CHAPTER V

A NEW COMPUTERISED SIMULATION FOR THE VOLTAGE DEVELOPED IN AN INHOMOGENEOUS PEIZOELECTRIC QUARTZ BAR DUE TO FINITE BENDING

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5.1. Introduction :

Considerable interest has been shown in the past in the fabrication of electroacoustical devices and electronic devices as the basic transducing materials because of their piezoelectric properties. It is evident that this interest has been engenerated by the demands placed on the designers of the sophisticated transducers in several of the present day high-technology industries. Bimorph, a composite transducing element is often used recently to reduce mechanical impedence without lowering the output voltage. But for practical purposes, the goal of the designs is to achieve a material of high output voltage and greater ruggedness with minimum weight for an electro-mechanical appliance. Orchard[23] suggests that the foregoing properties could be well introduced if we form the material in a thin layer using quartz as a concrete aggregate. Since the material layers thus formed have varying proportions of Quartz crystals, the composite bar is obviously nonhomogeneous. Moreover one of the most advantage of forming Piezoelectric Quartz texture in the form of aggregates of any shape is its flexibility for forming in any shape by simple technical processes of grinding and casting. Within the different segments of the same piezoelectric one can vary the electric polarisation and or mechanical force according to their requirement.

5.2. Basic Equations :

Let us consider a uniform narrow rectangular cross-section of a curved bar (Fig 5.2a & 2b) composed of different layers consisting as an aggregate of Quartz cement mixture.[23] Bending is effected in the plane of the curvature by couples "M" applied at the ends of the bar Naturally the stress distribution is the same in all radial cross-sections, so that the stress components do not depend on θ, but are functions of 'r' only. Owing to this symmetry the shearing stress does not exist.

The basic equations of this electro-mechanical problem consists of three parts, namely

(i) The equations of electricity

(ii) The equations of elasticity

(iii) The constitutive equations for the piezoelectric materials, to set up interaction amongst electric, clastic and Piezoelectric relations.

(i) The equation of electric field is governed by the well-known Maxwell's and Gauss's divergence equations, stated, mathematically as

FIG.5.2(a)

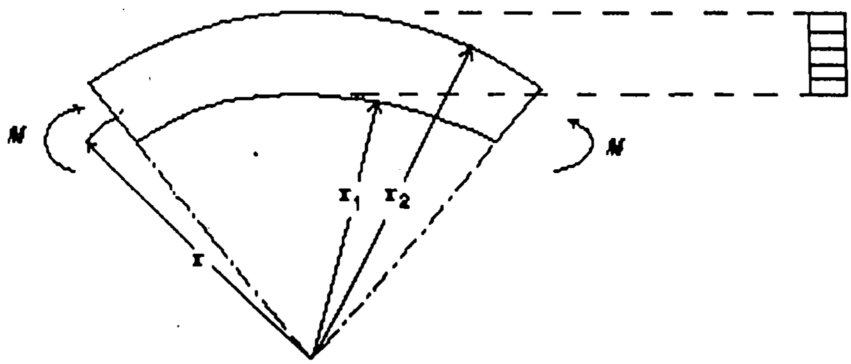


FIG.5.2(b)

$$\text{rot } \vec{E} = 0 \qquad \ldots \quad \ldots(5.2.1)$$

$$\text{Div } \vec{D} = 0 \qquad \ldots \quad \ldots(5.2.2)$$

Where $\vec{E}$ = Electric field intensity vector

$\vec{D}$ = Electric Induction vector

From equation (5.2.1) one can easily form

E = grad ϕ, where ϕ ≡ Electric potential function

$E_r = \frac{d\phi}{dr}$ in the present problem.

If r_1 and r_2 are the radii of the lower and upper faces of the bar and if the faces are coated with a cnducting material with uniform potential ϕ, the electrical boundary condition becomes -

$$V_2 - V_1 = \int_{r=r_1}^{r=r_2} V \qquad \ldots \quad \ldots(5.2.3)$$

The integral is taken along the path from the lower to upper face of the boundary of the bar.

(ii) The equations of elasicity starts from, the stress equation of equilibrium, in the following form

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_o}{r} = 0 \qquad \ldots \quad \ldots(5.2.4)$$

Where σ_r = the normal stress in radial direction

σ_θ = the normal stress in circumferential direction at a point (r,θ)

To set up the interaction between the electric, elastic and Piezoelectric relatin, we make use of the following constitutive relations for the present investigation.

$$S_r = S_{11}\delta_r + S_{12}\delta_\theta + d_{11}E_r \qquad \ldots\ \ldots(5.2.5)$$

$$S_\theta = S_{12}\delta_r + S_{11}\delta_\theta - d_{11}E_r \qquad \ldots\ \ldots(5.2.6)$$

$$S_{r\theta} = 0 \qquad \ldots\ \ldots(5.2.7)$$

$$D_r = d_{11}\ (\delta_r - \delta_\theta\ ) + \xi_{11}E \qquad \ldots\ \ldots(5.2.8)$$

$$D_\theta = 0 \qquad \ldots\ \ldots(5.2.9)$$

Where S_r, S_θ, $S_{r\theta}$ ≡ Strain components

S_{11}, S_{12} ≡ Elastic compliances at constant field

d_{11} ≡ The piezodectric·strain parameter

ξ_{11} ≡ Dielectric permitivity at constant stress.

For solving these problems one should have the electrical and mechanical boundary conditions of the problem. The electrical boundary conditions has already been in equation (5.2.3), we are to set up mechanical boundary conditions now. Taking the width of the rectangular cross-section as unity, the mechanical boundary conditions are-

$$\delta_r = 0 \text{ for } r = r_1 \text{ and } r = r_2 \qquad \ldots\ \ldots(5.2.10)$$

$$\int_{r_1}^{r_2} \sigma_\theta dr = 0 \qquad \ldots\ \ldots(5.2.11)$$

$$\int_{r_1}^{r_2} r\sigma_\theta dr = -M \qquad \text{... ...(5.2.12)}$$

5.3. Simulation of the exact mathematical model :

We have seen Piezoelectric property of the bar inherits the inhomogeneity of meterial. This inhomogencity is characterised by the variations of clastic, Piezoelectric and dielectric parameters from point to point in a static problem[14].In particular their variations,where radial symmetry is considered may be of the form[14].

$$s_{ij} = c_{ij} f(r) \qquad \text{... ...(5.3.1)}$$

$$d_{ij} = b_{ij} f(r) \qquad \text{... ...(5.3.2)}$$

$$\epsilon_{ij} = v_{ij} f(r) \qquad \text{... ...(5.3.3)}$$

for $i,j = 1, 2, 3$

Where C_{ij}, b_{ij}, v_{ij} are the material parameters and termed as elastic,piezoelectric and dielectric material constant of the bar under investigation. f(r) is the space position to take into consideration of the variation of these properties from point to point depthwise.

Since most of the real world problems are encountered to exhibit the depthwise, parabolic, linear and inverse variations in the physical parameters of the materials of the bar, we introduce the space function "f(r)" in the following form

$$f(r) = r^{2\alpha} \qquad \ldots \ \ldots (5.3.4)$$

where α = homogeneity factor[22].

To make the modelling usable for the practical deseign purpose one should have to assign the variations of α and r. To encounter the practical design problems with the mathematical modelling developed, we assign the variations of α = 0.25, 0.5, -0.25, -0.50 to encounter parabolic, linear and inverse variations for which the system in inhomogeneous and α = o for the homogeneous system. Since at α = 0 the material properties of the bar as given in the equations (5.3.1),(5.3.2),and (5.3.3) are independent of the space position, the mathematical model is homogeneous.

For matching f(r) for practical design purpose we are to assign 'r' also. For making the mathematical model usable in practice and since the material properties are varying point to point with respect to the space position 'r' it is evident to consider the variation of 'r' in a closed domain instead of assigning any specific value. Hence we introduce the variation of 'r' in the closed domain $2 \le r \le 2.5$.

