Chapter **2**

INTRODUCTION TO NONLINEAR DYNAMICS AND CHAOS

2.1 Introduction:

Tell me, O Muses who dwell on Olympos, and observe proper order
for each thing as it first came into being.
Chaos was born first and after her came Gaia
the broad-breasted, the firm seat of all
the immortals who hold the peaks of snowy Olympos,...

- *Hesiod,* Theogony, *lines 114-118*

This chapter covers the basic concepts of chaos.

2.2 Dynamic System, State and State-space:

Everything in this world exists in motion. There is nothing static or unchangeable. Some matters of this material world may appear to be static but those are also changing. Ever since this fact is recognized the study of dynamics has been a major pursuit. At first, all investigations were piecemeal. Newtonian scientists were studying the dynamics of the moving bodies , biologists were studying the changes in living organisms, chemists were studying the chemical properties of the materials etc. Gradually, it has been recognized that though the objective of these studies are different, there are common elements in all changes. Therefore, a body of knowledge is gradually emerged which is Dynamical System in general. A system whose status changes with time is called a *Dynamical System*.[1].

The status of a Dynamical System at any instant and the change in status of the system with time is uniquely expressed by a minimum number of properly identified variables known as *State Variables*. The study of the dynamics of a dynamical system is essentially an investigation of how these state variables change with respect to time. Mathematically, this is expressed as the rate of change of these state variables to their current values in terms of a system of first order differential equations. Thus, if state variables are given by $\{x_i, i = 1,2,.........,n\}$ then the state-space model of the system is expressed as in the form of a set of first order differential equations as follows:

$$\dot{x}_1 = \frac{dx_1}{dt} = f(x_1, x_2, ...x_n)$$

$$\dot{x}_2 = \frac{dx_2}{dt} = f(x_1, x_2, ...x_n)$$

$$\cdot$$

$$\cdot$$

$$\dot{x}_n = \frac{dx_n}{dt} = f(x_1, x_2, ...x_n)$$

In general, $\dot{x}_i = f_i(x_1, x_2, ..., x_n)$ (2.1)

or in vector form, $\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$ (2.2)

The ability to express $\dot{\mathbf{X}}$ purely as a function of $\mathbf{X}$ is what identifies x_i as state variables.

Some systems change discretely. Such a situation may arise when a system is actually changing continuously but is observed only at certain intervals. Most power electronic circuits can be modeled this way. There can be inherently discrete systems as in digital electronic systems or populations of various species. In such cases the state variables at the (n+1)-th instant are expressed as a function of those at the n-th instant:

$$\mathbf{x}_{\mathbf{n+1}} = f(\mathbf{x}_{\mathbf{n}}) \qquad (2.3)$$

The equations of forms 2.2 or 2.3 with a given set of initial conditions can be solved either analytically or numerically and the solutions give the future states of the system as functions of time.

The dynamics of a system can be visualized by constructing a space with the state variables as coordinates. This is called the state space or phase space. The state of the system at any instant is represented by appoint in the space. Starting from any given initial condition, the state-point moves in the state space and this movement is completely determined by the state equations. The path of the state-point is called the orbit or the trajectory of the system that starts from the given initial conditions. The trajectories are obtained as the solutions of the differential equations(2.2) or iterates of the map(2.3).

2.3 Autonomous and non-autonomous system:

If the system equations do not have any externally applied time-varying input or other time variations, the system is said to be ***autonomous***. In such systems the right hand side of (2.2) does not contain any time-dependent term. A typical example is the simplified model of atmospheric convection, known as the Lorenz system:

$$\begin{aligned} \dot{x} &= -3(x-y) \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - z \end{aligned} \tag{2.4}$$

where r is a parameter. Systems with external inputs or forcing functions or time variations in their definition, are called ***non-autonomous*** systems. In such systems, right hand side of (2.2) contains time dependent terms. As a typical example one can consider a pendulum with an oscillating support, the equations of which are

$$\begin{aligned} \dot{x} &= y + 5 \\ \dot{y} &= -y - y\sin x + r\sin \omega t \end{aligned} \tag{2.5}$$

Likewise, power electronic circuits with clock-driven control logic are non-autonomous systems.

2.4 Vector fields:

In studying the dynamical behavior of a given system, one has to compute the trajectory starting from a given initial condition. We have seen that this can be done numerically. However, it is generally not necessary to compute all possible trajectories (which may be a cumbersome exercise) in order to study a given system. It may be noted that the left hand side of (2.2) gives the rate of change of the state variables. This is a vector, which is expressed as a function of the state variables. The equation (2.2) thus defines a vector at every point of the state space. This is called the *vector field*. A solution starting from any initial condition follows the direction of the vectors, i.e., the vectors are tangent to the solutions. The properties of a system can be studied by studying this vector field. To give an example, the vector field for the system $\ddot{x}-(1-x^2)\dot{x}+x=0$ is shown in Fig. 2.1.

