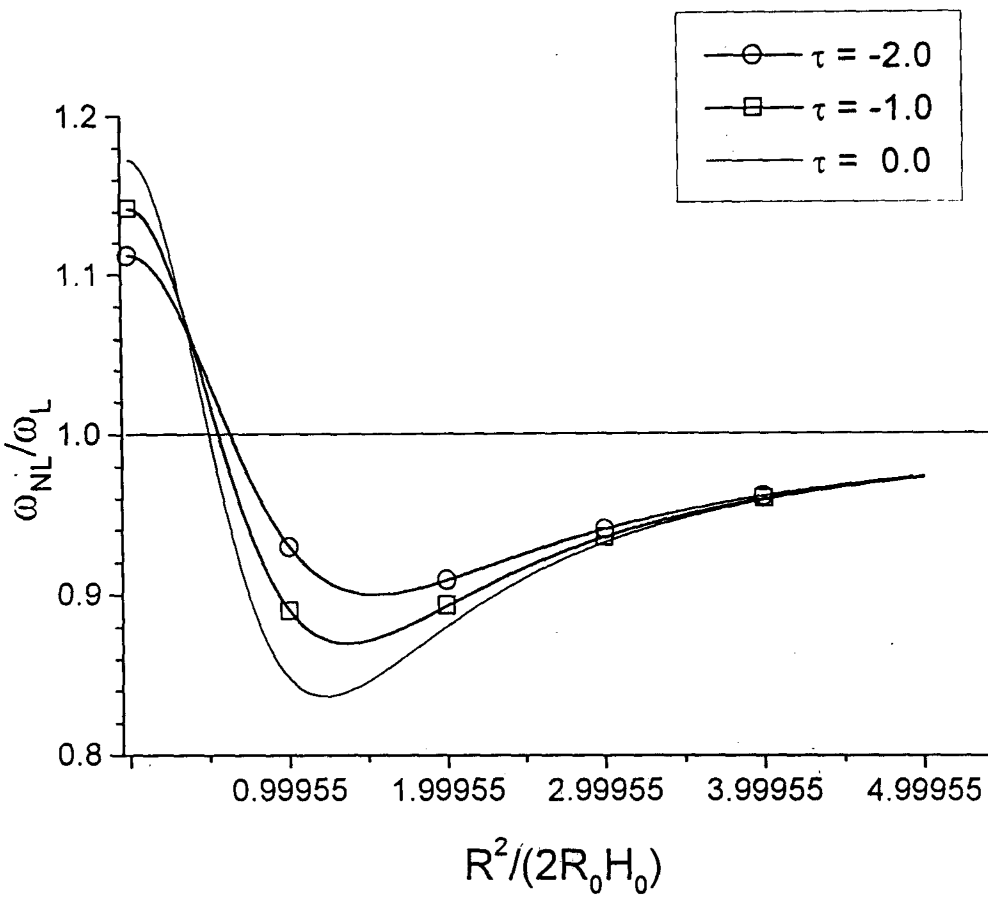

Fig. 5.3(a) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{R^2}{2R_0H_0}$ **for different values of** τ **at** $\frac{A}{H_0} = 1.0$.

[Berger's Method]

5.5.2 The Effect of thickness parameter (τ)

This section, again, refers to the Figures 5.2(a) and 5.2(b). By virtue of Equation (5.2) the shell thickness increases towards the edges as the value of τ decreases algebraically which is in tune with the design concepts and accordingly algebraically decreasing values of τ are considered. The shell stiffness, both linear as well as nonlinear, increases with decrease of τ algebraically. At very low value of $\frac{R^2}{2R_0H_0}$ when a shallow spherical shell tends to behave like a circular plate, $\frac{\omega_{NL}}{\omega_L}$ increases with increase of τ. As τ increases shell stiffness, both linear as well as nonlinear, decreases; but the linear stiffness, which is independent of amplitude of vibrations, decreases faster compared to the nonlinear stiffness due to additional stiffness associated with large deflection and $\frac{\omega_{NL}}{\omega_L}$ increases with increase of τ. At higher values of $\frac{R^2}{2R_0H_0}$ i.e. in the strain-softening zone, as the value of τ decreases algebraically the shell thickness increases towards the edges which in turn increases stiffness and hence decreases bending deformations of shell as well as reduces the shell contraction by lowering meridional stress level. As a result resistance of shallow spherical shell to snap buckling increases with decrease of τ. This phenomenon makes the nonlinear frequency to increase and the latter approaches the linear frequency i.e. $\frac{\omega_{NL}}{\omega_L}$ increases with decrease of τ.