The Gaussion divergence equation (5.2.2) in two dimensional polar co-ordinates owing to the radial symmetry yields.

$$\frac{\partial D_r}{\partial r} = 0 \qquad \ldots \ \ldots (5.3.5)$$

Where D = Constant = D_o(say)

Equation (5.2.8) with (5.3.5) yields

$$D_o = d_{11}(\delta_r - \delta_\theta) + \epsilon_{11}E_r \qquad \ldots\ \ldots(5.3.6)$$

Therefore $$E_r = \frac{D_o - d_{11}(\delta_r - \delta_\theta)}{\epsilon_{11}} \qquad \ldots\ \ldots(5.3.7)$$

The radial and tangential components of displacements can be taken as

$$U = r(1 - \psi) \quad \text{and } v = 0 \qquad \ldots\ \ldots(5.3.8)$$

where ψ is a function of r[21]

Hence the radial and circumferential strain components now become

$$S_r = \frac{d_{11}}{dr} = 1 - \psi - r\frac{d\psi}{dr} \qquad \ldots\ \ldots(5.3.9)$$

And $$S_\theta = \frac{u}{dr} = 1 - \psi \qquad \ldots\ \ldots(5.3.10)$$

Equations (5.2.5) and (5.2.6) along with equations (5.3.7),(5.3.1),(5.3.2),(5.3.3),(5.3.9),(5.3.10) can be in more compact form as

$$\lambda_1\sigma_r + \lambda_2\sigma_\theta = A_1 \qquad \ldots\ \ldots(5.3.11)$$

and $$\lambda_2\sigma_r + \lambda_1\sigma_\theta = A_2 \qquad \ldots\ \ldots(5.3.12)$$

Where $$\lambda_1 = C_{11} - \frac{b_{11}^2}{v_{11}}$$

$$\lambda_2 = C_{12} - \frac{b_{11}^2}{v_{11}}$$

$$\left.\begin{aligned} A_1 &= \left[1 - \psi - r \frac{d\psi}{dr} - \frac{b_{11}}{v_{11}} D_o \right] \Big/ f(r) \\ A_2 &= \left[1 - \psi + \frac{b_{11}}{v_{11}} - \frac{b_{11}}{v_{11}} D_o \right] \Big/ f(r) \end{aligned}\right] \quad \ldots \ \ldots (5.3.13)$$

Since $C_{11}, C_{12}, b_{11}, v_{11}$ all are the material constants, it is evident that λ_1 and λ_2 are the material properties. Since we are dealing with Quartz bar, the different values of the material constants are as follows[10] :-

$C_{11} = 13.16 \times 10^{-12}\ m^2/N$

$C_{12} = 1.53 \times 10^{-12}\ m^2/N$

$b_{11} = 2.2$ pico coulomb/N

$v_{11} = 4.5 \times 8.854 \times 10^{-12}$ F/M

$\lambda_1 = 1.3 \times 10^{-11}$

$\lambda_2 = 1.73 \times 10^{-12}$

Solutions of equation (5.3.11) and (5.3.12) yields

$$\sigma_r = \left[\frac{A_1\lambda_1 - A_2\lambda_2}{\lambda_1^2 - \lambda_2^2} \right] = 7.827\ E + 10\ A_1 - 1.04\ E + 10\ A_2 \quad \ldots \ \ldots (5.3.14)$$

$$\sigma_\theta = \left[\frac{A_2\lambda_1 - A_1\lambda_2}{\lambda_1^2 - \lambda_2^2} \right] = 7.827\ E + 10\ A_2 - 1.04\ E + 10\ A_1 \quad \ldots \ \ldots (5.3.15)$$

The describing differential equation from the equation (5.2.4) taking equations (5.3.4),(5.3.14) and (5.3.15) together, as

$$r^2 \frac{d^2\psi}{dr^2} + (3 - 2\alpha) r \frac{d\psi}{dr} + 2\alpha\left(\frac{\lambda_2}{\lambda_1} - 1\right) \psi$$

$$= 2\alpha \left(\frac{\lambda_2}{{}_1\lambda} - 1 \right) + 2(\alpha - 1)\left(\frac{\lambda_2}{\lambda_1} + 1 \right) \frac{b_{11}}{v_{11}} D_o \qquad \ldots \ \ldots(5.3.16)$$

For finding out the suitable form of ψ, satisfying equation (5.3.16), we are to find out complementary function ψ_c and particular function ψ_p. The complete solution will be

$$\psi = \psi_c + \psi_p$$

For finding out the complimentary function in inhomogeneous and homogeneous cases we introduce $\psi = r^m$, from which we have the roots of the auxiliary equation as

$$m_1, m_2 = (\alpha - 1) \pm \left[(\alpha - 1)^2 - 2\alpha\left(\frac{\lambda_2}{\lambda_1} - 1\right) \right]^{1/2}$$

Hence $\psi_c = Ar^{m_1} + Br^{m_2}$

Where A,B = constants which are to be evaluated from the boundary conditions.

For finding out the particular function in inhomogeneous case, we introduce

$$\psi = a_o + a_1 r + a_2 r^2$$

and substituting this value of ψ_{pn} in equation (5.3.16) and equating the like powers of 'r' on both sides, we have

$$a_1 = a_2 = 0$$

and $$a_o = 1 - \frac{(\alpha-1)}{\alpha} \times \frac{(\lambda_1+\lambda_2)}{(\lambda_1-\lambda_2)} \times \frac{b_{11}}{v_{11}} D_o$$

Hence $$\psi_{pN} = \left[1 - \frac{(\alpha-1)}{\alpha} \times \frac{(\lambda_1+\lambda_2)}{(\lambda_1-\lambda_2)} \times \frac{b_{11}}{v_{11}} D_o\right]$$

The complete solution of ψ for Inhomogeneous case, therefore,

$$\psi(r) = Ar^{m_1} + Br^{m_2} + \psi_{pN} \qquad \ldots \ldots (5.3.17)$$

The particular function for homogeneous case, when $\alpha = 0$, will be

$$\psi_{pH} = - \frac{(\lambda_1+\lambda_2)}{\lambda_1} \frac{b_{11}}{v_{11}} l_n r$$

The complete solution of ψ for homogeneous case, therefore,

$$\psi(r) = Ar^{m_1} + Br^{m_2} + \psi_{pH} \qquad \ldots \ldots (5.3.18)$$

Table (5.3.1) shows the different values of the parameters for $\psi(r)$ both in Homogeneous and In homogenous cases.

Combining Equations (5.3.17), (5.3.14) and (5.3.15), (5.3.13), the expression for δ_r and δ_θ for inhoogeneous case may be of the form

$$\delta_r = -\left[AK_1 r^{m_1} + BK_2 r^{m_2} + D_o K_3\right] r^{-2\alpha} \Big/ (\lambda_1^2 - \lambda_2^2) \qquad \ldots(5.3.19)$$

$$\delta_\theta = -\left[Al_1 r^{m_1} + Bl_2 r^{m_2} + D_o l_3\right] r^{-2\alpha} \Big/ (\lambda_1^2 - \lambda_2^2) \qquad \ldots(5.3.20)$$

After imposition of the boundary conditions of equations (5.2.10), (5.2.11) & (5.2.12) in (5.3.19) and (5.3.20) we have the following set of equations.

$$a_1 A + b_1 B + c_1 D_o = 0 \qquad \ldots\ \ldots(5.3.21)$$

$$a_2 A + b_2 B + c_2 D = 0 \qquad \ldots\ \ldots(5.3.22)$$

$$a_3 A + b_3 B + c_3 D_o = 0 \qquad \ldots\ \ldots(5.3.23)$$

where

$$a_1 = (\lambda_1 - \lambda_2 + \lambda_1 m_1) r_1^{m_1} = K_1 r_1^{m_1}$$

$$b_1 = (\lambda_1 - \lambda_2 + \lambda_1 m_2) r_1^{m_1} = K_1 r_1^{m_1}$$

$$c_1 = \frac{(\lambda_1 - \lambda_2)}{\alpha} \frac{b_{11}}{v_{11}} = K_3$$

$$a_2 = \left(\frac{r_2}{r_1}\right)^{m_1} ; \quad b_2 = b_1 \left(\frac{r_2}{r_1}\right)^{m_2} ; \quad c_2 = c_1$$

$$a_3 = \left(\lambda_1 - \lambda_2 - \lambda_2 m_1\right) \left(r_2^{m_1 - 2\alpha + 2} - r_1^{m_1 - 2\alpha + 2}\right) \Big/ (m_1 - 2\alpha + 2)$$

$$= l_1 \left(r_2^{m_1 - 2\alpha + 2} - r_1^{m_1 - 2\alpha + 2}\right) \Big/ \left(m_1 - 2\alpha + 2\right)$$

