Chapter **4**

MATHEMATICAL MODEL OF GENERALIZED ELECTRICAL MACHINES

4.1 Introduction:

The generalized theory of Electrical Machines is used to cover a wide range of electrical machines in a unified manner. A very important of this generalization is the application of the two axis theory in which, by means of appropriate transformations, any machine can be represented by the coils on the axes. Kron called it the two axis idealized machine, from which many other can be derived, a primitive machine with one coil on each axis on each element. Any machine can be shown to be equivalent to a primitive machine with an appropriate number of coils on each fixed axis. If the coils of the practical machine are permanently located on the axes, they correspond exactly to those of the primitive machine, but, if they are not, it is necessary to make a conversion from the variables of the practical machine to the equivalent axis variables of the corresponding primitive machine, or vice versa. Any particular system of description of a machine is known as a reference frame and a conversion from one reference frame to another is known as transformation.

In this chapter the concept of different reference frames, their transformation , generalized machine and its mathematical model, the dynamic model to describe the dynamics of the generalized machine etc. are discussed[5],[7].

4.2 Reference Frame:

From the voltage and current equations that describe the performance of conventional machines, it is found that some of the machine inductances are the functions of the rotor speed, whereupon the coefficients of the governing differential equations that describe the behavior of these machines are time varying except when the rotors are stalled. A change of variables is often used to reduce the complexity of those differential equations. There are several changes of variables that are used and it was originally thought that each change of variable was different and therefore they were treated separately. It was later learnt that all changes of variables used to transform real variables are contained in one. This general transformation refers machine variables to a frame of reference that rotates at an arbitrary angular velocity. All known real transformations are obtained from this transformation by simply assigning the speed of rotation of the reference frame.

In late 1920s, R. H. Park introduced the new approach to electric machine analysis. He formulated a change of variables which, in effect, replaced the variables (voltages, currents and flux linkages) associated with the stator windings of a synchronous machine with variables associated with fictitious windings rotating with the rotor. In other words, he transformed, or referred, the stator variables to a frame of reference fixed in the rotor. Park's transformation, which revolutionized the electrical machine analysis, has the unique property of eliminating all time-varying inductances from the voltage equations of the synchronous machine which occur due to electric circuit in relative motion and electric circuit with varying magnetic reluctance.

In late 1930s, H. C. Stanley employed a change of variables in the analysis of induction machines. He showed that the time varying inductances in the voltage equations of an induction machine due to electric circuit in relative motion could be eliminated by transforming the variables associated with the rotor windings (rotor variables) to variables associated with fictitious stationary windings. In this case, the rotor variables are transformed to a reference frame fixed in the stator.

G. Kron introduced a change of variables that eliminated the position or time varying mutual inductances of a symmetrical induction machine by transforming both the stator variables and the rotor variables to a reference frame rotating in synchronism with rotating magnetic field. This reference frame is commonly referred to as the synchronously rotating reference frame.

D. S. Brereton et al. employed a change of variables that also eliminated the time-varying inductance of asymmetrical induction machine by transforming the stator variables to a reference frame fixed in the rotor. This is essentially Park's transformation applied to induction machine.

Park, Stanley, Kron and Brereton et al. developed change of variables each of which appeared to be uniquely suited for a particular application. Consequently, each transformation was derived and treated separately in literature until it was noted in[1965] that all known real transformations used in induction machine analysis are contained in one general transformation that eliminates all time varying inductances by referring the stator and rotor variables to a frame of reference that may rotate any arbitrary angular velocity or remain stationary. All known real transformations may then be obtained by simply assigning the appropriate speed of rotation, which may in fact be zero, to this so called arbitrary reference frame. It also may e noted that this transformation is sometimes referred to as " generalized rotating real transformation." Later it was noted that the stator variables of a synchronous machine could also be referred to the arbitrary reference frame[6]. It also may be noted that the time-varying inductances of a synchronous machine are eliminated only if the reference frame is fixed in the rotor (Park's Transformation), consequently the arbitrary reference frame does not offer the advantages in the analysis of the synchronous machines that it does in the case of induction machines.

