CHAPTER VI

A NEW REFINED COMPUTERISED SIMULATION FOR STRESSES AND VOLTAGES DEVELOPED IN AN INHOMOGENEOUS PIEZOELECTRIC QUARTZ BAR DUE TO FINITE BENDING

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6.1. Introduction:

Piezoelectric effect was first discovered by the Curie brothers in 1880[28]. Piezoelectric properties of crystals are used to construct efficient transducers to work under different practical situations. For low frequency operation bimorph, a composite transducing material is often used to reduce the mechanical impedence without lowering the output voltage[25]. However, the aim of the designers is to achieve high output voltage and greater ruggedness with minimum weight in an electro-mechanical appliance.

Orchard suggests that the foregoing properties could be well introduced if we form the material in a thin layer using quartz as a concrete aggregate. Since the material layers thus formed have varying properties of quartz crystals, the composite bar is obviously non-homogeneous.

Moreover one of the most advantage of forming

piezoelectric quartz texture in the form of aggregate of any shape is its flexibility for forming in any desired shape by casting and grinding. Within the different segments of the same piezoelectric,one can vary the electric polarisation and or mechanical force according to their requirement.

The present investigation deals with such a model bar in the following aspects:

(i) The behavior of the output voltage both for Homogeneous and Non-Homogeneous material compositions.

(ii) The four types of non-homogeneity have been studied to suit the practical requirements. The output voltage and the stress for both the cases have been incorporated for a technically interesting comparison taking ruggedness and weight of the body into consideration together with the electric and piezoelectric parameters.

(iii) An exact and computerised mathematical model simulation along with the solutions have been developed and compared, showing the percentage error by statistical data handling technique following multivariable regression analysis.

6.2. Basic Equations.

Let us consider a uniform narrow rectangular cross-section of a curved bar Fig 6.2.1 composed of

FIG. 6.2.1(a)

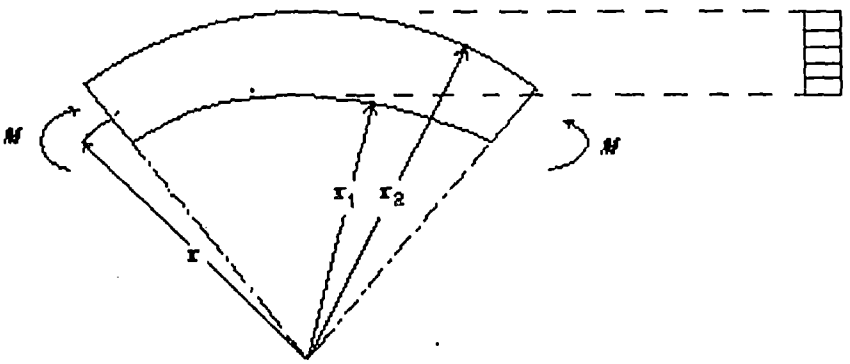


FIG. 6.2.1(b)

different layers consisting of an aggregate of quartz cement mixture[23]. Bending is effected in the plane of the curvature of the couples "M" applied at the ends of the bar. Naturally the stress distribution is the same in all radial cross-section, So that the stress components do not depend on "θ" but are functions of "r" only. Owing to this symmetry, the shearing stress does not exist.

The basic equations of this electro-mechanical problem consists of three parts, namely;

(i) The equations of electricity.

(ii) The equations of elasticity.

(iii) The constitutive equations for the piezoelectric materials, to set up interaction amongst electric,elastic and Piezoelectric relations.

(iv) The equation of electric field is governed by well known Maxwell's and Gauss's divergence equation, stated mathematically as:

$$\text{rot } \overrightarrow{E} = 0 \qquad \ldots\ldots(6.2.1)$$

$$\text{Div } \overrightarrow{D} = 0 \qquad \ldots\ldots(6.2.2)$$

Where $\overrightarrow{E}$ = Electric field intensity

$\overrightarrow{D}$ = Electric induction vector

From equation (6.2.1) one can easily form

$E = \text{grad } \phi$ where ϕ = Electric potential function

$$E = \frac{d\phi}{dr} \quad \text{in the present problem}$$

If r_1 and r_2 are the radii of the lower and upper faces of the bar and if the faces are coated with a conducting material with uniform potential ϕ, the electrical boundary condition becomes

$$V_1 - V_2 = \int_{r = r_1}^{r = r_2} E_r dr = V \qquad \ldots\ldots (6.2.3)$$

The integral is taken along the path from the lower to the upper face of the boundary of the bar.

(ii) The equation of elasticity starts from the stress equation of equilibrium in the following form.

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad \ldots\ldots (6.2.4)$$

where σ_r = normal stress in radial direction

σ_θ = normal stress in circumferential direction

To set up the interaction between the electric, elastic and Piezoelectric relation, we make use of the following constitutive relations for the present investigation.

$$S_r = S_{11}\sigma_r + S_{12}\sigma_\theta + d_{11}E_r \qquad \ldots\ldots (6.2.5)$$

$$S_\theta = S_{12}\sigma_r + S_{11}\sigma_\theta - d_{11}E_r \qquad \ldots\ldots (6.2.6)$$

$$S_{r\theta} = 0 \qquad \ldots\ldots (6.2.7)$$

$$D_r = d_{11}(\sigma_r - \sigma_\theta) + \epsilon_{11}E_r \qquad \ldots\ldots (6.2.8)$$

$$D_\theta = 0 \qquad \ldots\ldots (6.2.9)$$

Where $S_r, S_\theta, S_{r\theta}$ = The strain components

S_{11}, S_{12} = Electric compliances at constant field.

d_{11} = The piezoelectric strain parameters.

ϵ_{11} = Dielectric Permitivity at constant stress.