TABLE 5.3.1

VALUES OF $\psi(r)$ FOR HOMOGENEOUS AND INHOMOGENEOUS CASES

α	m_1	m_2	ψ_{pN}	ψ_{pH}	REMARKS
0.25	0.2479787	-1.74797	$1+0.213D_o$		
0.50	0.5568458	-1.556846	$1+0.071D_o$		INHOMOGEN EOUS
0.25	-0.1874378	-2.312562	$1-0.356D_o$		
0.50	-0.3239571	-2.676043	$1-0.213D_o$		
0	0	-2		$-0.062D_o$	HOMOGENE -OUS

$$b_3 = \left(\lambda_1-\lambda_2-\lambda_2 m_2\right)\left(r_2^{m_2-2\alpha+2}-r_1^{m_2-2\alpha+2}\right) \Big/ \left(m_2-2\alpha+2\right)$$

$$= l_2 \times \left(r_2^{m_2-2\alpha+2}-r_1^{m_2-2\alpha+2}\right) \Big/ \left(m_2-2\alpha+2\right)$$

$$c_3 = c_1\left(1 - 2\alpha\right)\left(r_2^{2(1-\alpha)}-r_1^{2(1-\alpha)}\right) \Big/ 2\left(1 - \alpha\right)$$

$$= c_1\left(1 - 2\alpha\right) \times l_3$$

$$K = 4E+12 \left(\lambda_1^2 - \lambda_2^2\right)$$

From equations (5.3.21),(5.3.22) & (5.3.23), we have

$$A = \frac{D_1}{D},\ B = \frac{D_2}{D},\ D_o = \frac{D_3}{D} \quad \ldots\ \ldots(5.3.24)$$

The value of D_i may be obtained by replacing the i th column by O,O,K where D stands for the non-singular determinant

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Table 5.3.2. shows all the above co-efficient in case of inhomogeneous Quartz bar for $r_1=2$ and $r_2=2.5$.

Incorporating the values of A, B, D_o from equation (5.3.24) on the equation (5.3.19) and 5.3.20) the stress

TABLE 5.3.2

HOMOGENEITY FACTOR (α)

$\alpha = 0.25$

a_1	b_1	c_1	a_2	b_2	c_2	a_3	b_3	c_3	A	B	D_o
1.721191 E-11	-3.409 987 E-12	3.2406 E-12	1.8191 17E-11	-2.30 864 E-12	3.2406 E-12	4.163 232 E-12	1.98173 E-12	0	16.654 73	14.80 852	-104.0 414

$\alpha = 0.5$

a_1	b_1	c_1	a_2	b_2	c_2	a_3	b_3	c_3	A	B	D_o
2.7227 65 E-12	-3.048 503 E-12	1.620 3 E-12	3.083 004 E-11	-2.153 837 E-12	1.6263 E-12	5.876 97 E-11	1.4701 87 E-12	0	12.331 08	49.65 131	-300.6 286

$\alpha = -0.25$

a_1	b_1	c_1	a_2	b_2	c_2	a_3	b_3	c_3	A	B	D_o
7.7571 E-12	-3.7831 E-12	-3.2406 E-12	7.4393 E-12	-2.258 1E-11	-3.2406 1E-12	-9.7120 E-12	3.0086 E-12	0	-73.08 895	15.22 87	-157.1 775

$\alpha = -0.50$

a_1	b_1	c_1	a_2	b_2	c_2	a_3	b_3	c_3	A	B	D_o
5.6389 E-12	-3.6799 E-12	-1.6203 E-12	5.2457 E-12	-2.0254 E-12	-1.6206 E-12	-2.6225 E-11	3.4269 E-12	0	-26.132	6.2109	-76.83 84.

$\alpha = 0$

a_1	b_1	c_1	a_2	b_2	c_2	a_3	b_3	c_3	A	B	D_o
6.7889 E + 10	2.2183 E + 10	-2.9325 E + 09	6.7889 E + 10	-1.4197 E + 10	-3.8766 E +09	7.6375 E + 10	1.98 E + 10	-8.4722 E + 09	797.625	639.85 36	-809.25 81

components σ_r and σ_θ are finally obtained in Inhomogeneous case as:

$$\sigma_r = - \left[A_1 r^p + B_1 r^q + C_1 \right]$$

$$\sigma_\theta = - \left[A_2 r^p + B_2 r^q + C_2 \right]$$

Where

$$A_1 = A\left(\lambda_1 - \lambda_2 + \lambda_1 m_1\right) \Big/ \left(\lambda_1^2 - \lambda_2^2 \right)$$

$$B_1 = B\left(\lambda_1 - \lambda_2 + \lambda_1 m_2\right) \Big/ \left(\lambda_1^2 - \lambda_2^2 \right)$$

$$C_1 = C_1 D_o$$

$$A_2 = A\left(\lambda_1 - \lambda_2 - \lambda_1 m_1\right) \Big/ \left(\lambda_1^2 - \lambda_2^2 \right)$$

$$B_2 = B\left(\lambda_1 - \lambda_2 - \lambda_2 m_2\right) \Big/ \left(\lambda_1^2 - \lambda_2^2 \right)$$

$$C_2 = \left(1-2\alpha\right) C_1$$

$P = m_1 - 2\alpha$ and $q = m_2 - 2\alpha$

Table 5.3.3 represents the different values of these constants for quartz bar in Inhomogeneous case.