A change of variables that formulates a transformation of 3-phase variables of stationary circuit elements to the arbitrary reference frame may be expressed as

$$\mathbf{f}_{qd0s} = \mathbf{K}_s \mathbf{f}_{abcs}$$

Where **f** =voltage, current, flux, flux linkage or charge

$$(\mathbf{f}_{qd0s})^T = \begin{bmatrix} f_{qs} & f_{ds} & f_{os} \end{bmatrix} \tag{4.1}$$

$$(\mathbf{f}_{abcs})^T = \begin{bmatrix} f_{as} & f_{bs} & f_{cs} \end{bmatrix}$$

$$\mathbf{K}_s = \frac{2}{3}\begin{bmatrix} \cos\theta & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \sin\theta & \sin\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{4.2}$$

$$\text{and } (\mathbf{K}_s)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 1 \\ \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta - \frac{2\pi}{3}\right) & 1 \\ \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) & 1 \end{bmatrix} \qquad \omega = \frac{d\theta}{dt} \tag{4.3}$$

Power is given by

$$P_{abcs} = v_{as}i_{as} + v_{bs}i_{bs} + v_{cs}i_{cs} \tag{4.4}$$

$$P_{dq0s} = \tfrac{3}{2}(v_{ds}i_{ds} + v_{qs}i_{qs} + v_{0s}i_{0s}) = P_{abcs} \tag{4.5}$$

Using this transformations, stationary circuit variables can be transformed to the arbitrary reference frame.

For 3-phase resistive circuit,

$$\mathbf{v}_{abcs} = \mathbf{r}_s\mathbf{i}_{abcs} \tag{4.6}$$

$$\mathbf{v}_{qd0s} = \mathbf{K}_s\mathbf{r}_s(\mathbf{K}_s)^{-1}\mathbf{i}_{dq0s} \tag{4.7}$$

$\mathbf{r}_s$ = resistance matrix of the machine.

When the machine is balanced or symmetrical and $\mathbf{r}_s$ is a diagonal matrix and its nonzero elements are equal then,

$$\mathbf{K}_s\mathbf{r}_s(\mathbf{K}_s)^{-1} = \mathbf{r}_s \tag{4.8}$$

When the machine is unbalanced or unsymmetrical and pase resistances are unequal then, arbitrary reference frame variables contain sinusoidal functions of θ except when $\omega = 0$

For inductive elements,

$$\mathbf{v}_{abcs} = p\lambda_{abcs} \qquad p \equiv \tfrac{d}{dt} \tag{4.9}$$

$$\mathbf{v}_{dq0s} = \mathbf{K}_s p[(\mathbf{K}_s)^{-1}\lambda_{dq0s}]$$

Or it can be written as

$$\mathbf{v}_{dq0s} = \mathbf{K}_s p[(\mathbf{K}_s)^{-1}\lambda_{dq0s}] + \mathbf{K}_s(\mathbf{K}_s)^{-1}p\lambda_{dq0s}$$

$$p[(\mathbf{K}_s)^{-1}] = \omega\begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & 0 \\ -\sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & 0 \end{bmatrix} \tag{4.10}$$

$$\text{So, } \mathbf{K}_s p[(\mathbf{K}_s)^{-1}] = \omega \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.11}$$

Therefore, $\mathbf{v}_{dq0s} = \omega\lambda_{dq0s} + p\lambda_{qd0s}$

This can be expanded as

$$\begin{aligned} v_{qs} &= \omega\lambda_{ds} + p\lambda_{qs} \\ v_{ds} &= -\omega\lambda_{qs} + p\lambda_{ds} \\ v_{0s} &= p\lambda_{0s} \end{aligned} \tag{4.12}$$

Reference Frame	**Reference Frame speed**	**Interpretation**	**variables**	**transformations**
Arbitrary reference frame	ω (unspecified)	stationary circuit variables referred to the arbitrary reference frame	$(\mathbf{f}_{qd0s})^T = \lfloor f_{qs} \quad f_{ds} \quad f_{os} \rfloor$	$\mathbf{K}_s$
Stationary reference frame	0	Stationary circuit variables referred to the stationary reference frame	$(\mathbf{f}^{\mathbf{s}}{}_{qd0s})^T = \lfloor f^s{}_{qs} \quad f^s{}_{ds} \quad f_{os} \rfloor$	$\mathbf{K}^{\mathbf{s}}{}_s$
Rotor reference frame	ω_r	Stationary circuit variables referred to a reference frame fixed in the rotor	$(\mathbf{f}^{\mathbf{r}}{}_{qd0s})^T = \lfloor f^r{}_{qs} \quad f^r{}_{ds} \quad f_{os} \rfloor$	$\mathbf{K}^{\mathbf{r}}{}_s$
synchronously rotating reference frame	ω_e	Stationary circuit variables referred to the synchronously rotating reference frame	$(\mathbf{f}^{\mathbf{e}}{}_{qd0s})^T = \lfloor f^e{}_{qs} \quad f^e{}_{ds} \quad f_{os} \rfloor$	$\mathbf{K}^{\mathbf{e}}{}_s$