5.5.3 The Effect of nondimensional amplitude $\left(\frac{A}{H}\right)$

Figures 5.4(a) and 5.4(b) depict the influence of nondimensional amplitude on relative nonlinear frequency for $\tau = 0.0$. At very low value of nondimensional geometrical parameter $\frac{R^2}{2R_0H_0}$ when a shallow spherical shell tends to behave like a circular plate, $\frac{\omega_{NL}}{\omega_L}$ increases with increase of $\frac{A}{H}$. The behavior is strain-hardening one. Reasons for such behavior are same as those of plates. The linear frequency is independent of amplitude of vibration. The additional stiffness due to large deflection i.e. deformation of the middle surface make the nonlinear stiffness and hence, the nonlinear frequency of vibrations of the shell to increase with increase of the amplitude of vibrations. This phenomenon explains the strain-hardening behavior of shell structures at very low value of $\frac{R^2}{2R_0H_0}$ (below the transition value) and this behavior is valid within the proportional limit of the structure material. At higher values of $\frac{R^2}{2R_0H_0}$ i.e. the zone where large deflection facilitates buckling of shallow spherical shell the nonlinear stiffness decreases faster compared to the linear stiffness with increase of $\frac{A}{H}$ and $\frac{\omega_{NL}}{\omega_L}$ decreases with increase of $\frac{A}{H}$ i.e. the behavior is strain-softening one. This phenomenon explains the behavior of shallow spherical at higher values of $\frac{R^2}{2R_0H_0}$ (above the transition value).