Incorporating these values of the constants in equation (5.3.7) we can have

$$E_r = \left[T_1 r^p + T_2 r^q + T_3 \right]$$

FIG. 5.3.1

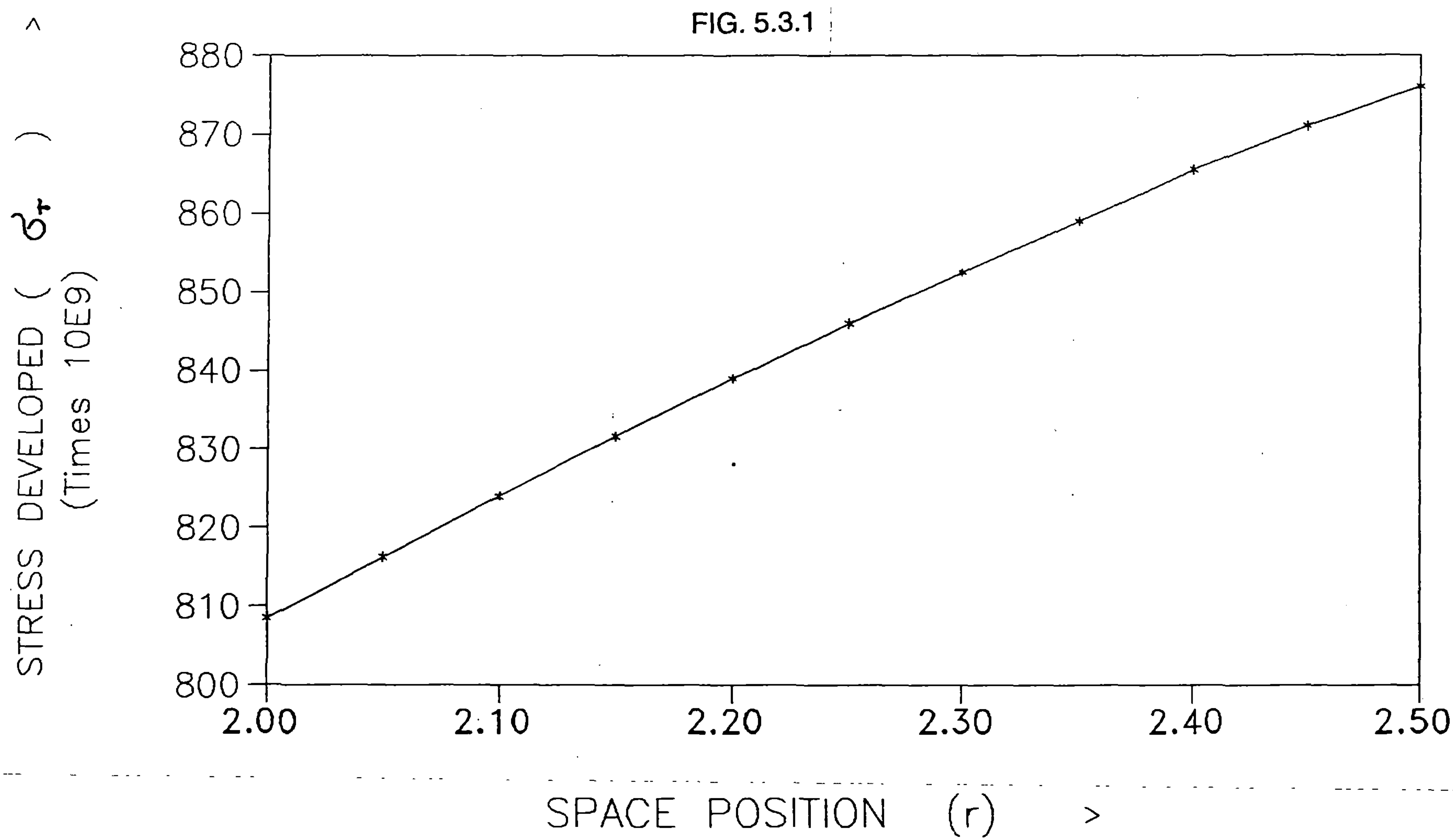
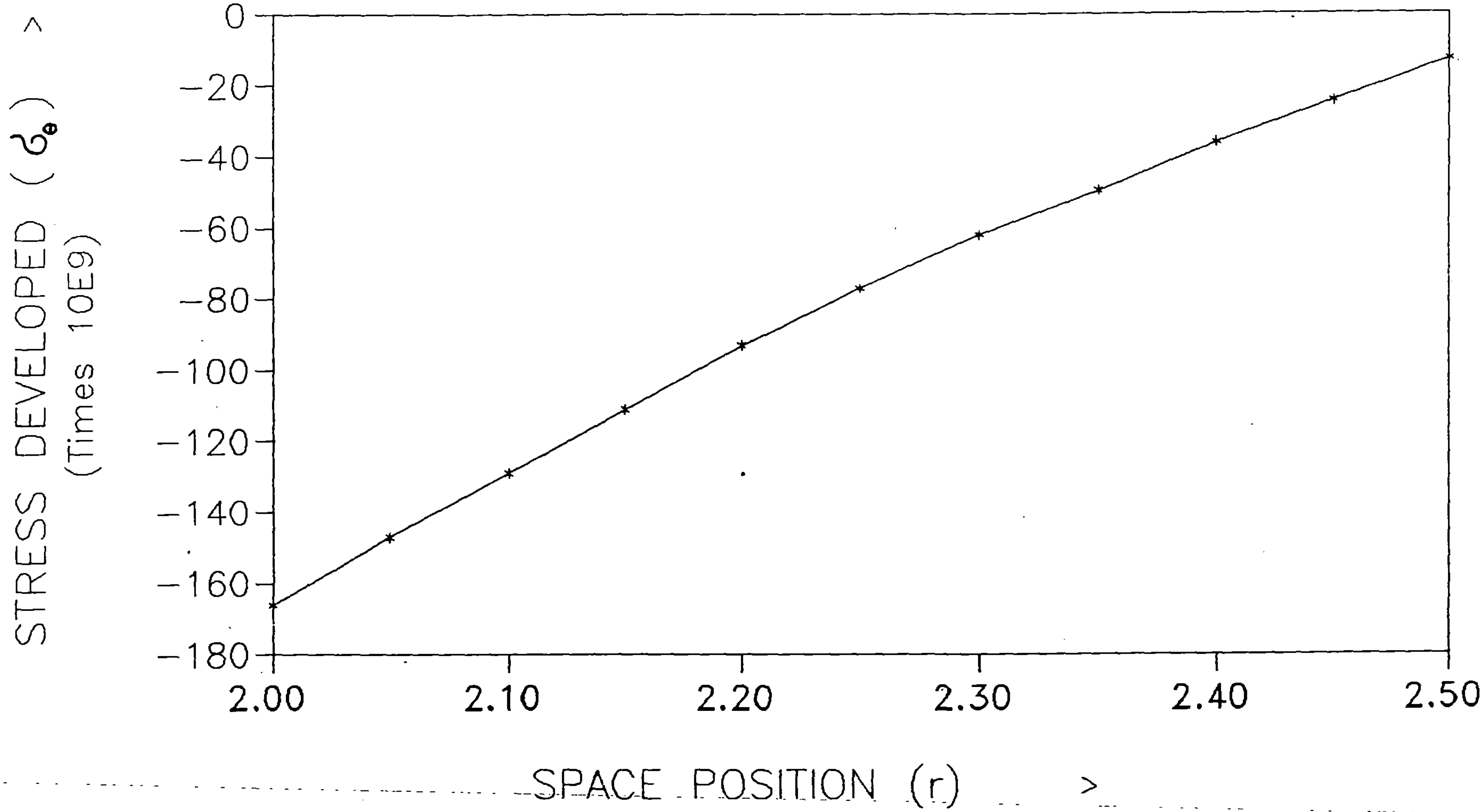


FIG. 5.3.1
880
870
860
850
840
830
820
810
800
2.00
2.10
2.20
2.30
2.40
2.50
STRESS DEVELOPED (σ_r) >
(Times 10E9)
SPACE POSITION (r) >

FIG. 5.3.2



STRESS DEVELOPED (σ_θ) >
(Times 10E9)
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
2.00
2.10
2.20
2.30
2.40
2.50
SPACE POSITION (r) >

TABLE 5.3.3

CONSTANTS FOR QUARTZ BAR, IN HOMOGENEOUS CASE

α	A_1	B_1	C_1	A_2	B_2	C_2	p	q
0.25 E+12	1.4541 E+12	-1.0217 E+12	-2.0309 E+12	1.0876 E+12	1.2751 E+12	-1.0155 E+12	-0.2520	-2.248
0.50	1.3748 E+12	-2.6824 E+12	-2.9343 E+12	7.6558 E+12	4.1764 E+12	0	-0.4431	-2.5568
-0.25	-3.8891 E+12	-1.724 E+12	3.0682 E+12	-5.1047 E+12	1.4001 E+12	4.6023 E+12	0.3126	-1.8126
0.50	-1.111 E+12	-8.7991 E+12	7.4997 E+12	-1.8623 E+12	5.9486 E+12	1.499 E+12	0.6760	-1.676

Where $T_1 = A\left(b_{11} m_1\right) \Big/ v_{11}\left(\lambda_1 + \lambda_2 \right)$

$T_2 = B\left(b_{11} m_2\right) \Big/ v_{11}\left(\lambda_{11} + \lambda_2\right)$

$T_3 = D_o\left(\lambda_1 + \lambda_2 + \frac{2b_{11}^2}{v_{11}} \right) \Big/ v_{11} \left(\lambda_1 - \lambda_2\right)$

$p = m_1 - 2\alpha$

$q = m_2 - 2\alpha$

Table 5.3.4 represents different values of constant for a Quartz bar in Inhomogeneous case.

From the electrical boundary condition of equation (5.2.3) we have the electrical voltage developed between the upper and lower surfaces of the Inhomogeneous baras

$$V = \left[V_1+V_2+V_3\right]$$

Where $V_1 = T_1 \dfrac{(r_2^{p+1}-r_1^{p+1})}{p + 1}$

$$V_2 = T_2 \frac{(r_2^{q+1}-r_1^{q+1})}{q + 1}$$

$$V_3 = T_3 \frac{(r_2^{1-2\alpha}-r_1^{1-2\alpha})}{1-2\alpha}$$

Table 5.3.5 represents different values for voltages r_1=2 and r_2=2.5 of a Quartz bar in Inhomogeneous case.

5.4. Solution For The Homogeneous Bar.

Since α is the factor which determines the

TABLE 5.3.4

CONSTANTS FOR QUARTZ BAR (INHOMOGENEOUS)

α	T_1	T_2	T_3	p	q
0.25	2.02573E+10	-1.2696E+11	-2.6705E+12	-0.2520	-2.2480
0.50	3.3878E+10	-3.7913E+11	-7.7165E+12	-0.4431	-2.5568
-0.25	6.7194E+10	-1.7273E+11	-4.0344E+12	0.3126	-1.8126
-0.50	4.1522E+10	-8.152E+10	-1.9723E+12	0.6760	-1.6760

TABLE 5.3.5

VOLTAGES FOR QUARTZ BAR (INHOMOGENEOUS CASE)

HOMOGENEITY FACTOR=α	VOLTAGE=V	REMARKS
0.25	-8.9370E+12	
0.50	-2.11E+12	INHOMOGENEOUS CASE
-0.25	-3.001E+12	
-0.50	-2.1934E+12	

homogeneity of the bar, it is natural that in a homogeneous piezoclectric bar, $\alpha=0$. Hence the material constants are independent of the space position. Proceeding in the same manner as that of the inhomogeneous case, one can have the following stress equations.

$$\sigma_r = -4M(H_1 r^{-2} + H_2 l_n r + H_3 \qquad \ldots \ldots (5.4.1)$$

$$\sigma_\theta = -4M(-H_1 r^{-2} + H_2 l_n r + H_4) \qquad \ldots \ldots (5.4.2)$$

Where $H_1 = r_1^2 r_2^2 l_n(\frac{r_2}{r_1})$

$H_2 = (r_1^2 - r_2^2)$

$H_3 = (r_2^2 l_n r_2 - r_1^2 l_n r_1)$

$H_4 = H_2 - H_3$

$N = (H_2)^2 - 4H_1 l_n(\frac{r_2}{r_1})$

The voltage generated between the upper and lower faces of the homogeneous bar can be expressed as

$$V = -\frac{4MH_2}{N(r_1 + r_2)}\left[5.96H_2 - 0.022 H_1\right] \qquad \ldots (5.4.3)$$

$$= -V_1 + V_2$$

Table 5.4.1 represents the different values of the constants of the Homogeneous bar with $r_1=2$ & $r_2=2.5$ for determining the voltage.