The 's' subscript denotes variables and transformations associated with circuits that are stationary in 'real life.' as opposed to rotor circuits that are free to rotate. Subscript 'r' is used to denote the variables and the transformation associated with rotor circuits. The raised index denotes the ds and qs variables and transformation associated with a specific reference frame except in case of the arbitrary reference frame that carries no raised index. Because the 0s variables are independent of ω and therefore not associated with a reference frame, a raised index is not assigned to f_{0s}. The transformation of variables associated with stationary circuits to a stationary reference frame was developed by E. Clarke, who used the notations $\lfloor f_\alpha \quad f_\beta \quad f_0 \rfloor$ rather than $\lfloor f^s{}_{qs} \quad f^s{}_{ds} \quad f_{os} \rfloor$. In Park's transformations to the rotor reference, he demonstrated the variables $\lfloor f_q \quad f_d \quad f_0 \rfloor$ rather than $\lfloor f^r{}_{qs} \quad f^r{}_{ds} \quad f_{os} \rfloor$. There appears to be no established notation for the variables in the synchronously rotating reference frame. The voltage equation for all reference frames may be obtained from those in the arbitrary reference frame. The transformations for a specific reference frame is obtained by substituting the appropriate reference frame speed for ω. In most cases the initial or time-zero displacement can be selected equal to zero; however, there are situations where the initial displacement of the reference frame to which the variables are being transformed will not be zero.

In some derivations and analysis it is convenient to relate variables of one reference frame to variables to another reference frame directly, without involving the abc variables in the transformation. In order to establish this transformation between any two frames of reference, let x denote the reference frame from which the variables are being transformed and let y denote the reference frame to which the variables are being transformed; then

$$\mathbf{f}^{y}_{qd0s} = {}^{x}\mathbf{K}^{y}\mathbf{f}^{x}_{qd0s}$$

$$\mathbf{f}^{x}_{qd0s} = \mathbf{K}^{\mathbf{x}}_{\mathbf{s}}\mathbf{f}_{abcs}$$

$$\mathbf{f}^{y}_{qd0s} = {}^{x}\mathbf{K}^{y}\mathbf{K}^{\mathbf{x}}_{\mathbf{s}}\mathbf{f}_{abcs}$$

$$\mathbf{f}^{y}_{qd0s} = \mathbf{K}^{\mathbf{y}}_{\mathbf{s}}\mathbf{f}_{abcs} \tag{4.13}$$

Therefore, ${}^{x}\mathbf{K}^{y}\mathbf{K}^{\mathbf{x}}_{\mathbf{s}} = \mathbf{K}^{\mathbf{y}}_{\mathbf{s}}$

from which ${}^{x}\mathbf{K}^{y} = \mathbf{K}^{\mathbf{y}}_{\mathbf{s}}(\mathbf{K}^{\mathbf{x}}_{\mathbf{s}})^{-1}$

$$
{}^{x}\mathbf{K}^{y} = \begin{bmatrix} \cos(\theta_y - \theta_x) & -\sin(\theta_y - \theta_x) & 0 \\ \sin(\theta_y - \theta_x) & \cos(\theta_y - \theta_x) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.14}
$$