For solving these equations one should have the electrical as well as mechanical boundary conditions of the problem. The electrical boundary condition has already been in equation (6.2.3), we are to set up mechanical boundary conditions now. Taking width of the rectangular cross-section as unity, the numerical boundary conditions are,

$$\sigma_r = 0 \quad \text{for } r = r_1 \quad \text{and } r = r_2 \qquad \ldots\ \ldots (6.2.10)$$

$$\int_{r_1}^{r_2} \sigma_\theta \, dr = 0 \qquad \ldots\ \ldots (6.2.11)$$

$$\int_{r_1}^{r_2} r.\sigma_\theta \, dr = -M \qquad \ldots\ \ldots (6.2.12)$$

6.3. Simulation Of The Exact Mathematical Model.

We have seen Piezoelectric property of the bar inherits the inhomogeneity of material. This inhomogeneity is characterised by the variations of elastic, Piezoelectric and dielectric parameters from point to point in a static problem[14]. In particular, this variations, where radial symmetry is considered, may be of the form,

$$S_{ij} = C_{ij} \, f(r) \qquad \ldots\ \ldots (6.3.1)$$

$$d_{ij} = b_{ij}\, f(r) \qquad \ldots\ \ldots (6.3.2)$$

$$\epsilon_{ij} = v_{ij}\, f(r) \qquad \ldots\ \ldots (6.3.3)$$

Where C_{ij} , b_{ij} , v_{ij} are the material properties and termed as elastic, Piezoelectric and dielectric material constants of the bar under investigation. f(r) is the space position to take into consideration of the variation of these properties from point to point depthwise.

Since most of the real world problems are encountered to exhibit the depthwise parabolic, linear and inverse variation of the bar we introduce the space function "f(r)" in the following form.

$$f(r) = r^{2\alpha} \qquad \ldots\ \ldots (6.3.4)$$

where α = homogeneity factor[22].

To make the modelling usable in the practical design purpose, one should have to assign the variation of α and r. To encounter the practical design problems with the mathematical modelling usable in practice and since the material properties are varying point to point with respect to the space position 'r'; it is evident to consider the variation of 'r' in a closed domain instead of assigning any specific value. Hence we introduce the variation of 'r' in the closed domain $2 \leq r \leq 2.5$.

The Gaussian divergence equation (6.2.2) in two dimensional polar co-ordinates owing to the radial symmetry yields,

$$\frac{\delta D_r}{\delta r} = 0 \qquad \ldots\ \ldots (6.3.5)$$

i.e. D_r = Constant = D_o (Say).

Equation (6.2.8) with (6.3.5) yields

$$D_o = d_{11}(\sigma_r - \sigma_\theta) + \epsilon_{11} E_r$$

$$\text{therefore,} \quad E_r = \frac{D_o - d_{11}(\sigma_r - \sigma_\theta)}{\epsilon_{11}} \qquad \ldots\ \ldots (6.3.6)$$

The radial and tangential components of displacement can be taken as,

$$u = r(1 - \psi) \qquad \text{and} \qquad v = 0 \qquad \ldots\ \ldots (6.3.7)$$

where ψ is a function of r

Hence the radial and circumferential strain components now become

$$S_r = \frac{du}{dr} = 1 - \psi - r\frac{d\psi}{dr} \qquad \ldots\ \ldots (6.3.8)$$

$$\text{and} \quad S_\theta = \frac{u}{r} = 1 - \psi \qquad \ldots\ \ldots (6.3.9)$$

Equations (6.2.5) and (6.2.6) along with equations (6.3.7), (6.3.1), (6.3.2) (6.3.3), (6.3.8), (6.3.9) can be written in more compact form as

$$\lambda_1 \sigma_r + \lambda_2 \sigma_\theta = A_1 \qquad \ldots\ \ldots (6.3.10)$$

$$\lambda_2 \sigma_r + \lambda_1 \sigma_\theta = A_2 \qquad \ldots\ \ldots (6.3.11)$$

$$\text{where} \quad \lambda_1 = C_{11} - \frac{b_{11}^2}{v_{11}}, \qquad \lambda_2 = C_{12} + \frac{b_{11}^2}{v_{11}}$$

$$\left.\begin{aligned} A_1 &= \left[1 - \psi - r\frac{d\psi}{dr} - \frac{b_{11}}{v_{11}} D_o\right] \Big/ f(r) \\ A_2 &= \left[1-\psi + \frac{b_{11}}{v_{11}} D_o\right] \Big/ f(r) \end{aligned}\right\} \quad \ldots(6.3.12)$$

Since C_{11}, C_{12}, b_{11}, v_{11} all are the material constants, it is evident that λ_1 and λ_2 are the material properties. Since we are dealing with quartz bar, the different values of the material constants are as follows.

$$C_{11} = 1.316 \times 10^{-12} \ m^2/N$$

$$C_{12} = 1.53 \times 10^{-12} \ m^2/N$$

$$\lambda_1 = 1.3 \times 10^{-11}$$

$$\lambda_2 = 1.73 \times 10^{-12}$$

$$v_{11} = 4.5 \times 8.854 \times 10^{-12} \ F/m$$

Solution of equation (6.3.10) and (6.3.11) yields

$$\sigma_r = \left[\frac{A_1\lambda_1 - A_2\lambda_2}{\lambda_1^2 - \lambda_2^2}\right] = 7.827E+10A_1 - 1.04E+10A_2 \quad \ldots\ldots(6.3.13)$$

$$\sigma_\theta = \left[\frac{A_2\lambda_1 - A_1\lambda_2}{\lambda_1^2 - \lambda_2^2}\right] = 7.827E+10A_2 - 1.04E+10A_1 \quad \ldots\ldots(6.3.14)$$