Table 5.4.2 shows the comparison of voltages developed in inhomogeneous cases of a Quartz bar.

TABLE 5.4.1

CONSTANTS FOR QUARTZ BAR (HOMOGENEOUS CASE) FOR DETERMINING VOLTAGE.

α	H_1	H_2	H_3	H_4	N	V_1	V_2	V
0	3.5785	2.25	2.9542	-0.7042	-22.83	-1.6436	-0.04299	1.687E +12

TABLE 5.4.2

COMPARISON OF VOLTAGES DEVELOPED

Voltage Developed in Homogeneous Quartz Bar	**Voltage Developed in Inhomogeneous Quartz Bar (V_{NH})**	$\frac{V_{NH}}{V_H}$	**Remarks**
1.689E+12 at α = 0	-8.937E+12 at α = 0.25	5.29	It is observed that the average voltage in Inhomogeneous case is 2.4 times then that of homogeneous case
	-2.11E+12 at α = 0.50	1.249	
	-3.001E+12 at α = -0.25	1.776	
	-2.193E+12 at at α = -0.50	1.296	

FIG. 5.4.1 GRAPH-Showing the voltages for homogeneous & inhomogeneous cases

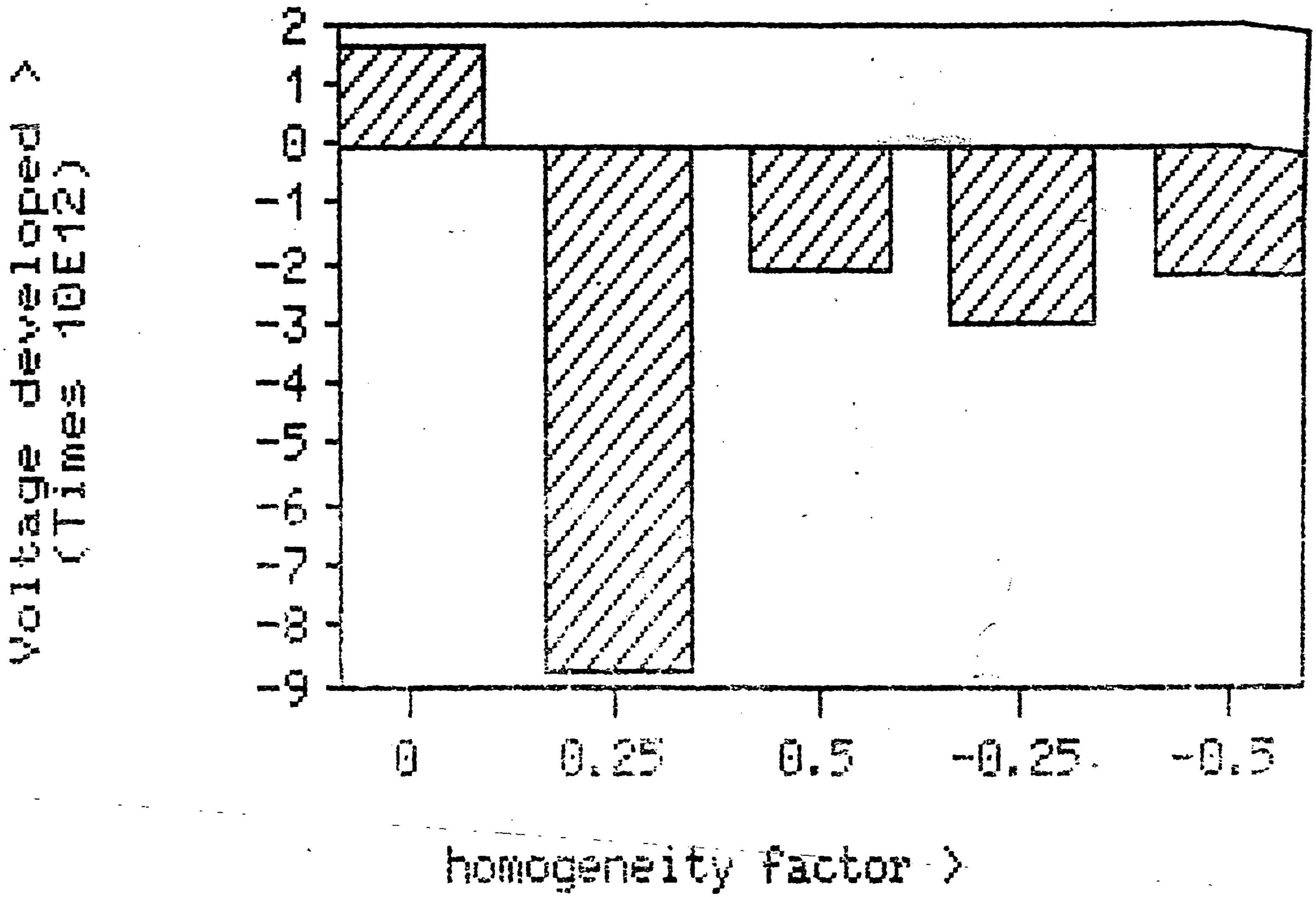


Fig. 5.4.1 shows the Graphical display of the various voltages in Homogeneous and Inhomogeneous bar material.

5.5. Computerised Simulation Of The Problem Under Investigation.

Almost without exception in engineering analysis situations where models are being constructed, the initial differential model set up to represent the behavior of a system will not be in any easily recognisable form. Consequently it is common practice to repose the model. So the describing differential model of equation (5.3.16) after reformulation reduces to :

$$\frac{d^3\psi}{dr^3} + \frac{(5-2\alpha)}{r}\frac{d^2\psi}{dr^2} + \frac{3+2\alpha(\frac{\lambda_2}{\lambda_1}-2)}{r^2}\frac{d\psi}{dr} = 0 \qquad (5.5.1)$$

Since, in any numerical simulation efforts, the breakdown of any higher-order differential equation into a relatively simple set is highly significant and the fact that all slopes are simultaneously simulated has significant appeal when working with engineering systems, [15] we introduce the following transform in equation (5.5.1)

(i) $\frac{d\psi}{dr} = x_1$,

(ii) $\frac{d^2\psi}{dr^2} = \frac{dx_1}{dr} = x_2$, (iii) $\frac{d^3\psi}{dr^3} = \frac{dx_2}{dr}$

The equation (5.5.1) after transformation reduces to

$$\frac{dx_2}{dr} = -\frac{(5-2\alpha)}{r}x_2 - \left[\frac{3+2\alpha(\frac{\lambda_2}{\lambda_1}-2)}{r^2}\right]x_1 \quad \ldots\ \ldots(5.5.2)$$

As we have encountered five different composite set of material of the bar at α=0.25,0.50,-0.25,-0.50,0, we obviously have the five different differential models. For the simulation of the model in the closed domain of 'r' i.e. $2 \leq r \leq 2.5$ we require the boundary conditions specified at both extremes. We adopt here the powerful "SHOOTING" method to ascertain the boundary conditions for these five sets. After introduction of the method we get the following set of differential models