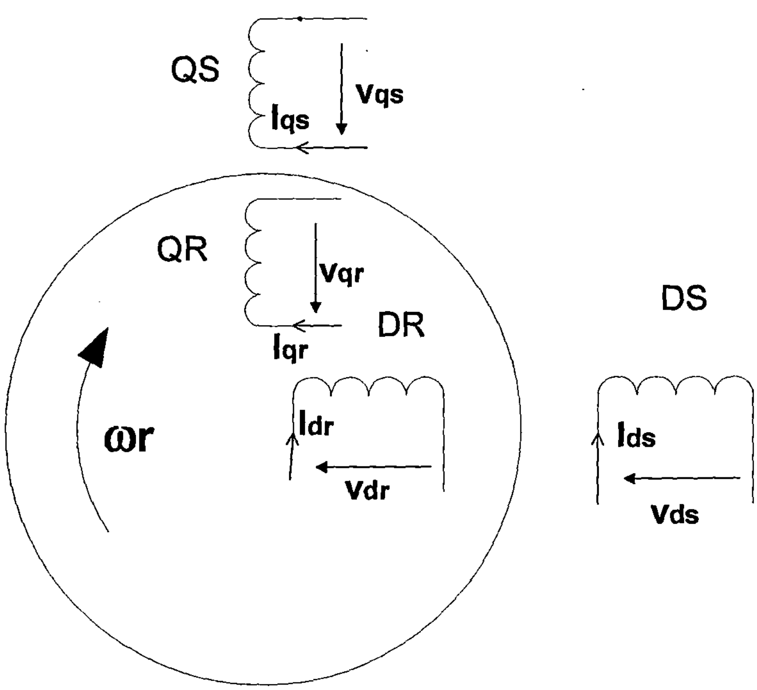
4.3 Generalized Machine:

For great majority of ac machines, including synchronous machine and induction machines, a great simplification is obtained by expressing the equations in a new reference frame and introducing certain fictitious currents and voltages which are different from but are related to the actual ones. The fictitious currents can have a physical meaning in that they can be considered to flow in fictitious windings acting along two axes at right angles called the direct and quadrature axes. In this way the two Reaction Theory of the ac machines has been developed, and the equations so obtained are found to correspond very closely to those of dc machines. The first step in its development was Blondel's 'Two Reaction Theory'of the steady state operation of salient pole synchronous machine. The method was examined in details by Doherty and Nickle, who published a series of five important papers on the same.. A paper by West on 'The Cross Field Theory of Alternating Current Machines' assumed without proof that a rotating cage winding is equivalent to a dc armature winding with two short circuited pair of brushes. A very valuable contribution to the subject was made by Park in asset of three papers. These papers not only develops the general two-axis equations of the synchronous machine, but they indicate how the equations can be applied to the practical problems. Park's transformations provide the most important fundamental concept in the development of Kron's Generalized Theory.

All Electrical Machines can be represented by a generalized model as proposed by Kron. This generalized model is basically a hypothetical machine and it's idea is based on some assumptions. These are :

1. As the distribution of current along the air gap periphery and flux is repeative in nature after every pole pair, a generalized machine is assumed to have only one pole pair in order to avoid the complicacy of understanding and conversion of electrical and mechanical angle. However, for torque and speed calculation, relevant quantities are to be suitably modified.

2. Each winding of the practical machine may have several parts. For generalized machine, it is assumed to have only a single coil. It makes the analysis simple.
3. The axis of the poles around which the field is wound, is called the direct axis and the axis 90^0 away from it, is called the quadrature axis. As per convention, direct axis is taken in horizontal direction and quadrature axis in vertical direction.
4. The positive direction of the current in any coil is towards the coil in the lead nearer to the center of the diagram. The positive direction of the flux linking a coil is radially outward along the axis of the coil.
5. The lower case v represents the externally impressed voltage of a coil and lower case i indicates the direction of current flowing as per the direction of the voltage.
6. The clockwise direction of the rotation of the machine is assumed as the positive direction of rotation.
7. The torque with clockwise sense is the is assumed as the positive direction.
8. The subscript d indicates direct axis, q as quadrature axis, s as stator and r indicates rotor.



Generalized Model of Electrical Machines

Fig.4.1

On the basis of the above assumptions the following voltage equations of the generalized machine, as shown in Fig.4.1, can be written.

$$v_{ds} = (r_{ds} + L_{ds}p)i_{ds} + M_d p i_{dr}$$

$$v_{qs} = (r_{qs} + L_{qs}p)i_{qs} + M_q p i_{qr}$$

$$v_{dr} = M_d p i_{ds} - M_q \omega_r i_{qs} + (r_{dr} + L_{dr}p)i_{dr} - \omega_r L_{qr} i_{qr}$$

$$v_{qr} = M_d \omega_r i_{ds} + M_q p i_{qs} + \omega_r L_{dr} i_{dr} + (r_{qr} + L_{qr}p)i_{qr} \qquad (4.15)$$

Here,

$v_{ds}, v_{qs}, v_{dr,} v_{qr}$ = direct and quadrature axis input voltages as shown in fig.4.1

In a realistic machine we may not come across the voltages. However, these may obtained from available stator and rotor variables using suitable transformations as described in (4.1) –(4.3)

$i_{ds}, i_{qs}, i_{dr,} i_{qr}$ = direct and quadrature axis input currents as shown in fig.4.1. These are also the transformed variables used for the analysis of generalized machine. In practical machine, we may get i_a, i_b, i_c which may be transformed using (4.1) –(4.3) to obtain these variables.