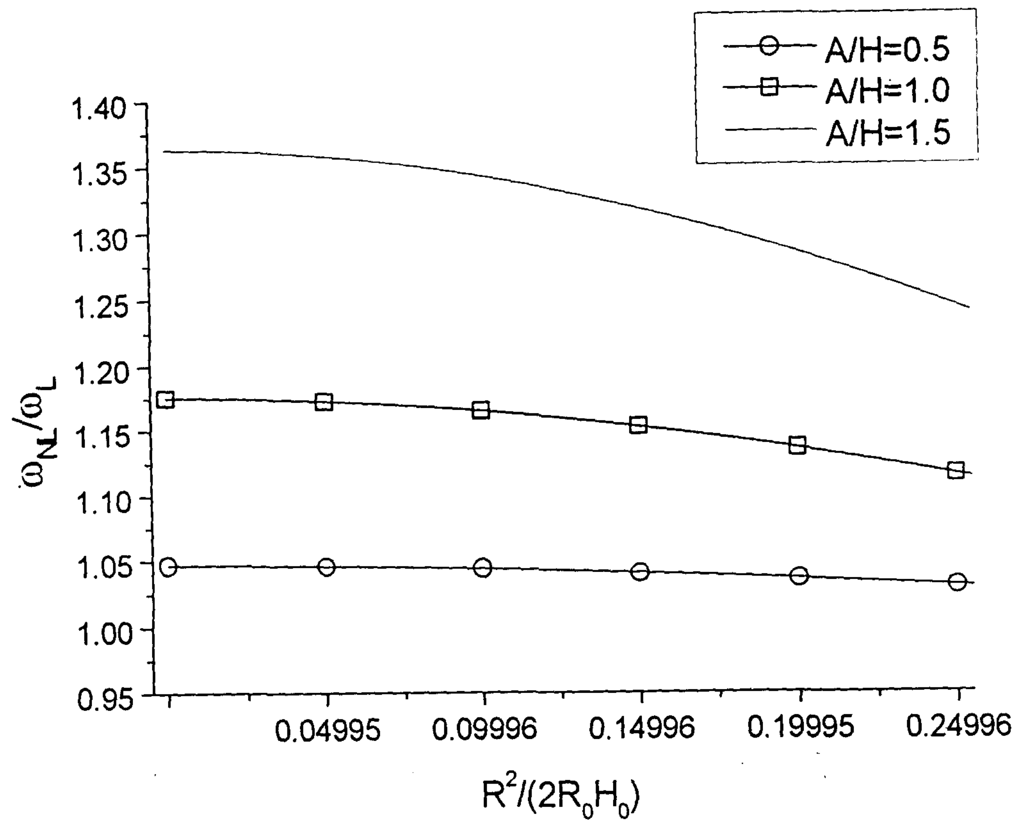

Fig. 5.4(a) $\dfrac{\omega_{NL}}{\omega_L}$ **vs.** $\dfrac{R^2}{2R_0H_0}$ **for different values of** $\dfrac{A}{H_0}$ **at** $\tau = 0.0$.

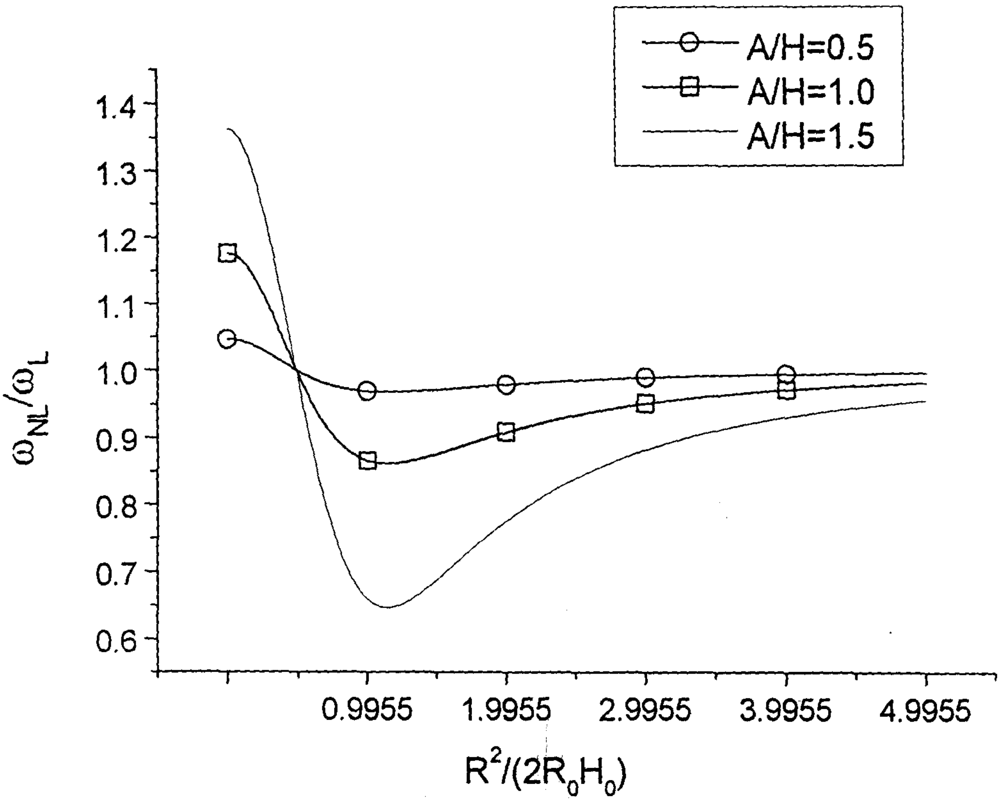

Fig. 5.4(b) $\frac{\omega_{NL}}{\omega_L}$ **vs.** $\frac{R^2}{2R_0H_0}$ **for different values of** $\frac{A}{H_0}$ **for** $\tau = 0.0$.

CHAPTER – VI

SUMMARY AND CONCLUSIONS

The nonlinear dynamic (free vibration) analysis of thin plate and shell structures of different geometrical shapes with or without thermal loading is presented. The structures are assumed to be made of homogeneous, isotropic and elastic materials. In some problems of plate and shell structures elastic properties of materials are assumed constant and temperature-independent. Thin isotropic circular plates, made of materials having temperature-dependent material properties, are also considered.

Continuum model has been used to derive basic governing differential equations as and whatever required, in the sense of von Karman classical large deflection theory or on the basis of Berger approximations i.e. the energy contribution due to second strain invariant of the middle surface is neglected in the total potential energy expression. The basic governing differential equations for the resulting system has been solved by Galerkin's error minimizing technique incorporating prescribed boundary conditions to obtain the dynamic characteristics, both linear as well as nonlinear.