CASE I when [$\alpha = 0.25$]

$$\frac{dx_2}{dr} = -\frac{4.5}{r}x_2 - \frac{2.0665}{r^2}x_1$$

With boundary conditions :

$$\psi(r) = 2.7497 \quad \text{at} \quad r = 2$$

$$\frac{d\psi}{dr} = -1.4009 \quad \text{at} \quad r = 2$$

$$\frac{d^2\psi}{d^2r} = +1.6703 \quad \text{at} \quad r = 2.5$$

$$\frac{d^2\psi}{dr^2} = \text{missing} \quad \text{at} \quad r = 2$$

CASE II. When [$\alpha = 0.50$], $2 \le r \le 2.5$

$$\frac{dx_2}{dr} = -\frac{4}{r} x_2 - \frac{1.1331}{r^2} x_1$$

With boundary conditions

$$\psi(r) = 14.4049 \quad \text{at} \quad r = 2$$

$$\frac{d\psi}{dr} = -8.0863 \quad \text{at} \quad r = 2$$

$$\frac{d^2\psi}{dr^2} = +6.7830 \quad \text{at} \quad r = 2.5$$

$$\frac{d^2\psi}{dr^2} = \text{Missing} \quad \text{at} \quad r = 2$$

CASE III When [$\alpha = -0.25$], $2 \le r \le 2.5$

$$\frac{dx_2}{dr} = -\frac{5.5}{r} x_2 - \frac{3.9334}{r^2} x_1$$

With boundary conditions :

$$\psi(r) = -3.625 \quad \text{at} \quad r = 2$$

$$\frac{d\psi}{dr} = 2.4706 \quad \text{at} \quad r = 2$$

$$\frac{d^2\psi}{dr^2} = 5.0671E-02 \quad \text{at} \quad r = 2.5$$

$$\frac{d^2\psi}{dr^2} = \text{Missing} \quad \text{at} \quad r = 2$$

CASE IV When [$\alpha = -0.50$], $2 \le r \le 2.5$

$$\frac{dx_2}{dr} = -\frac{6}{r} x_2 - \frac{4.8669}{r^2} x_1$$

With the boundary conditions :

$$\psi(r) = -2.338 \qquad \text{at} \quad r = 2$$

$$\frac{d\psi}{dr} = 2.0812 \qquad \text{at} \quad r = 2$$

$$\frac{d^2\psi}{dr^2} = -0.4909 \qquad \text{at} \quad r = 2.5$$

$$\frac{d^2\psi}{dr^2} = \text{Missing} \qquad \text{at} \quad r = 2$$

CASE V When $\alpha = 0, \quad 2 \le r \le 2.5$

$$\frac{dx_2}{dr} = -\frac{5}{r} x_2 - \frac{3}{r^2} x_1$$

With the boundary conditions :

$$\psi(r) = 9.5646E + 2 \qquad \text{at} \quad r = 2$$

$$\frac{d\psi}{dr} = -1.8519E + 12 \qquad \text{at} \quad r = 2$$

$$\frac{d^2\psi}{dr^2} = 1.0635E + 12 \qquad \text{at} \quad r = 2.5$$

$$\frac{d^2\psi}{dr^2} = \text{Missing} \qquad \text{at} \quad r = 2$$

Keeping these five distinct cases in mind we developed software "JTPBV" following shooting method along with the Newton's method. The algorithm of the software is given below : [15], [16], [17]

Step - I

The unspecified initial conditions of the system differential equations are guessed

Step - II

A set of variational equations are developed which indicate the sensitivities of the dependent vriables with respect to the general initial-values.

Step - III

The system variational equations are integrated forward as a set of simultaneous intial-value differential equations.

Step - IV

The initial conditions are corrected using the variations (sensitivities) calculated in Step III.

Step - V

Step II through IV are repeated with the corrected initial conditions, until the specified terminal values are achieved within a small criterion of 0.0001.

The sequence of software execution are as follows:

1. Title
2. Main Program
3. Output Options
4. Subroutine
5. Construction of Jacokian Matrix
6. Inversion of Jacobian Matrix
7. Calculation of corrected vector
8. Check for convergence
9. Correction of initial condition
10. Non Convergence option

11. Call to printing and plotting subroutine.

12. Return Options

Subroutines

1. Input Equations
2. Matrix Inversion
3. Input Integration Parameter
4. Differential equations
5. Integration methods
6. Print Table of Results
7. Plotting Options
8. Plotting

The variational equations which modelled the 5-different cases formed as:

CASE - I

```
4011 G [1] = Y [2]
4012 G [2] = Y [3]
4013 G [3] = [-4.5/x)* Y(3)-(2.066538/x^2) * Y(2)
4014 G [4] = Y(5)
4015 G [5] = Y(6)
4016 G [6] = (-4.5/x) * Y(6)-(2.066538/x^2) * Y(5)
4095 End.
```

CASE - II

```
4011 G [1] = Y [2]
4012 G [2] = Y [3]
4013 G [3] = [-4/x)* Y(3)-(1.1331/x^2) * Y(2)
```

```
4014 G [4] = Y(5)
4015 G [5] = Y(6)
4095  End.
```

CASE - III

```
4011 G [1] = Y [2]
4012 G [2] = Y [3]
4013 G [3] = [-5.5/x)* Y(3)-(3.9334/x^2) * Y(2)
4014 G [4] = Y(5)
4015 G [5] = Y(6)
4016 G [6] = (-5.5/x) * Y(6)-(3.9334/x^2) * Y(5)
```

CASE - IV

```
4011 G [1] = Y [2]
4012 G [2] = Y [3]
4013 G [3] = [-5.5/x)* Y(3)-(3.9334/x^2) * Y(2)
4014 G [4] = Y(5)
4015 G [5] = Y(6)
4016 G [6] = (-5.5/x) * Y(6)-(3.9334/x^2) * Y(5)
4095 End.
```

CASE - V

```
4011 G [1] = Y [2]
4012 G [2] = Y [3]
4013 G [3] = [-6/x)* (4.8669/x^2) * Y(2)
4014 G [4] = Y(5)
4015 G [5] = Y(6)
```

4016 G [6] = (-5/x) * Y(6)-(3/x^2) * Y(5)

At the end of the execution of the software and scanning we found a data set for case-I to case-IV of Inhomogeneous case study and for case-V of Homogeneous case study. The detail of the sets are given in Table 5.5.1 and Table 5.5.2.

For investigation of the dependence of ψ, the only dependent variable with the three independent variables namely r, α and D_o (in Inhomogeneous case) we set the multiple regession equation & developed the Software "JMVAR" using the following concepts [15], [16].

$$\psi = a + f_1(r) + f_2(\alpha) + f_3(D_o) \quad \ldots \ldots (5.5.3)$$

Where

$$\left.\begin{aligned} f_1(r) &= b_1 r + b_2 r^2 + b_3 r^3 \\ f_2(\alpha) &= b_4 \alpha + b_5 \alpha^2 + b_6 \alpha^3 \\ f_3(D_o) &= b_7 D_o + b_8 r_o^2 + b_9 D_o^3 \end{aligned}\right] \quad \ldots \ldots (5.5.4)$$

Let $x_1 = \psi$, $x_2 = r$, $x_3 = r^2$, $x_4 = r^3$.

$x_5 = \alpha$, $x_6 = \alpha^2$, $x_7 = \alpha^3$, $x_8 = D_o$.

$x_9 = D_o^2$, $x_{10} = D_o^3$

The multiple regression equation (5.5.3) will become

TABLE 5.5.1

DATA SET FOR INHOMOGENEOUS CASE (INHOMOGENEOUS).