The describing differential equation from the equations (6.2.4), may now be formed taking (6.3.5), (6.3.13) and (6.3.14) together as

$$r^2\frac{d^2\psi}{dr^2} + (3-2\alpha)\, r\frac{d\psi}{dr} + 2\alpha\left(\frac{\lambda_2}{\lambda_1} - 1\right)\psi$$

$$= 2\alpha\left(\frac{\lambda_2}{\lambda_1} - 1\right) + 2(\alpha-1)\left(\frac{\lambda_2}{\lambda_1} + 1\right)\frac{b_{11}}{v_{11}} D_o$$

$$\ldots\ldots(6.3.15)$$

6.4. Computerised Parabolic Simulation For Regression:

Almost without exception in engineering analysis situations where models are being constructed , the initial differential model set up to represent the behavior of a system will not be in any easily recognisable form. Consequently, it is common practice to repose the model [15] . So the describing differential model of equation (6.3.15) after reformulation reduces to

$$\frac{d^3\psi}{dr^3} + \frac{(5 - 2\alpha)}{r} + \left[3 + 2\alpha\left(\frac{\lambda_2}{\lambda_1} - 2\right)\right] \Big/ r^2 \times \frac{d\psi}{dr} = 0 \quad \ldots\ldots (6.4.1)$$

Since in any numerical simulation efforts, the breakdown of any higher -order differential equation into a relatively simple set is highly significant and the fact that all slopes are simultaneously simulated has significant appeal when working with engineering systems, [18] we introduce the following transform in equation (6.4.1)

(i) $\frac{d\varphi}{dr} = \chi_1$ (ii) $\frac{d^2\varphi}{dr^2} = \frac{dx_1}{dr} = \chi_2$,

(iii) $\frac{d^3\psi}{dr^3} = \frac{dx_2}{dr}$

The equation (6.4.1) after trnasformation reduces to

$$\frac{dx_2}{dr} = -\frac{(5 - 2\alpha)}{r} x_2 - \left[\frac{3 + 2\alpha\left(\lambda_2/\lambda_1 - 2\right)}{r^2}\right] x_1 \quad \ldots\ldots (6.4.2)$$

As we have encountered the parabolic variation for the composite material of the bar at $\alpha = 0.25$ we obviously have only one differential model.

For the simulation of the model in the closed domain of r i.e. $2 \leq r \leq 2.5$ we require the boundary conditions specified at both extremes. We adopt here the powerful "SHOOTING" method to ascertain the boundary conditions for this set. After introduction of the model we get the following set of differential models.

Case - I when $\alpha = 0.25$

$$\frac{dx_2}{dr} = -\frac{4.5}{r} x_2 - \frac{2.0665}{r^2} x_1$$

With boundary conditions:

$$\psi(r) = 2.7497 \quad \text{at} \quad r = 2$$

$$\frac{d\psi}{dr} = -1.4009 \quad \text{at} \quad r = 2$$

$$\frac{d^2\psi}{dr^2} = 1.6703 \quad \text{at} \quad r = 2.5$$

$$\frac{d^2\psi}{dr^2} = \text{Missing} \quad \text{at} \quad r = 2$$

Keeping this nature of variation of the describing transformed differential equation in mind, we developed a software "JTPBV" following the shooting method with the algorithm as follows[15], [16], [17].

Step - I.

The unspecified initial conditions of the system

differential equations are guessed.

Step - II.

A set of variational equations are developed which indicate the sensitivities of the dependent variable with respect to the general initial values.

Step - III

The system variational equations are integrated forward as a set of simultaneous initial value differential equations.

Step - IV

Step II through IV are repeated with the corrected initial conditions unless the specified terminal values are achieved within a small convergence criterion of 0.0001.

The sequence of the software execution are as follows:

1. Title.
2. Main Program.
3. Output options.
4. Subroutine.
5. Construction of Jacobian Matrix.
6. Inversion of Jacobian Matrix.
7. Calculation of Correction vector.
8. Check for Convergence.
9. Correction of initial condition.
10. Non convergence option.

11. Call to printing and plotting subroutine.

12. Return Options.

Subroutine

1. Input equations.

2. Matrix Inversion.

3. Input Integration Program.

4. Differential equation.

5. Integration method.

6. Print Table Results.

7. Plotting options.

8. Plotting.

The variational equations which modelled the investigation is found as:

```
4011G(1) = Y(2)

4012G(2) = Y(3)

4013G(3) = (-4.5/X) * Y(3) - (2.066538/x^2) * Y(2)

4014G(4) = Y(5)

4015G(5) = Y(6)

4016G(6) = (-4.5/x)* Y(6)-(2.066538/X^2)*Y(5)
```

At the end of the execution of Software and scanning we generte a data set to take into consideration of parabolic space variation with α=0.25 for inhomogeneous case study. The details of the data set as generated is given by Table-6.4.1

For investigation of the dependence of ψ, the dependent variable with the only independent variable "r",

TABLE 6.4.1

DETAILS OF GENERATED DATA SET

No.of OBS	DEPENDENT VARIABLE (ψ)	INDEPENDENT VARIABLE (r)	INDEPENDENT VARIABLE (α)	INDEPENDENT CONSTANT (D_0)
1.	2.749748	2	0.25	-104.0414
2.	2.683056	2.05	0.25	-104.0414
3.	2.622888	2.10	0.25	-104.0414
4.	2.568968	2.15	0.25	-104.0414
5.	2.521024	2.20	0.25	-104.0414
6.	2.478789	2.25	0.25	-104.0414
7.	2.441997	2.30	0.25	-104.0414
8.	2.410388	2.35	0.25	-104.0414
9.	2.383704	2.40	0.25	-104.0414
10.	2.361692	2.45	0.25	-104.0414
11.	2.344099	2.50	0.25	-104.0414

we set the parabolic regression equation and developed the Software "JEVAR".