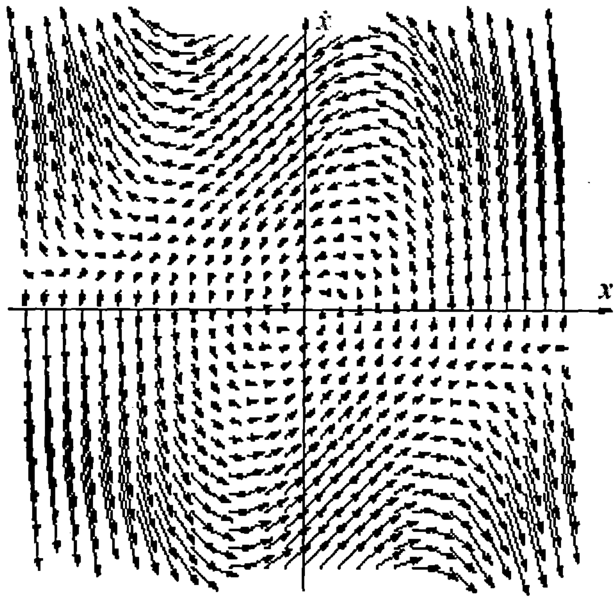

Fig.2.1: vector field for the system $\ddot{x}-(1-x^2)\dot{x}+x=0$

2.5 Local Behavior of Vector Fields Around Equilibrium Points:

The points where the $\dot{\mathbf{x}}$ vector has zero magnitude, i.e., where $\dot{\mathbf{x}}$ =f(x)=0, are called the *equilibrium points*. Since the velocity vector at the equilibrium point has magnitude zero, if an initial condition is placed there, the state-point will forever remain there. However, this does not guarantee that the equilibrium state will be stable, i.e., any deviation from it will die down. It is therefore important to study the *local* behavior of the system in the neighborhood of an equilibrium point. Since it is straightforward to obtain the solutions of a set of *linear* differential equations, the local properties of the state space in the neighborhood of an equilibrium point can be studied by locally linearizing the differential equations at that point. Indeed, most tools for the design and analysis of engineering systems concentrate only on the local behavior — because in general, the nominal operating point of any system is located at an equilibrium point, and if perturbations are small then the linear approximation gives a simple workable model of the dynamical system[2].

The local linearization is done by using the Jacobian matrix of the functional form at an equilibrium point. For example, if the state space is two dimensional, given by

$$\begin{aligned}\dot{x} &= f_1(x,y) \\ \dot{y} &= f_2(x,y)\end{aligned} \qquad (2.6)$$

then the local linearization at an equilibrium point (x^*, y^*) is given by

$$\begin{bmatrix} \delta\dot{x} \\ \delta\dot{y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} \qquad (2.7)$$

where $\delta x = x - x^*, \delta y = y - y^*$. The matrix containing the partial derivatives is called the Jacobian matrix and the numerical values of the partial derivatives are calculated at the equilibrium point. This is really just a (multivariate) Taylor series expanded to first order. Notice that in the linearized state space, the state variables are the *deviations* from the equilibrium point (x^*, y^*) . To avoid notational complexity, we'll drop the δ and will proceed with the equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad (2.8)$$

with the understanding that the origin is shifted to the equilibrium point. If the original system is non-autonomous, there will be time-dependent terms in the Jacobian matrix. In engineering literature it is customary to separate out the time-dependent and time-independent terms in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad (2.9)$$

where **A** and **B** are time-independent matrices, and the components of the vector **u** are the externally imposed inputs of the system. From (2.9) it is evident that the term **Bu** has influence on the location of the equilibrium point, while stability of the equilibrium point is given by the matrix . Therefore, while studying the stability of the equilibrium point, one considers the unforced system (2.8).

