CHAPTER III

A MATHEMATICAL MODELLING OF THE STRESSES IN A HETEROGENEOUS DIELECTRIC OF VARIABLE S.I.C. UNDER THE INFLUENCE OF ELECTRICAL AND MECHANICAL FIELDS

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3.1. Introduction :

It is well known that electrostrictive dielectrics under the influence of appropriate electric fields exhibits elastic properties [1], [4], [5]. Such electrostrictive effects are considered here in a body of dielectric of non-uniform Specific Inductive Capacity (S.I.C) varying as any function of space variables.

The space occupied by the dielectric between two concentric spherical surfaces maintains always a constant electric voltage difference at the surfaces. Both the spherical surfaces are subjected to normal mechanical pressures. The present topic is intended to consider the displacement and stresses of the dielectric from the point of view of mechanics of continuous media with the help of equations of Electricity and Elasticity together with the equations of electromechanical interaction. Use of such dielectrics in a spherical condenser of high capacities are well recognised.

3.2. Fundamental Equations:

We consider space between two concentric spherical surfaces of radii r_1 and r_2 $(r_2 > r_1)$ and also consider the surfaces at constant potentials V_1 and V_2 respectively. The space is occupied by a heterogeneous dielectric whose S.I.C varies as any function of the distance from the centre of the sphere. In addition to this the dielectric is held strained by internal and external mechanical pressures p_1 and p_2 applied normally on the boundary surfaces.

For the formulation of the problem the following equations of Electricity and Elasticity are taken into account.

The electric field is governed by the equations

[i] $\text{rot } \vec{E} = 0$

[ii] $\text{div } \vec{D} = 0$

[iii] $\vec{D} = K\vec{E}$ ... ...(3.2.1)

Where $\vec{D}, \vec{E}$ and K are the electric displacenent vector, electric intensity vector and S.I.C respectively.

If v be the electric potential then from the equation (3.2.1)

(i) $\vec{E} = -\text{grad} v$ ... ...(3.2.1a)

Using (3.2.1)(ii) and (3.2.1)(iii) one can have

$\text{div } (k \text{grad } v) = 0$ ... ...(3.2.2)

We take the spherical polar co-ordinates (r, θ, ϕ) as our co-ordinate of reference. Let the variable S.I.C, (K) obey the following law :

$K = k_o/r^2 \; \psi(r)$ ... ...(3.2.3)

where the integrability of the function $\psi(r)$ is assumed so that one can use

$$\psi(r) = \phi'(r) \qquad \ldots \ldots (3.2.3a)$$

Due to spherical symmetry the equation (3.2.2) in the light of (3.2.3) and (3.2.3a) becomes

$$\frac{d}{dr}\left[\frac{K_o}{r^2 \phi'(r)} \; r^2 . \frac{dv}{dr} \right] = 0$$

It indicates clearly the invariability of the expression.

$$\left[\frac{K_o}{\phi'(r)} . \frac{dv}{dr} \right]$$

The electric potential at any point may then be generally given by

$$V = A_1 \phi(r) + A_2 \qquad \ldots \ldots (3.2.4)$$

Where A_1 and A_2 are the constants of integration which can be evaluated from the prescribed boundary conditions,

$V = V_1$ at the surface $r = r_1$

$V = V_2$ at the surface $r = r_2$ $\qquad \ldots \ldots (3.2.5)$

Following equations (3.2.4) and (3.2.5) one can get A_1 and A_2 in known terms as.

$$A_1 = \frac{V_1 - V_2}{\phi(r_1) - \phi(r_2)}$$

$$A_2 = \frac{V_2\phi(r_1) - V_1\phi(r_2)}{\phi(r_1) - \phi_2(r_2)} \qquad \ldots\ \ldots(3.2.6)$$

The general form of the electric potential V under the prescribed boundary conditions may be found from equations (3.2.4) and (3.2.5)

$$V = \frac{V_1 - V_2}{\phi(r_1)-\phi(r_2)}\,\phi(r) + \frac{V_2\phi(r_1) - V_1\phi(r_2)}{\phi(r_1) - \phi_2(r_2)} \qquad \ldots\ \ldots(3.2.7)$$