Since we have fitted the generted data sets in the parabolic regression, the $\psi(r)$ takes the following:

$$\psi(r) = a_o + b_o r + C_o r^2 \quad \ldots\ldots(6.4.3)$$

We exploit the developed Software "JSVBAR" to find out the rgression co-efficients a_o, b_o and c_o to have the final form of $\psi(r)$.

The data set as generated in Table 6.4.1 through the Software "JTPBV" with the shooting algorithm and incorporating this data set into parabolic regression, the regression co-efficients are generated. Table 6.4.2 shows the regression co-efficients.

Hence $\psi(r)$ for homogeneous case reduces to

$$\psi(r) = 9.95654 - 5.842444r + 1.119409r^2 \quad \ldots\ldots(6.4.4)$$

After proper scanning of the data set as generated in Table 6.4.1, calculated value of $\psi(r)$ from equation (6.4.3) after regression, we generate the % Error in regression. Table 6.4.3 shows the % Error of Experimental data into regression.

Since it has been observed from Table 6.4.3, that the average % Error in regression is far below the normal practical acceptance of 3%, we take it as granted that form of $\psi(r)$ as generated in equation (6.4.4) is sufficiently accurate and hence the final form of $\psi(r), \sigma_\gamma, \sigma_\theta$ and V with $\alpha = 0.25$, $r_1 = 2$, $r_2 = 2.5$ reduces

TABLE 6.4.2.

REGRESSION CO-EFFICIENTS

REGRESSION CO-EFFICIENT	VALUE
a_o	9.95654
b_o	-5.84244
c_o	1.119409.

TABLE 6.4.3.

PERCENTAGE ERROR IN REGRESSION

No of OBS.	VALUE OF 'r'	GENERATED $\psi(r)$ THROUGH NUMERICAL TECHNIQUE	$\psi(r)$ THROUGH REGRESSION	% ERROR IN REGRESSION
1.	2	2.749748	2.749287	1.67602E-02
2.	2.05	2.683056	2.683846	-2.942182E-02
3.	2.10	2.622888	2.624	-4.239542E-02
4.	2.15	2.568968	2.569753	-3.054283E-02
5.	2.20	2.521024	2.521162	-3.092508E-02
6.	2.25	2.478789	2.478047	2.992269E-02
7.	2.30	2.441997	2.440591	5.758372E-02
8.	2.35	2.410388	2.408731	6.87445E-02
9.	2.40	2.383704	2.382468	5.184048E-02
10.	2.45	2.361692	2.361803	-4.704884E-63
11.	2.50	2.344099	2.346735	-1.124304E-01

