# CHAPTER IV

# A MATHEMATICAL MODELLING OF THE ELECTROMECHANICAL DEFORMATION OF AN ELECTROSTRICTIVE HETEROGENEOUS DIELECTRIC BAR OF VARIABLE S.I.C. IN THE PRESENCE OF ELECTRICAL AND MECHANICAL FIELD.

# CHAPTER IV

## A MATHEMATICAL MODELLING OF THE ELECTROMECHANICAL DEFORMATION OF AN ELECTROSTRICTIVE HETEROGENEOUS DIELECTRIC BAR OF VARIABLE S.I.C IN THE PRESENCE OF ELECTRICAL AND MECHANICAL FIELDS.

### 4.1. Introduction:

Electrostrictive dielectric under the influence of appropriate electrical fields exhibits elastic properties. Such electrostrictive effect in a composite prismatic bar made of n different types of dielectric materials with different Specific Inductive Capacities (S.I.C) which too depends upon the spatial positions, is studied from the point of view of mechanics of continuous media with the help of equations of elasticity and electricity, together with the constitutive equations of electromechanical interaction.

Heterogeneity in the S.I.C. of a composite dielectric is an essential feature. Such composite dielectric are used in condensers of high capacity. In practice, these type of condensers are termed as plane Multilayercapacitor.

### 4.2. Fundamental Equations:

We consider that S.I.C. of each dielectric depends upon the longitudinal position from the fixed end of the prism. Materials in the bar are assumed to be elastically isotropic.

The Z-axis in Fig.4.2.1 is directed along the axis of the bar. The end Z=0 is kept fixed and a charge 'q' is spread over the end. The end $Z=l_n$ is earthed and is under a constant mechanical Force $F_o$ acting along Z-axis. The cross-section $Z=l_1$ is the surface of separation of the first two dielectric materials and $Z=l_{n-1}$ is the surface of separation of the last two dielectric materials.

For the formulation of the model the following equations of electricity and elasticity are taken into consideration (4.2.2), (4.2.3).

The electric fielde governed by the equations

(i) rot $\vec{E} = 0$

(ii) Div $\vec{D} = 0$

(iii) $\vec{D} = \vec{K}\vec{E}$ ... ...(4.2.1)

where $\vec{D}$, $\vec{E}$, and K are the electric displacement vector, electric intensity vector and S.I.C. respectively.

If, 'v' be the electric potential, then from the equation (4.2.1) the electric intensity vector may be written as

$$\vec{E} = -\text{grad } v. \quad \text{... ...(4.2.1a)}$$

Using (ii) and (iii) one can have

$$\text{div } (K \text{ grad } v) = 0 \quad \text{... ...(4.2.2 )}$$

where K is the S.I.C of the dielectric.

Since S.I.C of each material of the composite dielectric is heterogeneous and it may vary as any function of Z.

With reference to the above let us model the SIC as follows:

$$K_r = k_r \psi_r (2) \quad \text{in } l_{r-1} < Z < l_r \quad \ldots(4.2.3)$$

where r = 1,2,3, .....,n

$l_o = 0$, vide Fig. 4.2.1

where $K_o's$ are the material constants.

For the convenience of the model and to suit the practical requirements, one can even choose

$$Z = \left[\frac{d}{dZ} \phi_r(Z)\right]^{-1/2} \quad \ldots \ldots(4.2.4)$$

The relevent constitutive equations of the prismatic dielectric bar are

$$T_{ii} = \lambda e + 2 G S_{ii} + af^2 + b E_1^2$$

$$T_{ij} = 2 a S_{ij} + b E_i E_j$$

$$(i \neq J) \quad (i, J \equiv x,y,z) \quad \ldots \ldots(4.2.5)$$

where T's are the stress components

S's are the strain components

$E_i$ is the component of Electric intensity E.

a,b are the scalar quantities known as electrostrictive constants.