The components of the electric intensity can be obtained from the equation (3.2.1a) as follows:

$$E_r = \frac{\partial v}{\partial r} = \frac{V_1 - V_2}{\phi(r_1) - \phi(r_2)}\,\phi'(r)$$

$$E_\theta = -\frac{\partial v}{r\partial\theta} = 0$$

$$E_\phi = -\frac{\partial v}{r\sin\theta\,\partial\phi} = 0 \qquad \ldots\ \ldots(3.2.8)$$

As we are only interested in the radial displacement 'U' of the dielectric, the strain components are given by

$$S_{rr} = \frac{du}{dr},\ S_{\theta\theta} = S_{\phi\phi} = \frac{u}{r},\ S_{\theta\phi} = S_{r\phi} = S_{r\theta} = 0$$

$$\ldots\ \ldots(3.2.9)$$

The constitutive relations of the electrostrictive material in terms of displacement and electric intensity are

$$\left.\begin{aligned} T_{rr} &= \left(\lambda + 2G\right)\frac{du}{dr} + 2\lambda\frac{u}{r} + \left(a + b\right)E_r^2 \\ T_{\theta\theta} = T_{\phi\phi} &= \lambda\frac{du}{dr} + 2\left(\lambda + G\right)\frac{u}{r} + a\,E_b^2 \\ T_{r\theta} &= 0,\quad T_{\theta\phi} = 0,\quad T_{r\phi} = 0 \end{aligned}\right] \quad \ldots(3.2.10)$$

T's are stress components,'a' and 'b' are electrostrictive scalar quantities, λ and G are the material constants. To obtain 'U' we make use of the equations of equilibrium together with the mechanical boundary conditions, namely.

$$T_{rr} = -\,p_1 \text{ on the surface } r = r_1$$

$$= P_2 \text{ on the surface } r = r_2 \quad \ldots(3.2.11)$$

The relevent stress equations of equilibrium is given by

$$\frac{\partial T_{rr}}{\partial r} + \frac{2}{r}\left[T_{rr} - T_{\theta\theta}\right] = \theta \quad \ldots\ \ldots(3.2.12)$$

Using the equation (3.2.10), the equation (3.2.12) becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) - 2u = -\frac{2r^2}{(\lambda + 2G)}\left[(a+b)E_r\frac{\partial E_r}{\partial r} + \frac{b}{r}E_r^2\right]$$

$$\ldots\ \ldots(3.2.13)$$

3.3. Solution Of The Equation.

To have a full knowledge of the radial displacement we transform the nonhomogeneous differential equation (3.2.13) setting

$$r = \exp \eta, \qquad \ldots \; \ldots (3.3.1)$$

So that the equation (3.2.13) is transformed to

$$(D^2 + D - 2)u = - \frac{2}{(\lambda + 2G)} \exp \eta \left[(a+b) E_r \frac{\partial E_r}{\partial \eta} + b E_r^2 \right]$$

$$= F(\eta) \text{ (say)}, \qquad \ldots \; \ldots (3.3.2)$$

where $D = \dfrac{d}{d\eta}$

The solution of the equation (3.3.2) is given by

$$u = A_3 (\exp \eta) + A_4 (\exp \eta)^{-2} + \frac{F(\eta)}{D^2 + D - 2} \qquad \ldots (3.3.3)$$

, which is the most general form of the deformation. A_3 and A_4 are arbitrary constants.

The electric intensity in most of the practical cases varies sinusoidally or exponentially and to suit the practical purposes we choose

$$F(\eta) = \text{Sin}(m\eta + E); \; \exp(m\eta + E); \; \text{Cos}(m\eta + E)$$

$$\ldots \; \ldots (3.3.4) \; (a,b,c)$$

in which E is the phase shift constant and 'm' is the attenuation constent.