Numerical results to study the effects of different parameters, as occurred in the analysis, on nonlinear dynamic behavior of such structures have been presented and compared with the available published results or observed behavior wherever possible in some cases. The present study reveals some interesting nonlinear dynamic behavior which may prove useful to the designers.

The Chapter I is introduction which includes scope, previous works, outline of present work and organization.

The Chapter II studies nonlinear free vibrations of axisymmetric thin isotropic circular plates with clamped immovable edges subjected to thermal loading characterized by constant surface temperatures T_u and T_b, measured from stress free temperature, for upper surface and lower surface respectively. Basic governing differential equations in the von Karman sense have been derived in terms of displacement components with the inclusion of thermal loading. These equations are solved by Galerkin error minimizing technique incorporating clamped immovable edge conditions. A parametric study of such structures is also presented.

It is observed that clamped circular plates exhibit strain-hardening type of nonlinearity for immovable edge conditions valid within the proportional limit of plate material. The additional stiffness due to large deflection i.e. stretching of the middle surface increases the nonlinear stiffness and hence, the nonlinear frequency of the plate with increase of amplitude. On the other hand, the linear frequency is independent of amplitude of vibration. This phenomenon explains the strain-hardening behavior of thin circular plate structures. This behavior is more pronounced in presence of thermal loading.

The average surface temperature $\left(\dfrac{T_u + T_b}{2}\right)$ part of thermal loading induces compressive stress in the circular plate with clamped immovable edges. As a result, increase of average surface temperature $\left(\frac{T_u + T_b}{2}\right)$ reduces stiffness and hence frequency of vibration; both linear as well as nonlinear. Due to the presence of the additional stiffness associated with large deflection the linear frequency decreases more than the nonlinear one proportionately and as a result $\frac{\omega_{NL}}{\omega_L}$ increases. At certain stage linear frequency becomes zero i.e. thermal instability of the structure occurs based on linear theory; but nonlinear frequency is still non-zero which makes $\frac{\omega_{NL}}{\omega_L}$ infinity. So, thermal instability is delayed due to additional stiffness associated with large deflection as per nonlinear theory. However, the constant linear thermal gradient part $\left(\frac{T_u - T_b}{H}z\right)$ along the thickness direction has no effects on nonlinear dynamic behavior of such structures for the assumed thermal loading characterized by unequal constant surface temperatures. This is due to the fact that the constant linear thermal gradient part does not contribute to the thermal stress resultants since the material properties are assumed independent of temperature. Further, the constant linear thermal gradient along the thickness direction through out the plate induce constant pure bending moment uniformly distributed along the edges of the plate and no shear will be induced in the thickness direction. As a result the linear thermal gradient part does not influence relative nonlinear frequency of vibrations in the transverse direction of such structures.

In presence of thermal loading, increase of slenderness parameter reduces stiffness and hence frequency of vibration; both linear as well as nonlinear; but the linear frequency decreases more than the nonlinear one proportionately due to the presence of the additional stiffness associated with large deflection and as a result $\frac{\omega_{NL}}{\omega_L}$ increases. At certain stage linear frequency becomes zero due to presence of thermal loading i.e. thermal instability occurs based on linear theory; but nonlinear frequency is still non-zero which makes $\frac{\omega_{NL}}{\omega_L}$ infinity.

In absence of thermal loading, besides amplitude (A) and plate thickness (H) natural frequencies, both linear and nonlinear, depend on the radius of the plate by the same proportion and increase of slenderness parameter $\left(\frac{R}{H}\right)$ may be viewed as increase of the radius at constant plate thickness (H) . So, with increase of slendemess parameter natural frequencies, both linear and nonlinear, decrease in the same proportion and relative nonlinear frequency is independent of slenderness parameter $\left(\frac{R}{H}\right)$ at constant amplitude (A) of vibrations and constant plate thickness (H).