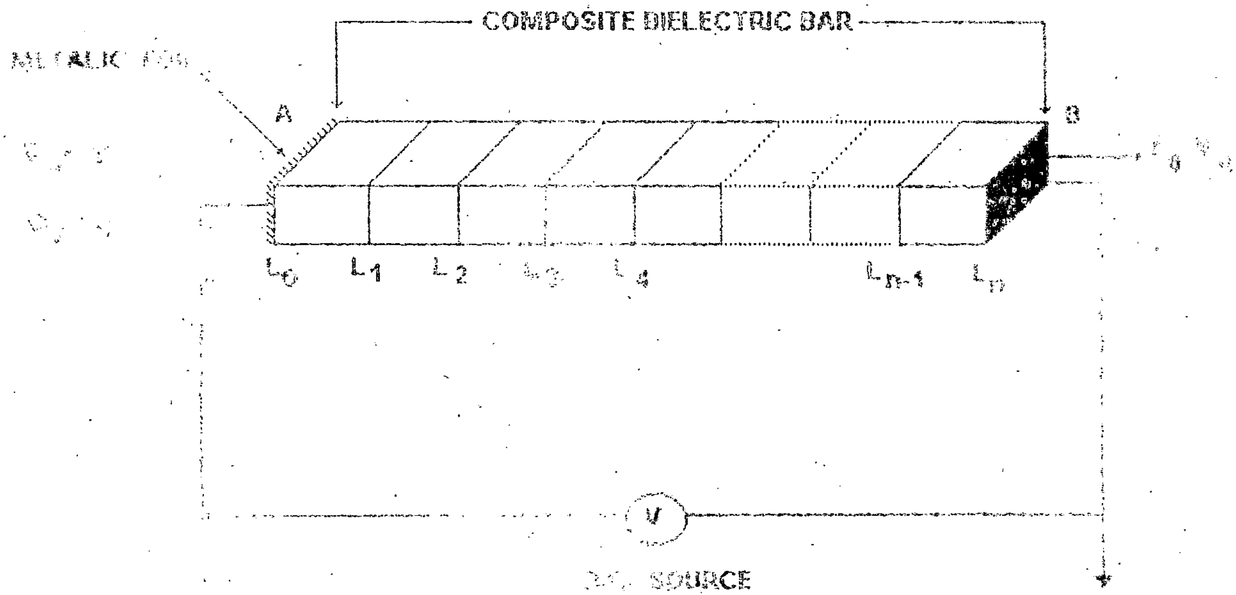


FIG:4.2.1 DEFORMATION OF A, COMPOSITE BAR

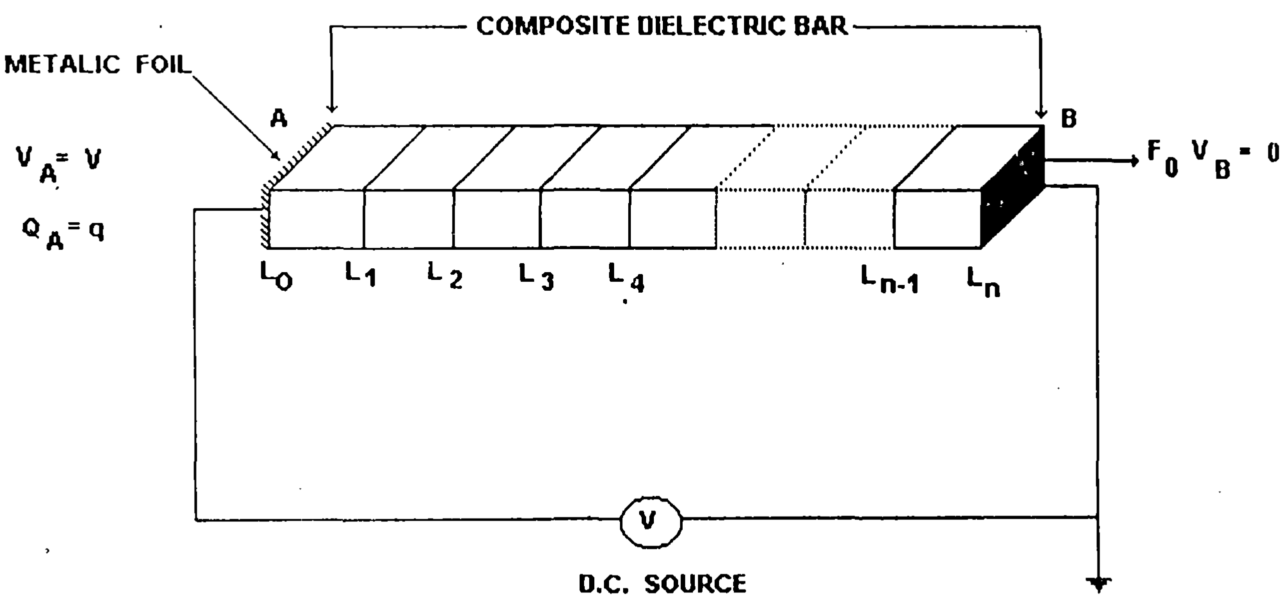


FIG:4.2.1 DEFORMATION OF A. COMPOSITE BAR

$e = \vec{Z}\, S_{ii}$

$f^2 = \sum E_i^2$ $\quad\lambda$ and G are the elastic constants

The bar under consideration is subjected to longitudinal deformation only. The displacement components u,v,w must be of the following form

$$u = 0, \quad v = 0, \quad \text{and} \quad W = W(Z) \qquad \ldots\ \ldots(4.2.6)$$

The strain components can then be modelled as

$$S_{xx} = S_{yy} = S_{xy} = S_{yz} = S_{zx} = 0$$

$$\text{and } S_{ZZ} = \frac{dW}{dZ} \qquad \ldots\ \ldots(4.2.7)$$

Since the field is in longitudinal direction only, one can now remodel the equation, (4.2.2) with the help of (4.2.3) and (4.2.4) as

$$\frac{d}{dZ}\left[K_r\left\{\frac{d}{dZ}\,\phi_r(Z)\right\}^{-1/2}\frac{\delta v}{\delta Z}\right] = 0 \text{ in } l_{r-1} < Z < l_r \qquad \ldots(4.2.8)$$

The equation (4.2.8) clearly states

$$\frac{dv}{dZ} = \frac{A_r}{K_r}\left\{\frac{d}{dZ}\,\phi_r(Z)\right\}^{1/2} \text{ in } l_{r-1} < Z < l_r \qquad \ldots(4.2.9)$$

where $A'_r$s are independent of Z and also independent of X and Y for the system under consideration.