$r_{ds}, r_{qs}, r_{dr,} r_{qr}$ = resistances of the windings as shown in fig.4.1

$L_{ds}, L_{qs}, L_{dr,} L_{qr}$ = resistances of the windings as shown in fig.4.1

M_d, M_q = direct and quadrature axis mutual inductances as shown in fig.4.1

The values of the parameters depends on the machine and the shape of the rotor. However, for a wide range of motors they remain within a range. For generalized machine, it is also assumed that they are linear. In practical machines these are to some extent nonlinear due to saturation, temperature rise and other factors.

In the matrix form,

$$\begin{bmatrix} v_{ds} \\ v_{qs} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_{ds}+L_{ds}p & 0 & M_d p & 0 \\ 0 & r_{qs}+L_{qs}p & 0 & M_q p \\ M_d p & -M_q\omega_r & r_{dr}+L_{dr}p & -\omega_r L_{qr} \\ \omega_r M_d & M_q p & \omega_r L_{dr} & r_{qr}+L_{qr}p \end{bmatrix} \begin{bmatrix} i_{ds} \\ i_{qs} \\ i_{dr} \\ i_{qr} \end{bmatrix} \tag{4.16}$$

Or $[v]=[Z][i]$

$[Z]=[R]+[L]p+[G]\omega_r$

$$\text{Where } [Z]=\begin{bmatrix} r_{ds}+L_{ds}p & 0 & M_d p & 0 \\ 0 & r_{qs}+L_{qs}p & 0 & M_q p \\ M_d p & -M_q\omega_r & r_{dr}+L_{dr}p & -\omega_r L_{qr} \\ \omega_r M_d & M_q p & \omega_r L_{dr} & r_{qr}+L_{qr}p \end{bmatrix} \tag{4.17}$$

$$[R]=\begin{bmatrix} r_{ds} & 0 & 0 & 0 \\ 0 & r_{qs} & 0 & 0 \\ 0 & 0 & r_{dr} & 0 \\ 0 & 0 & 0 & r_{qr} \end{bmatrix}$$

$$[L]=\begin{bmatrix} L_{ds} & 0 & M_d & 0 \\ 0 & L_{qs} & 0 & M_q \\ M_d & 0 & L_{dr} & 0 \\ 0 & M_q & 0 & L_{qr} \end{bmatrix}$$

$$[G]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -M_q & 0 & -L_{qr} \\ M_d & 0 & L_{dr} & 0 \end{bmatrix}$$

$P_i=[i]^T[v]=v_{ds}i_{ds}+v_{qs}i_{qs}+v_{dr}i_{dr}+v_{qr}i_{qr}$

$$\text{Or, } P_i=[i]^T[R][i]+[i]^T[L]p[i]+\omega_r[i]^T[G][i] \tag{4.18}$$

$[i]^T[R][i]=i_{ds}^2r_{ds}+i_{qs}^2r_{qs}+i_{dr}^2r_{dr}+i_{qr}^2r_{qr}$ it represents the copper loss

$[i]^T[L]p[i]$ It actually represents the change in stored energy .

The third term $\omega_r[i]^T[G][i]$ represents Power being converted to mechanical form.

T_e=Electrical torque developed $= \frac{P}{\omega_r} = [i]^T [G][i]$

$$T_e = i_{qr} M_d i_{ds} - M_q i_{dr} i_{qs} + (L_{dr} - L_{qr}) i_{qr} i_{dr} \qquad (4.19)$$

When $(L_{dr} = L_{qr})$ third term of the torque becomes zero. This is actually the reluctance torque. In absence of saliency this term becomes zero.

The dynamic equation of the drives is also applicable at all conditions:

$$T_e - T_L = J \frac{d\omega}{dt}$$

Therefore the dynamics of any machine can be studied using the voltage equation, torque equation and dynamic equation all together. The nature of load torque plays an important role in the dynamics.