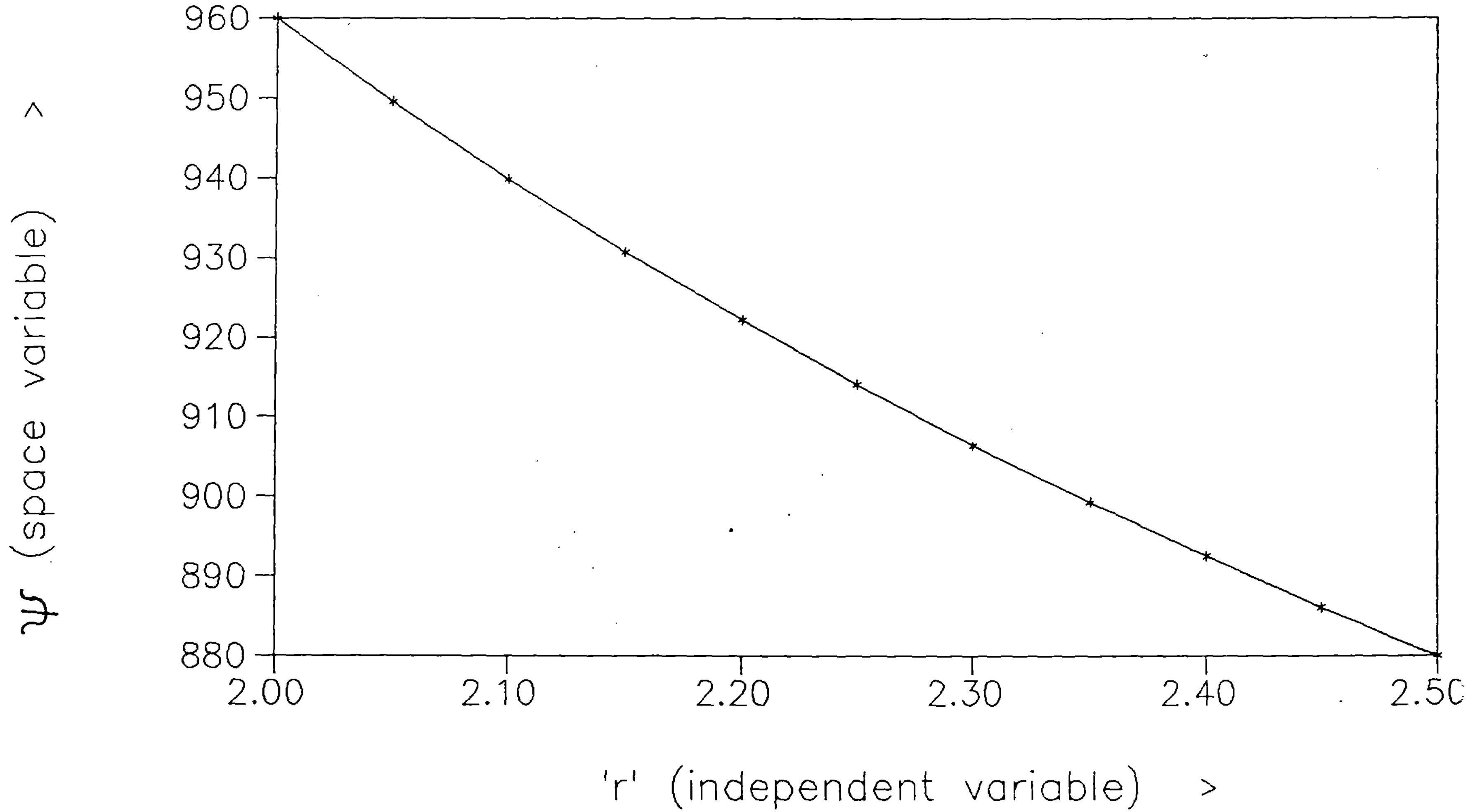
NO. OF OBS	DEPENDENT VARIABLE (ψ)	INDEPENDENT VARIABLE (r)	INDIPENDENT VARIABLE (α)	INDEPENDEN VARIABLE (D_o)	REMARK
1.	2.749748	2.00	0.25	-104.0414	Data for Inhomogeneous case study
2.	13.06631	2.20	0.50	-300.6286	
3.	-2.522325	2.40	-0.25	-157.1775	
4.	-1.315056	2.50	-0.50	-76.83836	

TABLE 5.5.2

DATA SET FOR HOMOGENEOUS CASE

NO. OF OBS.	**DEPENDENT VARIABLE(ψ)**	**INDEPENDENT VARIABLE(ψ)**	**REMARK**
1.	956.463	2.00	Date set for Homogeneous Case study
2.	947.5091	2.05	
3.	939.1283	2.10	
4.	931.2695	2.15	
5.	923.8869	2.20	
6.	916.9399	2.25	
7.	910.3922	2.30	
8.	904.2113	2.35	
9.	898.3678	2.40	
10.	892.8352	2.45	
11.	887.5901	2.50	

FIG. 5.5.2



960
950
940
930
920
910
900
890
880
2.00
2.10
2.20
2.30
2.40
2.50
Ψ (space variable) >
'r' (independent variable) >

$$x_1 = a + b_1x_2 + b_2x_3 + b_3x_4 + b_4x_5$$

$$+ b_5x_6 + b_6x_7 + b_7x_8 + b_8x_9 + b_9x_{10} \quad \ldots \quad \ldots (5.5.5)$$

The net regression co-efficients b_1 to b_9 can be determined form the following set of 9 Simultaneons equations

$$\Sigma\ (x_2^2)b_1 + \Sigma\ (x_2x_3)b_2 + \Sigma\ (x_2x_3b_4) + \ldots\ldots + \Sigma\ (x_2x_{10})b_9$$

$$= \Sigma\ (x_1x_2)$$

$$\Sigma\ (x_2x_3)b_1 + \Sigma\ (x_3)^2\ b_2 + \Sigma\ (x_3x_4)b_3 + \ldots\ldots + \Sigma\ (x_3x_{10})b_9$$

$$= \Sigma\ (x_1x_3)$$

$$\Sigma\ (x_2x_{10})b_1 + \Sigma\ (x_3x_{10})b_2 + \Sigma(x_4x_{10})b_3 + \ldots\ldots + \Sigma\ (x_{10})^2b_9$$

$$= \Sigma\ (x_1x_{10})$$

In general $\Sigma\ x_px_q = \Sigma\ x_px_q - nM_pM_q$

Where n = no. of observations

p and q can have all the values 1 to p and 1 to q

$$M_p = \frac{\Sigma x_p}{n}, \qquad M_q = \frac{\Sigma x_q}{n}$$

Since all the co-efficients from b_1, to b_2 are known from the above 9 sets of simultaneous equations, the constant 'a' can then be evaluated as:

$$a = M_1 - b_1M_2 - b_2M_3 - b_3M_4 - b_4M_5 - b_5M_6$$

$$- b_6M_7 - b_7M_8 - b_8M_9 - b_9M_{10}$$

Taking the data set as generated in Table 5.5.2 and incorporating these data set into multiple regression, the regression co-efficients become, as given in Table 5.5.3 and Table 5.5.4.

Hence for Inhomogeneous case $\psi(r)$, σ_r, σ_θ and v finally take the form

$$\psi(r) = -12.1587 + 7.149708r + 0.684085r^2 - 0.3946133r^3 + 5.411582\alpha + 37.266865\alpha^2 + 72.1915\alpha^3 + 0.03202176D_o + 0.0000575D_o^2 - 0.00000015D_o^3 \quad \ldots \ldots (5.5.6)$$

$$\sigma_r = \Big[0.89330773 - 1.045232r - 0.1534035r^2 - 0.1277r^3 - 0.3673692\alpha - 2.5218\alpha^2 - 4.09\alpha^3 - 0.0065035D_o + 0.0000039028D_o^2 + 0.0000000135D_o^3 \Big] r^{-2\alpha} \times 10^{12}$$

$$\sigma_\theta = \Big[0.8931073 - 0.410765r - 0.0321607r^2 + 0.01444r^3 - 0.36733\alpha - 2.52965\alpha^2 - 4.9003\alpha^3 - 0.00155433D_o + 0.0000039028D_o^2 - 0.00000001359D_o^3 \Big] r^{-2\alpha} \times 10^{12} .$$

$$V = \Big[0.002527D_o \frac{(r_2^{1-2\alpha} - r_1^{1-2\alpha})}{(1 - 2\alpha)} + 0.035 \frac{r_2^{2-2\alpha} - r_1^{2-2\alpha}}{(2 - 2\alpha)} + 0.00781 \frac{(r_2^{4-2\alpha} - r_1^{4-2\alpha})}{(4-2\alpha)} - 0.015 \times 10^{-7} D_o^3 \Big] \times 10^{12}$$

Table 5.5.4 shows clearly the accuracy of the method developed by the comparison of results of

TABLE 5.5.3

REGRESSION CONSTANTS OF MULTIVARIOUS REGRESSION, (INHOMOGENEOUS CASE)

a	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
-12.15 681	7.149 710	0.684 085	- 0.394 613	5.4115 80	37.268 650	72.191520	3.202 18E-02	- 5.749 66E - 05	- 1.531 50E - 07

TABLE 5.5.4

COMPARISON OF RESULTS OF ANALYTICAL SOLUTION AND COMPUTERISED MULTIVARIUS REGRESSION IN CASE OF INHOMOGENEOUS STUDY.