2.6 Eigenvalues and eigenvectors:

Notice that in (2.8), **A** *operates* on the vector **x** to give the vector $\dot{\mathbf{x}}$. This is basic function of any matrix-mapping one vector into another vector. Generally the derived vector is different from the source vector, both in magnitude and direction. But there may be some special directions in the state space such that if the vector **x** is in that direction, the resultant vector $\dot{\mathbf{x}}$ also lies along the same direction. It only gets stretched or squeezed. Any vector along these special directions are called *eigenvectors* and the factor by which any eigenvector expands or contracts when it is operated on by the matrix **A** , is called the *eigenvalue*.[3]

To find the eigenvectors, we need to find their eigenvalues first. When the matrix **A** operates on the vector **x** , and if **x** happens to be an eigenvector, then we can write

$$\mathbf{Ax}=\lambda\mathbf{x}$$

where λ is the eigenvalue. This yields

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$$

where **I** is the identity matrix of the same dimension as **A** . This condition would be satisfied if the determinant $|\mathbf{A}-\lambda\mathbf{I}|=\mathbf{0}$.Thus

$$\left|\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A22 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0 \tag{2.10}$$

$$\Rightarrow \lambda^2 - (A_{11} + A_{22})\lambda + (A_{11}A_{22} - A_{12}A_{21}) = 0$$

This is called the *characteristic equation*, whose roots are the eigenvalues. Thus, for a 2x2 matrix, one gets a quadratic equation — which in general yields two eigenvalues. For each eigenvalue there is one *direction* of eigenvector, and any vector in that direction is an eigenvector. The direction of the eigenvector is determinate but the magnitude is indeterminate.

If the eigenvalues are real and negative, the system is stable in the sense that any perturbation from an equilibrium point decays exponentially and the system settles back to the equilibrium point. Such a stable equilibrium point is called a *node*. If the real parts of the eigenvalues are positive, any deviation from the equilibrium point grows exponentially, and the system is unstable.

If one eigenvalue is real and negative while the other is real and positive, the system is stable along the eigenvector associated with the negative eigenvalue, and is unstable away from this. Such an equilibrium point is called a *saddle*, and a system with a saddle equilibrium point is globally unstable. The vector fields of the three types of systems are shown in Fig. 2.2.

Complex eigenvalues always occur as complex conjugate pairs. If $\lambda=\sigma+j\omega$ is an eigenvalue, $\lambda=\sigma-j\omega$ is also an eigenvalue. Let $\mathbf{v}$ be an eigenvector corresponding to the eigenvalue $\lambda=\sigma+j\omega$. This is a complex-valued vector. It is easy to check that $\bar{\mathbf{v}}$, the conjugate of the vector $\mathbf{v}$, is associated with the eigenvalue, $\lambda=\sigma-j\omega$.

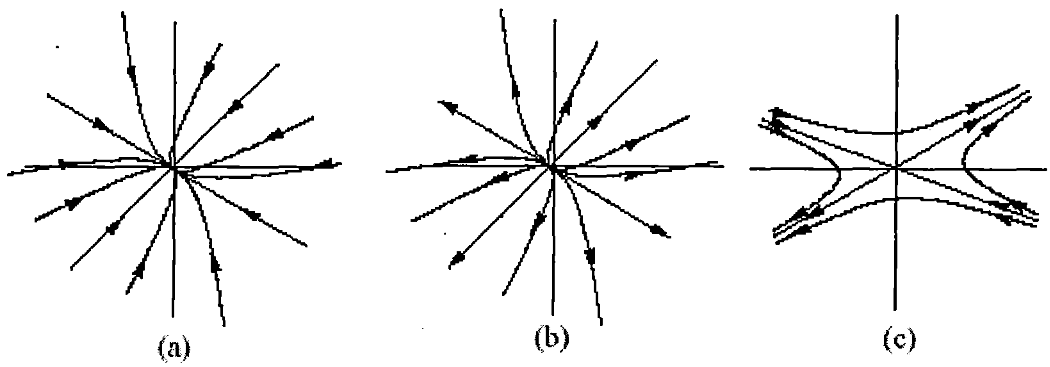

Fig.2.2: vector Field of the Linear Systems with Real Eigenvalues- (a) both eigenvalues negative, (b) both eigenvalues positive, (c) one eigen values negative and other positive.

In general, if the eigenvalues are purely imaginary, the orbits are elliptical. For initial conditions at different distances from the equilibrium point, the orbits form a family of geometrically similar ellipses which are inclined at a constant angle to the axes, but having the same cyclic frequency. When the eigenvalues are complex, with σ nonzero, the sinusoidal variation of the state variables will be multipled by an exponential term $e^{\sigma t}$

If σ is negative, this term will decay as time progesses. Therefore the waveform in time-domain will be a damped sinusoid, and in the state space the state will spiral in towards the equilibrium point. If σ is positive, the term $e^{\sigma t}$ will increase with time, and so in the state space the behavior will be an outgoing spiral as shown in Fig 2.3.