FIG. 6.4.3

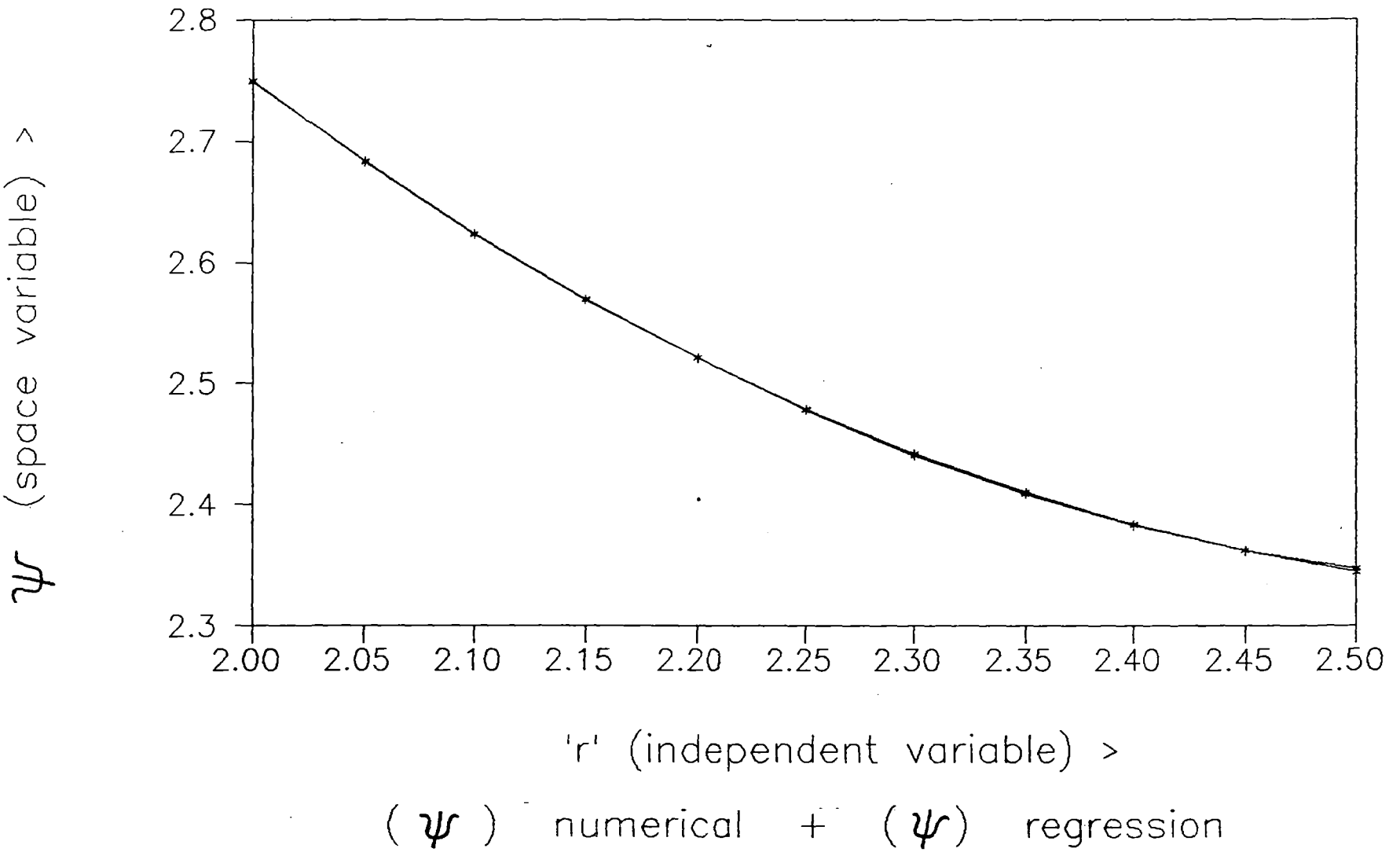
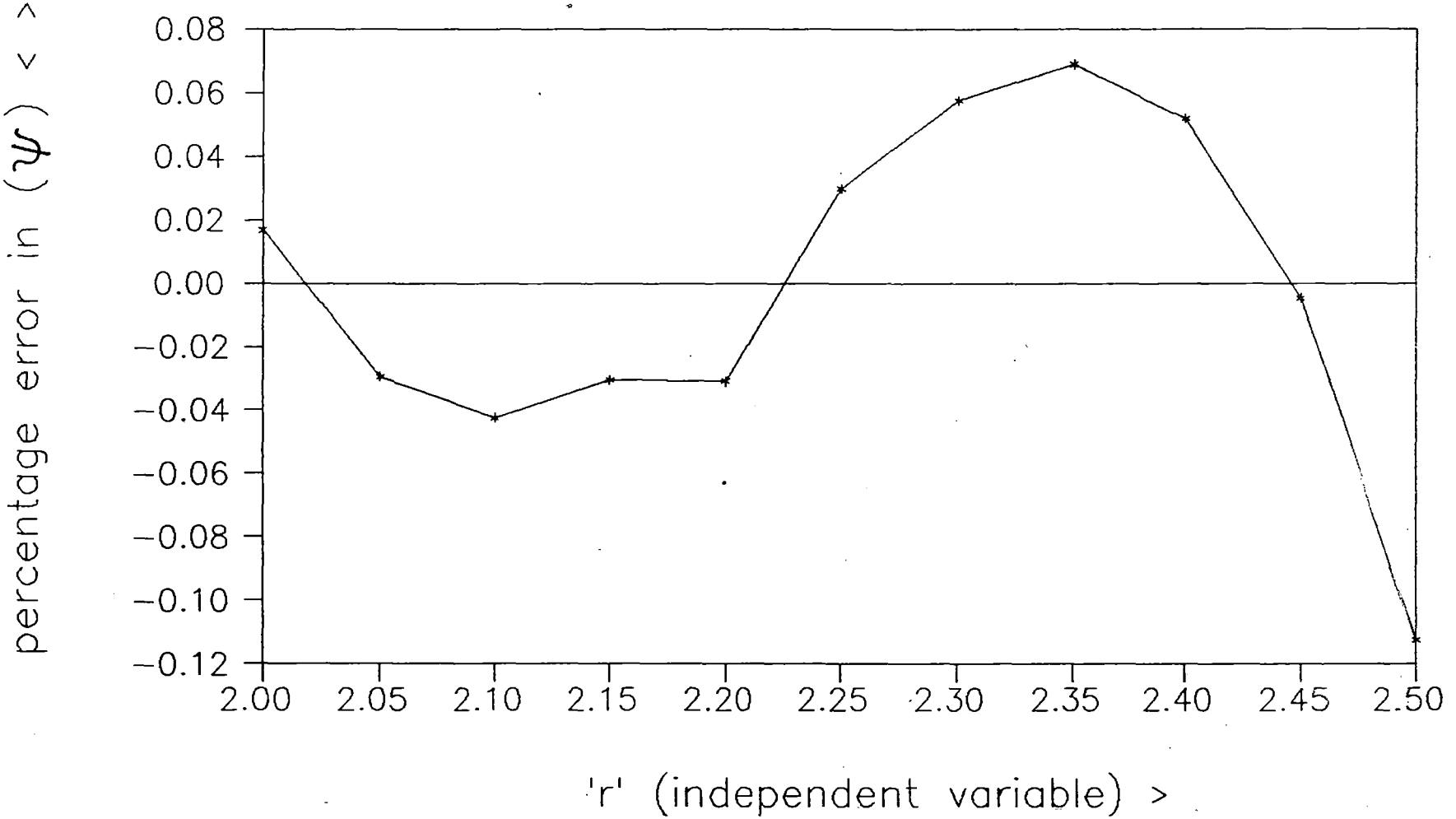

2.8
2.7
2.6
2.5
2.4
2.3
2.00
2.05
2.10
2.15
2.20
2.25
2.30
2.35
2.40
2.45
2.50
ψ (space variable) >
'r' (independent variable) >
(ψ) numerical + (ψ) regression

FIG. 6.4.3a


percentage error in (ψ) < >
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
2.00
2.05
2.10
2.15
2.20
2.25
2.30
2.35
2.40
2.45
2.50
'r' (independent variable) >

to

$$\psi(r) = 9.95654 - 5.84244r + 1.119409r^2 \quad \ldots\ldots(6.4.5)$$

$$A_1 = \left[1 - \psi - r \frac{d\psi}{dr} - \frac{b_{11}}{\nu_{11}} D_o \right]$$

$$A_1 = \left[-8.95654 + 11.6488r - 3.358227r^2 - 0.0552D_o \right] r^{-2\alpha} \quad (6.4.6)$$

$$A_2 = \left[1 - \psi + \frac{b_{11}}{\nu_{11}} D_o \right]$$

$$= \left[-8.95654 + 5.84244r - 1.119409r^2 + 0.0552D_o \right] r^{-2\alpha} \quad (6.4.7)$$