The electrical boundary condition at the fixed end $z = 0$ when imposed on equation (4.2.9) for $r = 1$ results,

$$\left[-K_1\frac{\partial v}{\partial Z}\right]_{z=o} = -q = \left[\frac{K_1}{\left\{\frac{d}{dZ}\phi_1(Z)\right\}^{1/2}} \frac{A_1\left\{\frac{d}{dZ}\phi_1(Z)\right\}^{1/2}}{K_1}\right]_{Z=0}$$

Then clearly the arbitrary constants,

$$A_1 = -q \qquad \ldots \ \ldots (4.2.10)$$

Again the condition of continuity of electric intensity E at the boundary surfaces at $Z = l_1, l_2 \ldots, l_{n-1}$ which are supposed to be sufficiently bounded so as to ensure the continuity and displacement relates to

$$\left[-K_r\left\{\frac{d}{dZ}\phi_r(Z)\right\}^{-1/2} \frac{A_r\left\{\frac{d}{dZ}\phi_r(Z)\right\}^{1/2}}{K_r}\right]_{Z=l_r}$$

$$= \left[-K_{r+1}\left\{\frac{d}{dz}\phi_{r+1}(z)\right\}^{-1/2} \frac{A_{r+1}\left\{\frac{d}{dZ}\phi_{r+1}(Z)\right\}^{1/2}}{K_{r+1}}\right]_{Z=l_{r+1}}$$

This set along with the equation (4.2.10) creates an interesting conclusion that

$$A_1 = A_2 = A_3 = \ldots\ldots = A_{n-1} = -q \qquad \ldots (4.2.11)$$

The component of the electric intansity $E_z$ can then be modelled with help of equation (4.2.9) and (4.2.11) as

$$E_z = -\frac{\partial v}{\partial z} = \frac{q}{K_r}\left\{\frac{d}{dz}\phi_r(z)\right\}^{1/2} \text{ in } (l_{r-1}, l_r) \qquad \ldots (4.2.12)$$

Also $E_x = E_y = 0$ ...(4.2.13)

Equations (4.2.5),(4.2.7), (4.2.12), (4.2.13) sets the stress equations of equilibrium[13] only to

$$\frac{\partial}{\partial Z}(T_{ZZ}) = 0$$

which leads to

$$T_{ZZ} = (\lambda + 2G)\frac{dw}{dZ} + (a+b)\,E_Z^2 = \text{Constant for} \qquad \ldots(4.2.14)$$

all values of Z

The mechanical boundary condition when applied on (4.2.14) gives us

$$T_{ZZ} = F_o \quad \text{at} \quad Z = l_n$$

Equation (2.2.12) and (2.2.14) along with this mechanical boundary condition lead to

$$(\lambda_r + 2\,G_r)\frac{dw_r}{dZ} + p_r\frac{d\phi_r}{dZ} = F_o \text{ in } (l_{r-1}, l_r) \qquad \ldots(4.2.15)$$

Where $$p_r = \frac{q^2(a+b)}{K_r^2} \qquad \ldots\ \ldots(4.2.16)$$

For a particular material placed in the 'r'th position, $\lambda_r$ and $G_r$ are constants and Equation (4.2.15) on integration stands as

$$W_r(Z) = \frac{1}{\lambda_r + 2\,G_r}\left[F_oZ - P_r\,\phi_r(z) + C_r\right] \qquad \ldots(4.2.17)$$

$C_r'$s are the constants to be evaluated from the mechanical boundary conditions, which are,

$$W_1 = 0 \text{ at } Z = 0 \text{ at the fixed end} \qquad \ldots(4.2.18)$$

and $$W_r = W_{r+1} \text{ at } Z = l_r \qquad \ldots \ldots(4.2.19)$$

The condition of continuity at common surfaces Equations (4.2.17) and (4.2.18) evaluates the arbitrary constant $C_1$ in known terms.

$$C_1 = p_1 \phi_1 (o) \qquad \ldots \ldots(4.2.20)$$

The condition (4.2.19) when applied in the displacement equation (4.2.17) gives

$$C_{r+1} = \frac{\lambda_{r+1} + 2G_{r+1}}{\lambda_r + 2G_r} \left[ F_o l_r - p_r \phi_r (l_r) + C_r \right]$$

$$+ P_{r+1} \phi_{r+1} (l_r) - F_o l_r \qquad \ldots \ldots(4.2.21)$$

Since $C_1$ is known from equation (4.2.20), $C_2$ can be found from equation (4.2.21) in known terms for r=1 and ultimately the displacement at any position can be fully evaluated.