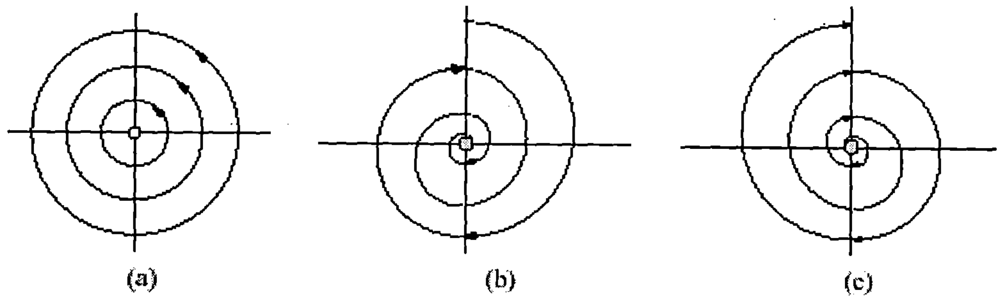

Fig.2.3: The structure of the vector field in the state space for (a) imaginary eigenvalues, (b) complex eigenvalues with negative real part, and (c) complex eigenvalues with positive real part.

2.7 Attractors in nonlinear systems:

To illustrate some typical features of nonlinear systems, we take the system given by $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$ known as the *van der Pol* equation. Fig. 6.13 shows the vector field of this system with state variables x and y=$\dot{x}$. If the parameter μ is varied from a negative value to a positive value, a fundamental change in the property of the vector field occurs. The stable equilibrium point becomes unstable and the field lines spiral outwards. But it does not become globally unstable as the field lines at a distance from the equilibrium point still point inwards. Where the two types of field lines meet,

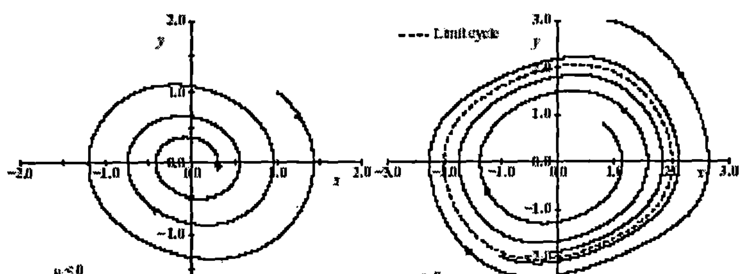

Fig.2.4: The vector fields for $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$, (a) for μ<0, (b) for μ>0
The dashed line shows the limit cycle.

there develops a stable periodic behavior. This is called a limit cycle. It is a *global* behavior whose existence can never be predicted from linear system theory. One point is to be noted here. There is a fundamental difference between the periodic behaviors in a linear system with purely imaginary eigenvalues and Fig.2.4. In the first case a different periodic orbit (though of constant period) is attained for initial conditions at different radii, while in case of the limit cycle, trajectories starting from different initial conditions converge on to the same periodic behavior. The limit cycle appears to attract points of the state space. This is an example of an *attractor*. Thus in a two-dimensional nonlinear system one can come across periodic attractors as in Fig.2.4. If the state space is of higher dimension, say three, there can be more intricate attractors. To understand this point, suppose a third-order dynamical system is going through oscillations and when we plot one of the variables against time, it has a periodic waveform as shown in Fig.2.5, corresponding to a state-space trajectory that shows a single loop.

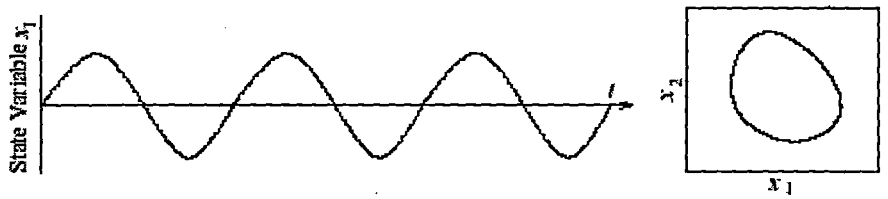

Fig2.5: The time plot (left) and the state space trajectory (right) for a period-1 attractor.

When some parameter is varied, the waveform can change to the type shown in Fig. 2.6, which has twice the period of the earlier periodic waveform. In order for such orbits to exist, the state must three dimensions. (Note that the figure actually shows a projection of a 3-D state space onto two dimensions — a real state-space orbit cannot cross itself because there is
a unique velocity vector $\dot{\mathbf{x}}$ associated with every point in the state space.)