$$\sigma_r = \frac{\lambda_1}{\lambda_1^2 - \lambda_2^2} [A_1] - \frac{\lambda_2}{\lambda_1^2 - \lambda_2^2} [A_2]$$

$$= \left[-0.6117 + 0.8565r - 0.2510r^2 - 0.004874D_o \right] r^{-2\alpha} \times 10^{12}$$

$$\ldots\ldots(6.4.8)$$

$$\sigma_\theta = \frac{\lambda_1}{\lambda_1^2 - \lambda_2^2} [A_2] - \frac{\lambda_2}{\lambda_1^2 - \lambda_2^2} [A_1]$$

$$= \left[-0.6117 + 0.341r - 0.054r^2 + 0.004872 D_o \right] r^{-2\alpha} \times 10^{12}$$

$$\ldots\ldots(6.4.9)$$

$$E_r = \frac{D_o}{E_{11}} - \frac{d_{11}}{\nu_{11}} (\sigma_r - \sigma_\theta)$$

$$= 0.025 \times 10^{12} r^{-2\alpha} D_o - 0.0552 \times 10^{12} r^{-2\alpha} \times \left[0.5155 r - 0.197r^2 - 0.009748 D_o \right]$$

$$= \left[0.0255 D_o - 0.0285r + 0.011r^2 \right] r^{-2\alpha} \times 10^{12}$$

$$\ldots\ldots(6.4.10)$$

$$V = \int_{1}^{r_2} E_r \, dr$$

$$= \left[0.0255 \, D_o \frac{\left(r_2^{1-2\alpha} - r_1^{1-2\alpha}\right)}{(1 - 2\alpha)} - 0.0285 \frac{\left(r_2^{2-2\alpha} - r_1^{2-2\alpha}\right)}{(2-2\alpha)} + 0.011 \frac{\left(r_2^{3-2\alpha} - r_1^{3-2\alpha}\right)}{(3-2\alpha)} \right] \times 10^{12} \quad \ldots(6.4.11)$$

at $\alpha = 0.25$, $r_2 = 2.5$, $r_1 = 2$

$$V = \left[- 0.885 - 0.0214 + 0.0186 \right] \times 10^{12}$$

$$= - 0.8878 \times 10^{12}$$

6.5. Refined Computerised Algorithm for the improvement of the regression Co-efficient to reduce the % Error.

Let us now introduce another method of improvement of the regression co-efficients as obtained in equation (6.4.3) where $\psi(r)$ is being fitted as,

$$\psi(r) = a_o + b_o r + c_o r^2 \quad \ldots \ldots (6.5.1)$$

which can be represented symbolically as

$$\psi = f(r, a, b, c) \quad \ldots \ldots (6.5.2)$$

We have already generated values of a,b,c by parabolic regression as a_o, b_o, c_o, and we are interested to refine this evaluation of a_o, b_o, c_o to reduce the % Error in regression. Let us assume that for the refinements of a_o, b_o, c_o we are to add some corrections to these values

thus obtained as K_1, K_2, K_3 so that the new form of the co-efficients after the introduction of the correction will be,

$$a = a_o + k_1$$

$$b = b_o + k_2$$

$$c = c_o + k_3$$

We are to evaluate K_1, K_2, and K_3 with proper sign to select the over estimation and or under estimation. K_1, K_2, K_3, will be positive if a_o, b_o, c_o is underestimated and negative if a_o, b_o, c_o is over estimated. So refered co-efficients will be either of the following form.

For over estimation

$$a = a_o - k_1$$

$$b = b_o - k_2$$

$$c = c_o - k_3$$

For under estimation

$$a = a_o + k_1$$

$$b = b_o + k_2$$

$$c = c_o + k_3$$

Inserting the pre-determined values of a,b,c that is a_o, b_o, c_o we can have from equation (6.5.2)

$$\psi_{11} = f(r_{11}, a_o, b_o, c_o)$$

$$\psi_{12} = f(r_{12}, a_o, b_o, c_o)$$

$$\psi_{13} = f(r_{13}, a_o, b_o, c_o)$$

... ...

$$\psi_{1n} = f(r_{1n}, a_o, b_o, c_o)$$

Let us introduce the vaviations as $V_{11}.....V_{1n}$

Then $V_{11} = f(r_{11}\ a_o\ b_o\ c_o) - \psi_{11}$ ... ...(6.5.3)

i.e. $V_{11} + \psi_{11} = f(r_{11},\ a_o + k_1,\ b_o + k_2,\ c_o + k_3)$ and so on.

We treat the above equation as function of a,b,c.Then expressing it by Taylor's series,

$$V_{11} + \psi_{11} = f(r_{11},\ a_o,\ b_o,\ c_o) + k_1\left(\frac{\partial f_1}{\partial a}\right)_o + k_2\left(\frac{\partial f_1}{\partial b}\right)_o + k_3\left(\frac{\partial f_1}{\partial c}\right)_o + \text{Terms involving higher powers and product of } k_1,\ k_2,\ k_3 \quad \ldots\ \ldots(6.5.4)$$

Where $\left(\frac{\partial f_1}{\partial a}\right)_o$ mens $\left(\frac{\partial f}{\partial a}\right)_{\substack{r = r_{11} \\ a = a_o \\ b = b_o \\ c = c_o}}$

Therefore $V_{11} + \psi_{11} = \psi_{11} + K_1\left(\frac{\partial f_1}{\partial a}\right)_o + K_2\left(\frac{\partial f_1}{\partial b}\right)_o + K_3\left(\frac{\partial f_1}{\partial c}\right)_o$

Wherefrom we finally developed the normal equations and find out K_1, K_2 and K_3 .

To suit the above algorithm, a Software "REPGN" has been developed and the values of the corrections have been

found out as in Table 6.5.1.