Using equation (3.3.4)(a) on equation (3.3.3) we get

$$u = A_3(\exp \eta) + A_4 (\exp \eta)^{-2} + \frac{\mathrm{Sin}(m\eta + E)}{D^2 + D - 2}$$

$$A_3(\exp \eta) + A_4(\exp \eta)^{-2} + \frac{(2+m^2)\ \mathrm{Sin}(m\eta+E) - m(\mathrm{Cos}(m\eta+E)}{(m^2+2)^2 + m^2}$$

... ...(3.3.5)

To evaluate the constants of integration A_3 and A_4 of the equation (3.3.5) we shall use the mechanical boundary condition stated in the equations (3.2.11) along with the equations (3.2.8) (3.2.10).

$$(\lambda + 2G)\frac{du}{dr} + 2\lambda\frac{u}{r} + (a+b)A_1^2\left\{\phi'(r)\right\}^2 = -p_1 \text{ on } r = r_1$$

... ...(3.3.5)

$$(\lambda + 2G)\frac{du}{dr} + 2\lambda\frac{u}{r} + (a+b)A_1^2\left\{\phi'(r)\right\}^2 = -p_2 \text{ on } r = r_2$$

... ...(3.3.7)

Using equation (3.3.4) on equation (3.3.3)

$$u = A_5 r + \frac{A_6}{r^2} + \frac{F_o e^{m\log r}}{m^2 + m - 2} \quad \left[\begin{array}{l}\text{where } r = \mathrm{cxp} \\ \text{and } F_o = e^t\end{array}\right]$$

$$= A_5 r + \frac{A_6}{r^2} + \frac{F_o r^m}{m^2 + m - 2}$$

... ...(3.3.8)

To evaluate the constant of integration A_5 and A_6 we should use the equation (3.2.11)

$$(\lambda + 2G)\frac{du}{dr} + 2\lambda\frac{u}{r} + (a+b)A_1^2\left\{\phi'(r)\right\}^2 = -p_1 \text{ on } r = r_1$$

... ...(3.3.9)

$$(\lambda + 2G) \frac{du}{dr} + 2\lambda \frac{u}{r} + (a+b)A_1^2 \left\{ \phi'(r) \right\}^2 = - p_2 \text{ on } r = r_2$$

... ...(3.3.10)

Using (3.3.8),(3.3.9) and (3.3.10), we can have,

$$(3\lambda + 2G)A_5 (r_2^3 - r_1^3) + F_o \frac{m(\lambda+2G)}{m^2 + m - 2} \left[r_2^{m+2} - r_1^{m+2} \right]$$

$$+ \frac{2\lambda F_o}{m^2 + m - 2} \left[r_2^{m+2} - r_1^{m+2} \right]$$

$$+ (a+b) A_1^2 \left[\left\{\phi'(r_2)\right\}^2 r_2^3 - \left\{\phi'(r_1)\right\}^2 r_1^3 \right] = p_1 r_1^3 - p_2 r_2^3 .$$

$$\therefore \quad A_5 = \frac{p_1 r_1^3 - p_2 r_2^3}{(3\lambda + 2G)(r_2^3 - r_1^3)} - \frac{m_1 F_o}{(3\lambda + 2G)(r_2^3 - r_1^3)}$$

$$\times \left[r_2^{m+2} - r_1^{m+2} \right] - \frac{m_2 F_o}{(3\lambda + 2G)(r_2^3 - r_1^3)} \left[r_2^{m+2} - r_1^{m+2} \right]$$

Similarly,

$$A_6 = \frac{r_1^3 r_2^3 (p_1 - p_2)}{4G(r_2^3 - r_1^3)} - \frac{m_1}{4G(r_2^3 - r_1^3)} \left[r_2^{m+2} . r_1^3 - r_1^{m+2} r_2^3 \right]$$

$$- \frac{m_2}{4G(r_2^3 - r_1^3)} \left[r_2^{m+2} . r_1^3 - r_1^{m+2} r_2^3 \right]$$

$$- \frac{(a+b) A_1^2 r_1^3 r_2^3 \left[\left\{ \phi'(r_2) \right\}^2 - \left\{ \phi'(r_1) \right\}^2 \right]}{4G(r_2^3 - r_1^3)}$$

where $m_1 = \dfrac{m(\lambda + 2G)}{m^2 + m - 2}$ and $m_2 = \dfrac{2\lambda}{m^2 + m - 2}$

Hence the displacement 'U' is completely determined from the equation (3.3.8)