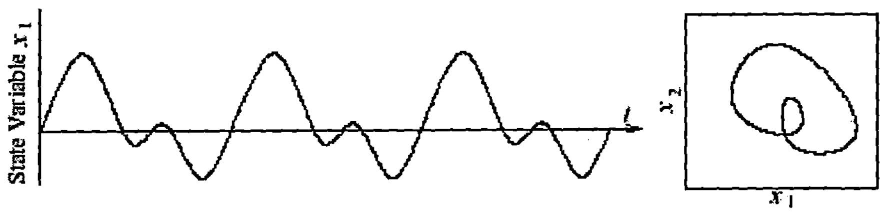

Fig.2.6: The appearance of period-2 waveform in the time domain and in state space.

Sometimes the orbit has one periodicity superimposed on another, and we have a torus-shaped attractor in the state space. This is called a quasiperiodic attractor. Fig.2.7 gives a graphic illustration.

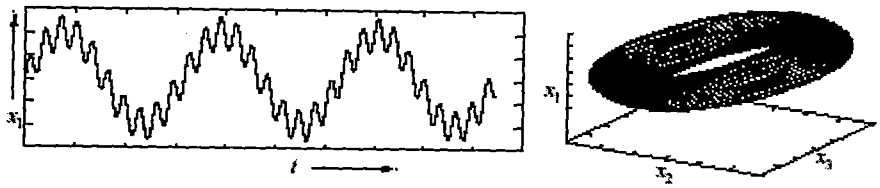

Fig. 2.7: The appearance of a quasiperiodic attractor in time domain and in state space.

One interesting possibility opens up in systems of order 3 or greater : bounded aperiodic orbits, as shown in Fig. 2.7. In such a case the system state remains bounded — within a definite volume in the state space, but the same state never repeats. In every loop through the state space the state traverses a new trajectory. This situation is called *chaos* and the resulting attractor is called a *strange attractor*. When such a situation occurs in an electrical circuit or a mechanical systems, the system undergoes apparently random oscillations.

2.8 Bifurcation:

*A **bifurcation*** is defined as a point where the flow is unstable.Aqualitative change in the dynamics which occurs as a system parameter is changed is called *bifurcation.* Conceptually, it is when there is a change in dynamic behavior, i.e., when a fixed point branches into two fixed points, or when a system changes from a sink to a saddle. This change does not happen over the course of time, but due to a change in parameters. Studying the bifurcations is helpful in determining whether a system is purely random, or an actual chaotic system. Bifurcations happen at regular intervals, which is the determinism inherent in an otherwise random system[8]. There are several kinds of bifurcations: the Hopf bifurcation, pitchfork bifurcation, explosive bifurcation and fold bifurcation. A pitchfork bifurcation branches from one fixed point into two fixed points and one unstable point. Using μ as the parameter that changes, we see a pitchfork bifurcation in the Fig2.8:

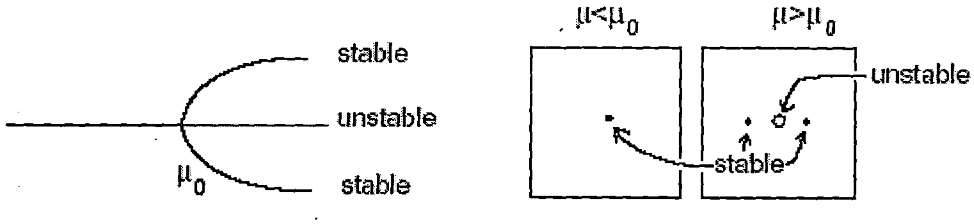

Fig. 2.8: Pitchfork bifurcation.

The phenomenon of a system evolving into a limit cycle from a fixed point is called a Hopf bifurcation, Fig. 2.9. As the system approaches the critical value μ_0, the trajectories take longer and longer to enter the equilibrium of the final state, until it takes an infinite amount of time to the equilibrium state. A secondary Hopf bifurcation when a system branches from a limit cycle to a torus. All other bifurcations transform the system from an (*n*- 1) dimensional torus to an *n*-dimensional torus.

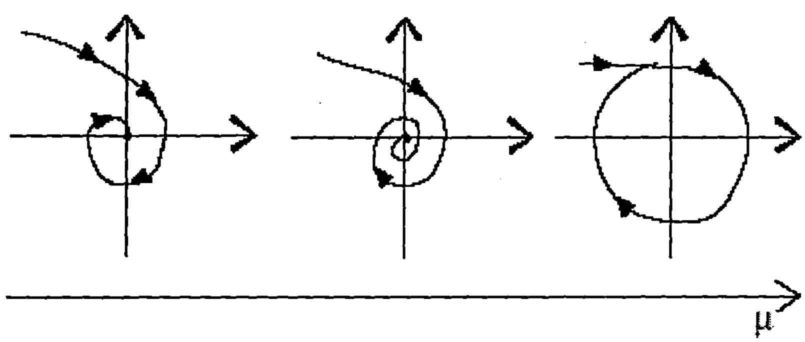

Fig. 2.9: Hopf bifurcation.