Chapter III presents nonlinear free vibrations of thin elastic plates of various geometric shapes with clamped immovable edges viz. triangular and parabolic plates in presence of thermal loading. The thermal loading is again characterized by constant surface temperatures T_u and T_b, measured from stress free temperature, for upper surface and lower surface respectively. Decoupled nonlinear governing differential equations on the basis of Berger approximation (i. e. neglecting second strain invariant e_2) have been used. These equations have been have been solved by Galerkin's error minimizing technique. This Chapter also include parametric studies.

The triangular plates and the parabolic plates with clamped immovable edges exhibit strain-hardening type of nonlinearity i.e. $\frac{\omega_{NL}}{\omega_L}$ increases with increase of nondimensional amplitude $\left(\frac{A}{H}\right)$ valid within the proportional limit of the plate material and such effects are greatly influence by plate geometry and thermal loading.

In case of triangular plates with clamped immovable edges, as the deviation from the geometric configuration of triangular plates having aspect ratio around 1.0 and skew angle around 30^0 increases, any two supports relatively come closer or one of the perpendicular from an apex to the opposite side becomes shorter and as a result the stiffness of the plate increases. In other words, as the deviation from the geometric configuration of triangular plates having aspect ratio around 1.0 and skew angle around 30^0 decreases, the stiffness of the plate and hence, the frequency decreases. However, the rate of decrease of linear frequency is more than nonlinear frequency because of the additional stiffness of the plate due to large deflection and the relative nonlinear frequency increases. As a consequence, a triangular plate with the geometric configuration having aspect ratio around 1.0 and skew angle around 30^0 is more prone to develop dynamic instability due to reasons discussed above. Such effects are more pronounced at higher values of average surface temperature $\frac{(T_u + T_b)}{2}$ and slenderness parameter $\left(\frac{ab}{H^2}\right)$ in presence of thermal loading.

Similarly, for parabolic plates with clamped immovable edges it is observed that with increase of aspect ratio relative nonlinear frequency increases up to a maximum around the value of aspect ratio 1.4 and then decreases. This is due to the fact that as the deviation from the geometric configuration of a parabolic plate having aspect ratio around 1.4 increases, opposite supports relatively come closer and as a result the stiffness of the plate in the transverse direction increases. In other words, as the deviation from the geometric configuration of a parabolic plate having aspect ratio around 1.4 decreases, the stiffness of the plate i.e. the frequency decreases. However, the rate of decrease of linear frequency is more than nonlinear frequency because of the additional stiffness associated with large deflection. The effect is more pronounced at higher values of average surface temperature $\frac{(T_u + T_b)}{2}$ and/or slenderness parameter $\left(\frac{ab}{H^2}\right)$ in presence of thermal loading. So, a parabolic plate having aspect ratio around 1.4 is more prone to develop dynamic instability with or without thermal loading due to the reasons discussed above.

In absence of thermal loading, besides amplitude (A) of vibrations, plate thickness (H), aspect ratio $\left(\frac{a}{b}\right)$ and skew angle (ϕ) natural frequencies, both linear and nonlinear, also depend on the plate dimensions in the same proportion and increase of slenderness ratio may be achieved by increasing plate dimensions at constant plate thickness (H). So, with increase of slenderness ratio natural frequencies, both linear and nonlinear, decrease in the same proportion and the relative nonlinear frequency remains constant at constant amplitude (A) of vibrations and plate thickness (H) in absence of thermal loading.

The thermal loading, characterized by unequal constant surface temperatures, and the slenderness parameter influence the free vibrations of triangular and parabolic plates in the same fashion as in case of circular plates.

The Chapter IV investigates the nonlinear dynamics of thin circular elastic plates under thermal loading considering nonhomogeneity arising due to temperature dependency of material properties such as thermal conductivity, modulus of elasticity, Poisson's ratio and coefficient of thermal deformations under thermal loading with temperature gradient along the thickness direction. The basic governing differential equations have been derived in the von Karman sense in terms of displacement components and solved with the help of Galerkin procedure. Further, to solve the steady-state heat conduction problem temperature dependency of thermal conductivity has been considered.