Data as tabulated in Table 6.5.1 will obviously revise $\psi(r)$ also. Now with the refinement of the regression co-efficients the $\psi(r)$ will take the form as:

$$\psi(r) = R_o + R_1 r + R_2 r^2 \quad \ldots\ldots(6.5.5)$$

Taking the values of refined $\psi(r)$, the stresses σ_r, σ_θ, E_r and V took the form as

$$\psi(r) = 9.808906 - 5.710682\ r + 1.09015\ r^2 \quad (6.5.6)$$

$$\sigma_r = \left[- 0.602 + 0.837r - 0.245\ r^2 - 0.004874\ D_o \right] r^{-2\alpha} \times 10^{12} \quad (6.5.7)$$

$$\sigma_\theta = \left[- 0.602 + 0.333r - 0.0527r^2 + 0.004874\ D_o \right] r^{-2\alpha} \times 10^{12} \quad \ldots\ldots(6.5.8)$$

$$E_r = \left[0.0255\ D_o - 0.0278\ r + 0.011\ r^2 \right] r^{-2\alpha} \times 10^{12} \quad \ldots\ldots(6.5.9)$$

$$V = \left[0.0255\ D_o \frac{(r_2^{1-2\alpha} - r_1^{1-2\alpha})}{(1 - 2\alpha)} - 0.0278\ \frac{(r_2^{2-2\alpha} - r_1^{2-2\alpha})}{(2 - 2\alpha)} + 0.011\ \frac{(r_2^{3-2\alpha} - r_1^{3-2\alpha})}{(3 - 2\alpha)} \right] \times 10^{12} \quad \ldots\ldots(6.5.10)$$

At $\alpha = 0.25$, $r_2 = 2.5$, and $r_1 = 2$

$$V = - 0.8878 \times 10^{12}$$

It has been observed from the Table 6.5.2. that the average % Error is still reduced far below the normal practice of acceptance of 3%. We take it as granted that the form of $\psi(r)$ as generated is more or less the exact solution of the describing differential equation.

TABLE 6.5.1

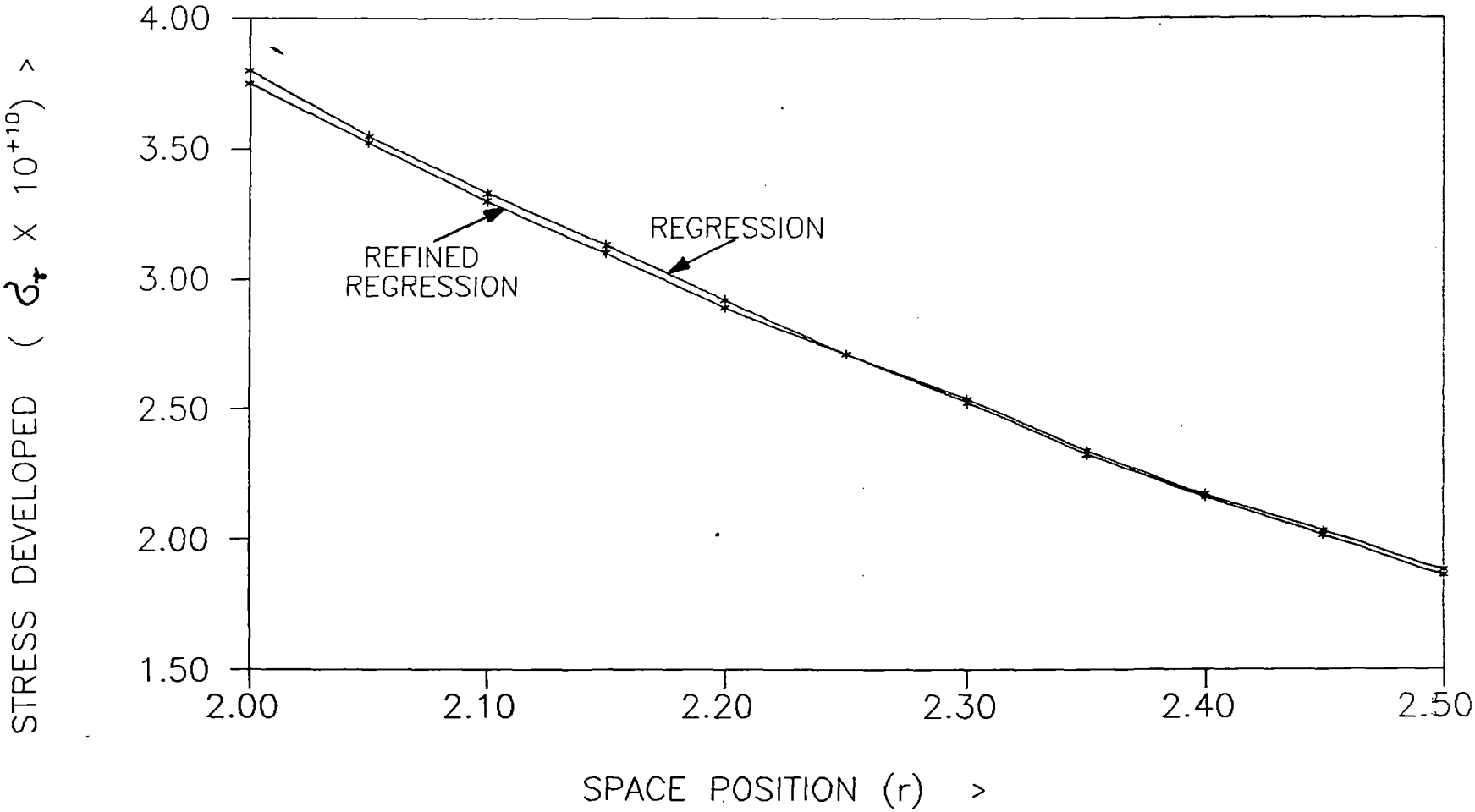
INITIAL REGRESSION CO-EFFICIENTS			REFINED REGRESSION CO-EFFICIENTS					
a_0	b_0	c_0	k_1	k_2	k_3	R_0	R_1	R_2
9.95654	-5.842444	1.199409	-0.147634	0.131762	-0.02246	9.808906	-5.710682	1.090154