A Hopf bifurcation breeds a new limit cycle, whereas a flip bifurcation turns one limit cycle into two. Successive bifurcations give birth to more limit cycles.

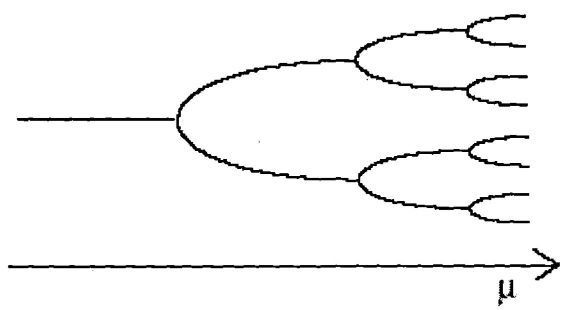

Fig. 2.10: Flip bifurcation.

Bifurcation occurs when a fixed point loses stability.Condition of stability of a fixed point, i.e., Eigenvalues should remain inside the unit circle. The classification of bifurcations depends on where an eigenvalue crosses the unit circle. Smooth systems can lose stability in three possible ways.

(a) A period doubling bifurcation: eigenvalue crosses the unit circle on the negative real line,

(b) A saddle-node or fold bifurcation: an eigenvalue touches the unit circle on the positive real line,

(c) A Hopf or Naimark bifurcation: a complex conjugate pair of eigenvalues cross the unit circle.

Explosive bifurcations are when bifurcations lead to chaotic attractors, like μ = 4:0 in the logistic map, which is an example of period doubling as well. The logistic map is as follows[79]:

$$x_{n+1} = \mu x_n(1 - x_n) \tag{2.10}$$

The system has one fixed point until μ = 2:98, then the system undergoes bifurcation, having two periods, rather than just one. When a system bifurcates and doubles the amount of stable points, the system undergoes ***period doubling***. The system bifurcates again around μ = 3:445. As μ is increased, the intervals between period doubling become shorter and shorter until at ʹ = 4:0, when the system becomes completely chaotic. At this point, the system is non-periodic, where it has an infinite amount of periods. This evolution of periodic doubling is a ***route to chaos*** as shown in Fig. 2.8.

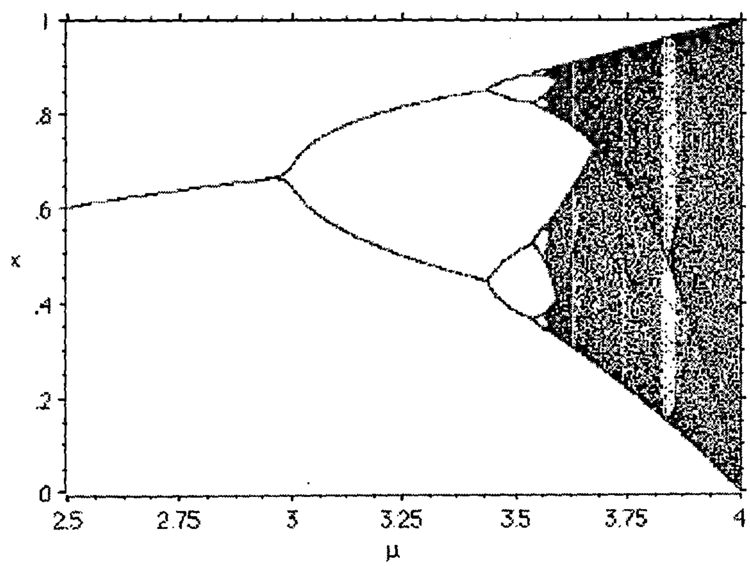

Fig. 2.10: Bifurcation diagram of the logistic mapping.