Parametric studies have been presented for thin circular elastic plates made of Titanium Alloy $(Ti-6Al-4V)$ to understand the influences of different parameters; temperature level is being kept below the temperature which causes thermal instability. The modulus of elasticity decreases with increase of temperature and hence, the stiffness of the plate reduces which results in reduction of nonlinear frequency of vibration. The coefficient of thermal deformation increases with increase of temperature in the temperature range under consideration. Since the presence of thermal loading reduces plate stiffness, the increase of coefficient of thermal deformation intensifies effects of thermal loading and hence reduces plate stiffness resulting in reduction of nonlinear frequency. The effect of temperature dependency of Poisson's ratio on nonlinear frequency is negligible. Finally, the combined effect of temperature dependency of modulus of elasticity, coefficient of thermal deformation and Poisson's ratio is to reduce the nonlinear frequency; but the reduction is not appreciable compared to the effects of large deflection and direct heating for such structures.

This study reveals that temperature dependency of thermal conductivity does not alter temperature distribution across the thickness significantly within the temperature level considered. However, the temperature level across the thickness increases marginally due to higher value of thermal conductivity at increased level of temperature and as a result temperature-dependent thermal conductivity does not influence the natural frequency significantly for such structures. Though numerical results for natural frequency are not presented, it decreases the natural frequency slightly due to marginal increased level of temperature across the thickness. Again, the nonlinear frequency decreases with increase of the temperature difference between the surfaces when the material properties including the thermal conductivity are temperature-dependent; although the magnitude of reduction is very small compared to the effects of large deflection as well as direct heating.

The Chapter V studies the nonlinear free vibrations of axisymmetric thin shallow spherical shell of variable thickness having tangentially clamped immovable edges without thermal loading. Analysis has been performed by using both (i) coupled governing differential equations derived in the von Karman sense in terms of displacement components as well as (ii) decoupled nonlinear governing differential equations on the basis of Berger approximation (i. e. neglecting second strain invariant e_2) derived from energy expression applying Hamilton's principle and Euler's variational equations. The governing differential equations are solved by Galerkin error minimizing technique incorporating clamped immovable edge conditions. Axisymmetric thin shallow spherical shells exhibit both strain-hardening as well as strain-softening types of behavior. As the value of geometrical parameter $\frac{R^2}{2R_0H_0}$ approaches very low value i.e. a very shallow thin spherical shell tends to behave like a circular plate and exhibit strain-hardening type of nonlinearity i.e. $\frac{\omega_{NL}}{\omega_L}$ increases with increase of $\frac{A}{H}$; linear frequency being independent of $\frac{A}{H}$. With increase of $\frac{R^2}{2R_0H_0}$, the behavior of shallow spherical shell changes from strain-hardening type to strain-softening type. This change of behavior of shallow spherical shell may be attributed to large deflection which facilitates snap buckling of shallow spherical shell. During compression phase of vibrations, shallow spherical shell primarily contracts in size without appreciable bending; bending deformations being localized near the built-in edges and try to buckle in a direction opposite to the initial curvature, called snap buckling. Large deflection facilitates snap buckling of shallow shell and the force required to maintain equilibrium reduces i.e. nonlinear stiffness decreases. On the other hand, linear stiffness, which is independent of amplitude of vibrations, increases with increase of $\frac{R^2}{2R_0H_0}$. As a result, relative nonlinear frequency decreases to yield the lowest peak and this peak corresponds to the geometrical configuration of thin shallow spherical shell which is most liable to undergo snap buckling. With further increase of $\frac{R^2}{2R_0H_0}$ the resistance of shallow spherical shell to snap buckling increases and the shell stiffness, both linear as well as nonlinear, increases; the rate of increase of the nonlinear stiffness being slightly greater than the linear one due to additional stiffness associated with large deflection and the relative nonlinear frequency increases and finally it approaches unity. As the nondimensional geometric parameter $\frac{R^2}{2R_0H_0}$ for thin shallow shell increases the shallow spherical shell primarily contracts in size and the bending deformations, which can be neglected, remain localized near the edges while the remaining part of the shallow spherical shell remains spherical. As a result the contribution of additional stiffness associated with large deflection i.e. stretching of the middle surface tends to become insignificant and the nonlinear stiffness approaches the linear stiffness; hence $\frac{\omega_{NL}}{\omega_L}$ approaches unity.