TABLE 6.5.2

PERCENTAGE ERROR IN REFINED REGRESSION

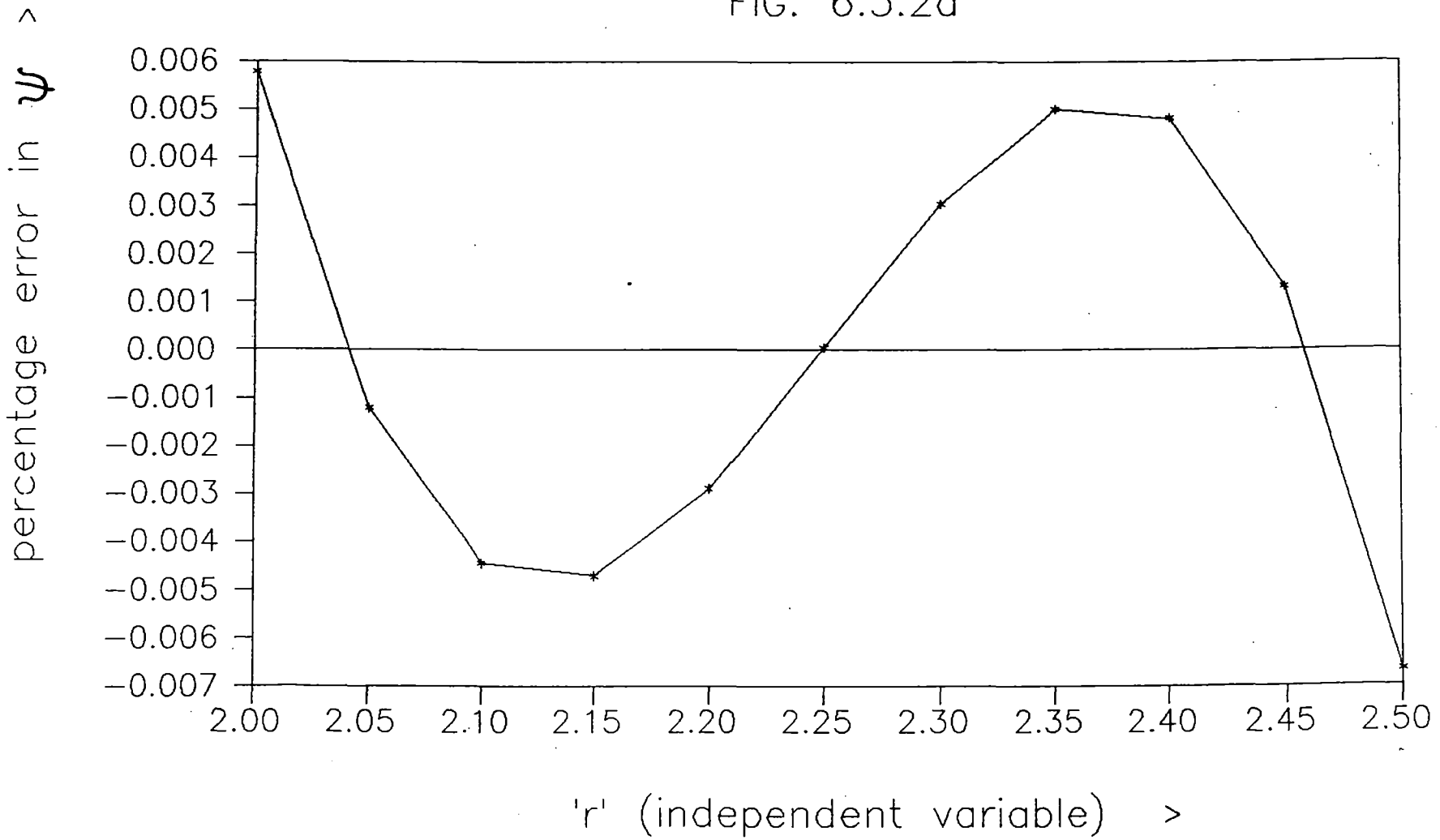
NO OF OBS	VALUE OF (r)	GENERATED $\psi(r)$ THROUGH NUME-RICAL TECHNIQUE	$\psi(r)$THROUGH REFINED TECHNIQUE	%ERROR IN REFINED REGRESSION
1.	2	2.749748	2.748159	5.780664E-03
2.	2.05	2.683056	2.683381	-1.2117E-03
3.	2.10	2.622888	2.624053	-4.44134E-03
4.	2.15	2.568968	2.570177	-4.70625E-03
5.	2.20	2.521024	2.521752	-2.88539E-03
6.	2.25	2.478789	2.478776	5.2901E-05
7.	2.30	2.4410997	2.441253	3.04810E-03
8.	2.35	2.4100388	2.409179	5.01686E-03
9.	2.40	2.3833704	2.382555	4.81997E-03
10.	2.45	2.361692	2.361384	1.30632E-03
11.	2.50	2.344099	2.345662	-6.66811E-03

FIG. 6.5.1



COMPARISON OF REGRESSION AND REFINED REGRESSION

FIG. 6.5.2a


percentage error in ψ >
0.006
0.005
0.004
0.003
0.002
0.001
0.000
−0.001
−0.002
−0.003
−0.004
−0.005
−0.006
−0.007
2.00
2.05
2.10
2.15
2.20
2.25
2.30
2.35
2.40
2.45
2.50
'r' (independent variable) >