2.9 Chaos:

The word ***chaos*** is defined three ways. The word originates from the ancient Greek word ***χάος***. According to Hesiod's *Theogony,* Chaos was the first god to come into existence. She was Void, what the universe was before order and logic were laid down. The second definition of chaos is the vernacular one: a condition of great disorder or confusion, which implies randomness. Lastly, chaos may be defined as complex behavior that displays randomness, yet arises deterministically. This contrasts the original Greek meaning of the word. The ancient Greek definition of chaos conveys indescribability and incomprehension. There is no way of knowing or predicting the outcome of behavior, even in a probabilistic sense. There is no order in this description, it is the antithesis of logic. This is the primary difference between the ancient Greek usage and the vernacular definition. The American Heritage Dictionary defines vernacular chaos as, \a condition or place of great disorder or confusion." This use describes the inability to correctly predict future behavior, which is a half of the definition of scientific chaos. The difference between this definition and the ancient Greek one is that the behavior is somewhat ordered. This means that the behavior is not completely disordered: there are probabilities for future behavior, rather than a complete lack of information. Quantum mechanics is an example of this kind of chaos. There is a finite amount of accuracy in measuring the system due to Heisenberg's Uncertainty Principle, and the wave function deals with probabilities. So statistics make the behavior more logical, but complete determinism of the behavior is impossible. This is where the new definition of chaos is different. The new science of chaos is defined as stochastic behavior occurring in a deterministic system. Picking this definition apart, ***stochastic behavior*** is random behavior due to random external forces. For example, a spinning top that is randomly forced exhibits stochastic behavior. Thus, stochastic behavior is behavior that has random attributes due to *indeterminate* factors. Conversely, chaotic behavior has random attributes due to *determinant* factors. A ***system***, then, is a group of elements that form a complex whole. These elements may take the form of differential equations, or the factors that produce weather patterns. A system is chaotic if it is non-periodic, deterministic and exhibits sensitivity towards initial conditions. ***Non-periodic*** behavior does not follow a set pattern. If there is periodic behavior in a system, then future behavior may be determined. Non-periodicity is an outcome of randomness, a sign of a chaotic system. The second aspect to chaos is sensitivity towards initial conditions. A system is ***sensitive towards initial conditions*** when a slight difference in initial conditions exponentially grows over time. For example, let us drop a ball on a nail head: small differences in initial conditions result in vastly different behavior. To put it more mathematically, a system is chaotic if an initial difference of $\Delta f = x_0$ between two systems exponentially grows in the form of $\Delta f = x_0 e^{\lambda t}$ with time. This basic definition of chaos lead to the discovery of chaos in 1961, by Edward Lorenz. While working on a system of equations that is now called the Lorenz system, he ran the computer modeling program twice with the same initial conditions, except that one was to accurate to six digits, while the other was accurate to three digits. At first the two behaved identically, but after a short while they acted drastically different. He published his results in a meteorological journal in 1963, but was not recognized for his work for almost ten years until people asked questions about random, deterministic behavior[42],[43].

2.10 Poincáre Section:

While the phase plotting shows the general behavior of a system, it does not show whether a system is repetitious in a messy way, or truly chaotic. The *Poincáre Section* plot is a way to discern these two phenomena. This is done by taking an n-1 dimensional slice from an n dimensional system. See the fig below for a graphical representation of a Poincáre Section in a three dimensional chaotic system with two periods.

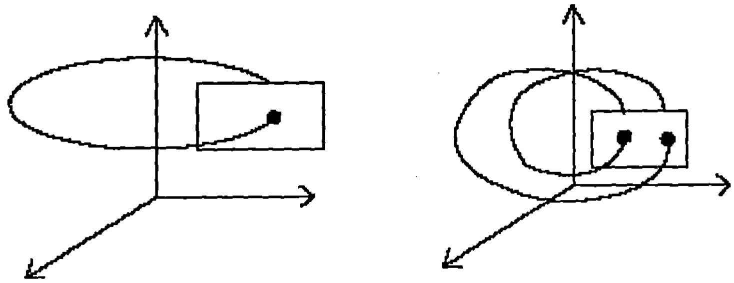

Fig. 2.11: Poincáre Section plot

2.11 Lyapunov Exponents:

A *Lyapunov exponent* measures the exponential rate of growth (or decay) of one variable in a system. A chaotic system exhibits sensitivity towards initial conditions, where the difference grows exponentially. Thus, if a system has a positive Lyapunov exponent, it is chaotic. The Lyapunov exponent of a function $f(x)$ may be found by taking two trajectories, x and x', where $x'_0 = x_0 + \varepsilon$, and ε is some small amount. Let the difference d $=|x - x'|$. Looking at the rate of expansion over N iterations, we find:

$$d_N = \varepsilon^{N\lambda}$$

where λ is the Lyapunov exponent., may be found by:

$$\lambda = \underset{N\to\infty}{Lt} \frac{1}{N} \sum_{i=1}^{N-1} \ln\left|\frac{f(x_i) - f(x_{i-1})}{dt}\right| \qquad (2.11)$$

In order to determine chaos, all the exponents must be looked at. If one of the Lyapunov exponents is positive, the system is chaotic. In phase space, this represents the system's volume growing over time. It takes more information to determine its original state. If the sum is zero, then the system is stable. There is no loss of information over time. If the sum is less than zero then the system is merely dissipative. Another tool for verifying chaos is to look at the frequency distribution of the data[78].