The shell thickness increases towards the edges as the value of τ decreases algebraically which in turn increases shell stiffness both linear as well as nonlinear. At very low value of $\frac{R^2}{2R_0H_0}$ when a shallow spherical shell tends to behave like a circular plate, $\frac{\omega_{NL}}{\omega_L}$ increases with increase of τ. As τ increases algebraically shell stiffness, both linear as well as nonlinear, decreases; but the linear stiffness, which is independent of amplitude of vibrations, decreases faster compared to the nonlinear stiffness due to additional stiffness associated with large deflection and $\frac{\omega_{NL}}{\omega_L}$ increases with increase of τ. At higher values of $\frac{R^2}{2R_0H_0}$ i.e. the strain-softening zone, as the value of τ decreases algebraically the shell thickness increases towards the edges which in turn increases stiffness and hence decreases bending deformations of shell as well as reduces the shell contraction by lowering meridional stress level. As a result resistance of shallow spherical shell to snap buckling increases with decrease τ. Also, nonlinear frequency approaches linear frequency with decrease of τ since additional stiffness due to large deflection become negligible with decrease of bending deformations.

Further, at very low value of nondimensional geometrical parameter $\frac{R^2}{2R_0H_0}$ (below the transition value) when a shallow spherical shell tends to behave like a circular plate, $\frac{\omega_{NL}}{\omega_L}$ increases with increase of $\frac{A}{H}$. The behavior is strain-hardening one as mentioned earlier and this behavior is valid within the proportional limit of the structure material. At higher values of $\frac{R^2}{2R_0H_0}$ i.e. the zone where large deflection facilitates buckling of shallow spherical the nonlinear stiffness decreases faster compared to the linear stiffness with increase of $\frac{A}{H}$ and $\frac{\omega_{NL}}{\omega_L}$ decreases with increase of $\frac{A}{H}$ i.e. the behavior is strain-softening one. This phenomenon explains the behavior of shallow spherical at higher values of $\frac{R^2}{2R_0H_0}$ (above the transition value).

Finally, it can be mentioned that this study reveals some interesting dynamic behavior of thin isotropic plate and shell structures which may be useful to the designers, For example, the knowledge of dynamic behavior of such structures will help to achieve desired level of frequency of vibrations by properly proportioning the geometric elements. This will also help to utilize the advantage of extra strength due to strain-hardening and to reduce the failure probability by taking proper precaution against structural instability.

It is suggested that in a future work it will be very interesting to study nonlinear free vibrations of thin shallow spherical shell under thermal loading. Theoretical investigations considering various factors like material nonlinearity, initial imperfections, complex geometry of structures etc. will be very good work in this field. As thin plate and shell structures made from fiber-reinforced composite materials are increasingly widely used as major structural components in industries, there is lot of scope of investigations in this field.

Experimental investigations on large deflection static and dynamic behavior of plate and shell structures with or without thermal loading will be very useful to strengthen and develop theoretical investigations further.

NOTATIONS

Unless otherwise stated, the following notations have been used in this thesis

A = Amplitude of transverse vibrations.

a = Plate dimension.

B = Amplitude of in-plane radial displacement.

b = Plate dimension.

C_o, C_1, C_2, C_3 = Parameters; function of time.

C_b = Berger's constant which is independent of x and y; but involves time 't'.

C_1^T , C_2^T = Constants of integration.

c_p = Specific heat.

cn = Jacobi's elliptic function

D_c = Constant flexural rigidity of a plate or shell.

D_0 = Flexural rigidity of a plate or shell of variable thickness at the center.

D = Variable flexural rigidity of a plate or shell defined by appropriate coordinates.

E_c = Constant Young modulus of a plate or shell

$E\{T(r,\theta,z)\}$ = Temperature-dependent modulus of elasticity.

E_1, E_2, E_3, E_4, E_5, E_6 = Parameters defined by Equations (4.20a) to (4.20f)

e = 2.718282.

e_1 , e_2 = First and second strain invariants of the middle surface of a plate or shell.

F(t) = Function of time t.

F_1, F_2, F_3 = Parameters defined by Equations (4.28a) to (4.28c).

F' = A Functional.

f(t) = Function of time t.

H = Uniform thickness of a plate or shell.

H_o =Thickness of a plate or shell at the center.

h(r) = Thickness of a plate or shell at radial distance r.

i , j = Integers.

K_E = Kinetic energy of a vibrating plate or shell.