2.12 Frequency Distribution and Power Spectrum:

The frequency distribution of a system shows whether a system is periodic or not. A finite number of peaks corresponds to a number of periods, so if there are no distinguishable peaks, the system is non-periodic. So a chaotic system will have no distinguishable peaks. To find the frequency distribution, we must use Fourier analysis. Named after Joseph Fourier in the 1820's, Fourier analysis dictates that any signal may be represented by a series of sines and cosines. Given a set of $\{x\} = x_1, x_2, \ldots\ldots\ldots, x_{N-1}$, it is possible to take the Fourier transform and get another set of data, $\{X\} = X_1, X_2, \ldots\ldots\ldots, X_{N-1}$ where X is the Fourier transform. For any $0 < k < N\text{-}1$,

$$X_k = \sum_{j=0}^{N-1} x_j e^{-2\pi ijk/N} \tag{2.12}$$

which puts {X} on the complex plane. To see all the information, we look at the ***power spectrum*** P, where $P_i = |X_i|^2$. For a simple system of a 1Hz wave, the power spectrum is given in Fig. 2.12.

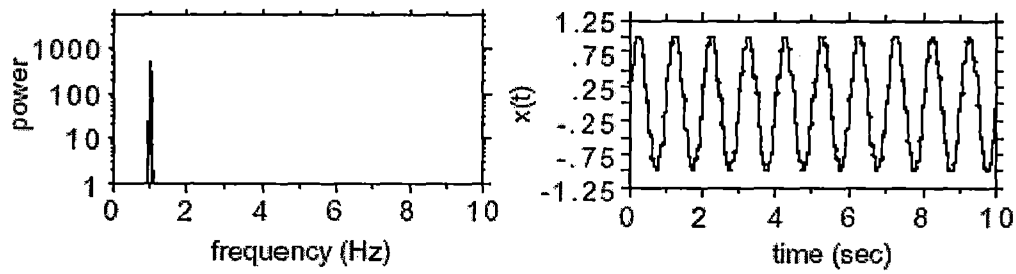

Fig. 2.12: Sinusoidal waveform power spectrum (left), or signal (right).

2.13 Dimension:

There are three kinds of dimensions: topological, Euclidean and fractal. Topological dimension is the continuity of the points, and Euclidean dimension is the dimension that the system is embedded in. Take the object below for example (Fig. 2.13). It is a disfigured plane, like a sheet of aluminum that has been struck several times with a hammer. A plane is a two dimensional object, giving it a topological dimension of $DT = 2$. However, this is embedded in three dimensions $(x;\ y;\ z)$, thus $DE = 3$. Therefore, the

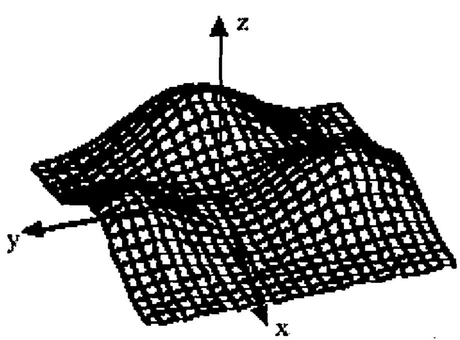

Fig.13: DT = 2 while DE = 3, thus 2 · D · 3

dimension of the object, D, must be between two and three dimensions, which is the ***fractal dimension***. The conventional method of determining dimension of a set of points is the Hausdorff- Besicovitch method, or ***box counting***. The H-B dimension is found by minimally covering the set of points with hypercubes y of different size. The difference comes from the dimension having identically sized cubes while the H-B dimension cubes may be differently sized. First we find a hypercube with side length L that encompasses that contains all points in the set. We may choose any length L as long as it binds the system. Numerically, it is better to choose the smallest possible one due to memory restrictions. Next, we fill the hypercube with hypercubes of side length $l = L/2$. $N(l)$ is the count of how many boxes contain a point inside. Doing this for $l_n = L/2^n$ for increasing n gives a dimension defined as follows:

$$D = \underset{n \to \infty}{Lt} \frac{\log(N(l_n))}{\log(l_n)} \tag{2.13}$$

Another way of finding dimension is using the Lyapunov exponents.

$$D = j + \sum_{i=1}^{j} \lambda_i / |\lambda_{j+1}| \tag{2.14}$$

where j is defined by the condition that:

$$\sum_{i=1}^{j} \lambda_i > 0 \text{ and } \sum_{i=1}^{j+1} \lambda_i < 0 \tag{2.15}$$