K = Positive constant.

k_c = The constant thermal conductivity.

k_0 = The value of thermal conductivity at $T = 0.0$.

k_1, k_2 = The temperature coefficients of thermal conductivity.

$k\{T(r,\theta,z)\}$ = The temperature-dependent thermal conductivity.

L = The Lagrangian = $K_E - P_E$

M_{rr} = Bending moment per unit length in radial direction.

M_T = Thermal stress couple (bending moment) per unit length.

$M_{\theta\theta}$ = Bending moment per unit length in circumferential direction

N_{rr} = Stress resultant per unit length in radial direction of a circular.

N_T = thermal stress resultant per unit length.

$N_{\theta\theta}$ =Stress resultant per unit length in circumferential direction.

N_T^* = A thermal parameter.

P_E = Total potential energy due to bending and stretching of the middle surface of a deflected plate or shell.

Q = Transverse shear force per unit length on circumferential plane of a plate or shell.

q = The internal heat generation per unit time per unit volume.

R = Radius of a circular plate or base plane of a shell.

R_0 = Radius of curvature of the middle surface of a shell.

r, θ, z = Cylindrical polar coordinates.

s_0 = A parameter defined by Equation (4.14)

s_1, s_2 = Constants defined by Equations (4.15) and (4.16) respectively.

T = Temperature field, defined by proper coordinates, over the area of a plate or shell.

T_b and T_u = Surface temperatures of plates.

T_o = The strain free reference temperature of a plate.

T_{NL} = The nonlinear time period.

U = Shape function for Inplane radial displacement.

u, v = Inplane displacements defined by appropriate coordinates.

W = Shape function for transverse deflection.

w = Deflection normal to the middle surface of plates or base plane of shell defined by appropriate coordinates.

x, y, z = Rectangular Cartesian coordinates.

$Z(r)$ = The elevation of the middle surface of the shell above the horizontal base plane defined by Equation (2.20).

α_t = Coefficient of linear thermal deformations.

$\alpha\{T(r,\theta,z)\}$ = Temperature-dependent coefficient of linear thermal deformations.

ε_r^0 and ε_θ^0 = Strains of the middle surface in the radial and circumferential directions respectively.

$\varepsilon_{r\theta}^0$ = shearing strain of the middle surface in the $r\theta$ plane

ε_{rr} and $\varepsilon_{\theta\theta}$ = Strains in the radial and circumferential directions at a distance z from the middle surface respectively.

$\varepsilon_{r\theta}$ = shearing strain at a distance z from the middle surface in the $r\theta$ plane

ε_x and ε_y = Strains of the middle surface of a plate in x and y directions respectively.

ϕ = Skew angle of skew coordinate system.

$\varphi(r)$ = A function of r defined by eqn. (2.13)

γ_{xy} = Shearing strain of the middle surface of a plate in Cartesian coordinates.

$\eta' = \dfrac{k}{\rho c_p}$ = thermal diffusivity

λ_i, λ_i' λ_{ij}, = parameters.

ν_c = Constant Poisson's ratio.

$\nu\{T(r,\theta,z)\}$ = Temperature-dependent Poisson's ratio.

$\pi = \dfrac{22}{7}$

ρ = Density of plate or shell material.

σ_{rr} and $\sigma_{\theta\theta}$ = Stresses at a distance z from the middle surface of the plate in the radial and circumferential directions respectively.

$\sigma_{r\theta}$ = Stresses at a distance z from the middle surface of the plate in the radial and circumferential directions respectively.

ς, η = Skew coordinates.

τ = A constant; called thickness parameter.

ω_{NL} = Nonlinear frequency of vibrations

ω_L = Linear frequency of vibrations.

$\psi(r)$ = A function of r, the particular solution for the term $\phi(r)$.

Ω = The Hamilton's integral.

∇^2 = Laplace operator

$= \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}$ in polar coordinates and

$= \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial}{\partial y^2}$ in Cartesian coordinates.

$\nabla^4 = \nabla^2 . \nabla^2$

Superscript ".."= Represents double derivative of the super-scripted variable with respect to time.

Superscript "."= Represents derivative of the super-scripted variable with respect to time.

Subscript "," represent derivative of the subscripted variable with respect to the subscript.

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