NO OF OBS	ANALYTICAL VALUE OF(ψ)	COMPUTERISED VALUE OF (ψ)	%ERROR COMPUTATION	AVERAGE % ERROR
1.	2.742748	2.750728	-3.5636E - 02	2.99315E-02
2.	13.06631	13.0661	-2.2845E - 03	
3.	-2.522325	-2.522768	-1.7562E - 02	
4.	-1.315056	-1.31590	-6.4243E - 02	

Analytical Solution and computerised Multivarious Regression in case of Inhomogeneous study.

For the investigation of the dependence of ψ the only dependent variable in the single independent variable 'r', in Homogeneous case, as $\alpha=0$ and D_o=Constant, we set the cubic regression equation and developed the Software "WCRA" using the following concepts [15],[16].

$$\psi(r) = a + b_1 r + b_2 r^2 + b_3 r^3 \qquad \ldots \quad \ldots (5.5.7)$$

The four regression constants a, b_1, b_2, b_3 are computed by the simultaneous solution of the three equations using the notations

$$u = r^2$$

$$V = r^3$$

$$\left(\sum r^2\right) b_1 + \left(\sum ru\right) b_2 + \left(\sum rv\right) b_3 = \sum r\,\psi$$

$$\left(\sum ru\right) b_1 + \left(\sum u^2\right) b_2 + \left(\sum uv\right) b_3 = \sum u\,\psi$$

$$\left(\sum rv\right) b_1 + \left(\sum uv\right) b_2 + \left(\sum v^2\right) b_3 = \sum v\,\psi$$

Where in general

$$M_v = \frac{\sum v}{n}$$

$$\sum uv = \sum uv - n\,M_u\,M_v$$

$$\sum rv = \sum uv - n\,M_r\,M_p$$

$$\sum v^2 = \sum v^2 - n\,M_v^2$$

$\sum v\,\psi = \sum v\,\psi - n\,M_p\,M_\psi$

Since all b_1, b_2, b_3 are being evaluated by equation (5.5.8) "a" can be found out

$$a = M_\psi - b_1M_r - b_2M_u - b_3M_v$$

Taking the data set as generated in Table 5.5.2 and after incorporating these data set into cubic regression, the regression co-efficients become as in Table 5.5.5.

Hence $\psi(r)$, σ_r σ_θ v for the Homogeneous case finally took the form as

$$\psi(r) = 1640.981 - 506.7127r + 82.18098r^2 + 8.335E - 04r^3 \qquad \ldots\ \ldots(5.5.9)$$

$$\sigma_r = \left[0.06788 - 0.06788a - 0.0574b_1r - 0.047b_2r^2 - 0.3028b_3r^3 + 0.0049D_o\right]r^{-2\alpha} \times 10^{12}$$

$$\sigma_\theta = \left[0.06788 - 0.06788a - 0.0574\,b_1r - 0.047\,b_2r^2 - 0.0366\,b_3r^3 + 0.0049\,D_o\right]r^{-2\alpha} \times 10^{12}$$

$$V = \left[0.01255D_o + 0.0055b_1 + 0.0248b_2 + 0.0847b_3\right] \times 10^{12}$$

Table 5.5.6 shows comparison of the results of Analytical solution and Computerised Cubic regression in case of Homogeneous Study.

The voltage developed in Inhomogeneous and Homogeneous case study are compared in Table 5.5.7.

The interesting aspects of this investigation is the magnitude and polarity developed of the output voltages for both Inhomogeneous and Homogeneous bar

TABLE 5.5.5

REGRESSION CO-EFFICIENTS

a	b_1	b_2	b_3
1640.981	- 506.7127	82.18098	- 8.2351E - 04

TABLE 5.5.6

COMPERISON OF THE RESULTS OF ANALYTICAL SOLUTION AND COMPUTERISED CUBIC REGRESSION IN CASE OF HOMOGENEOUS STUDY.

NO. OF OBS	ANALYTICAL VALUE OF ψ	COMPUTERISED VALUE OF ψ	% ERROR IN COMPUTATION	AVERAGE % ERROR
1.	956.46	956.2723	1.9935E - 02	
2.	947.5091	947.2942	-7.2662E - 03	
3.	939.1283	939.2942	-3.3820E - 01	
4.	931.2695	931.4215	-1.6326E - 02	
5.	923.8869	923.9597	-7.8814E - 03	
6.	916.9399	916.9086	3.4081E - 03	-2.9138E-02
7.	910.3922	910.2686	1.3576E - 02	
8.	904.2113	904.0394	1.9008E - 02	
9.	898.3678	898.2209	1.6353E - 02	
10.	892.8352	892.8135	2.4336E - 03	
11.	887.5901	887.8169	-2.5560E - 02	

GRAPH 5.5 : SHOWING VARIATION OF $\psi(r)$ BY ANALYTICAL AND COMPUTERISED METHOD

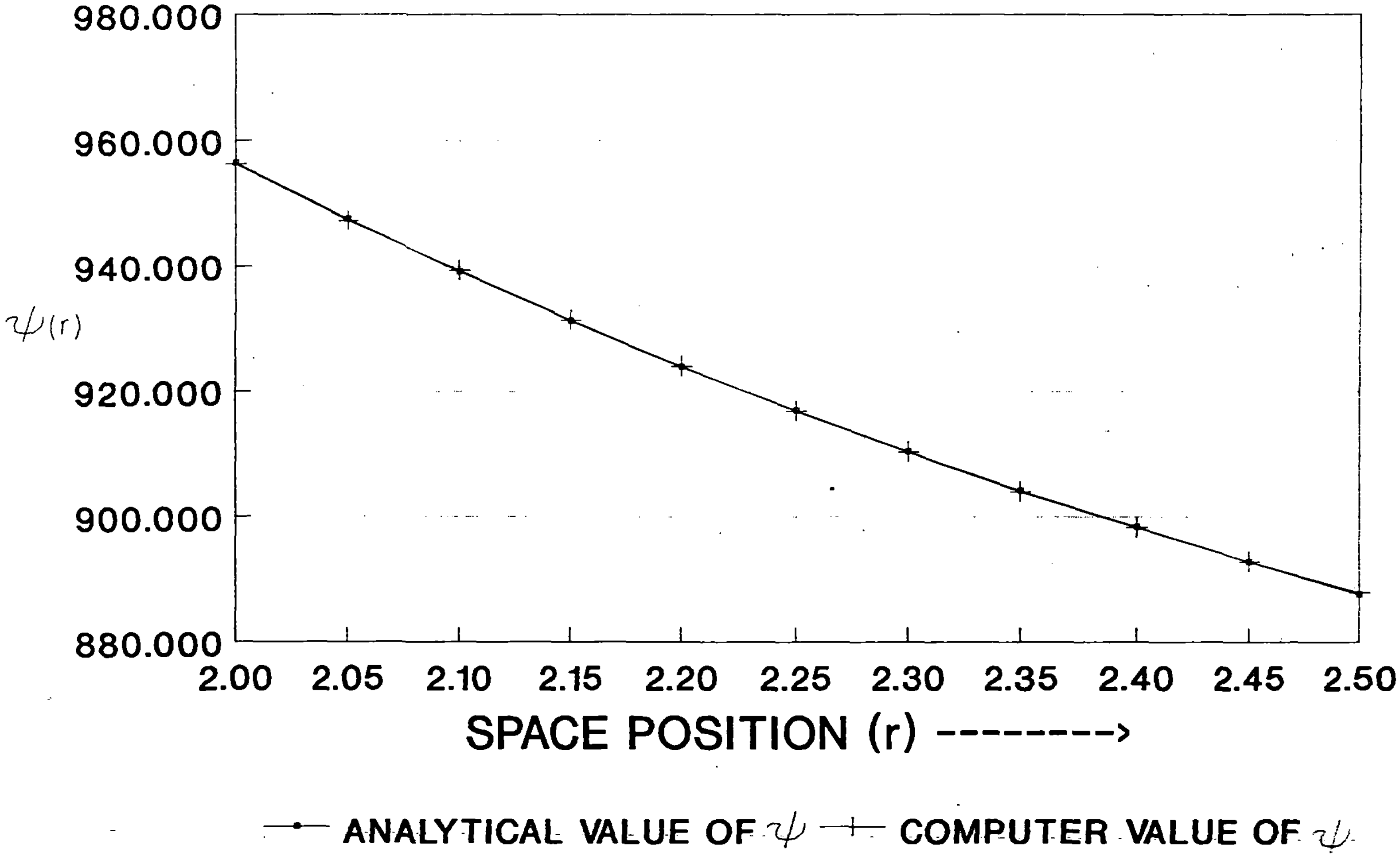


FIG. 5.5.6

FIG. 5.5.6a

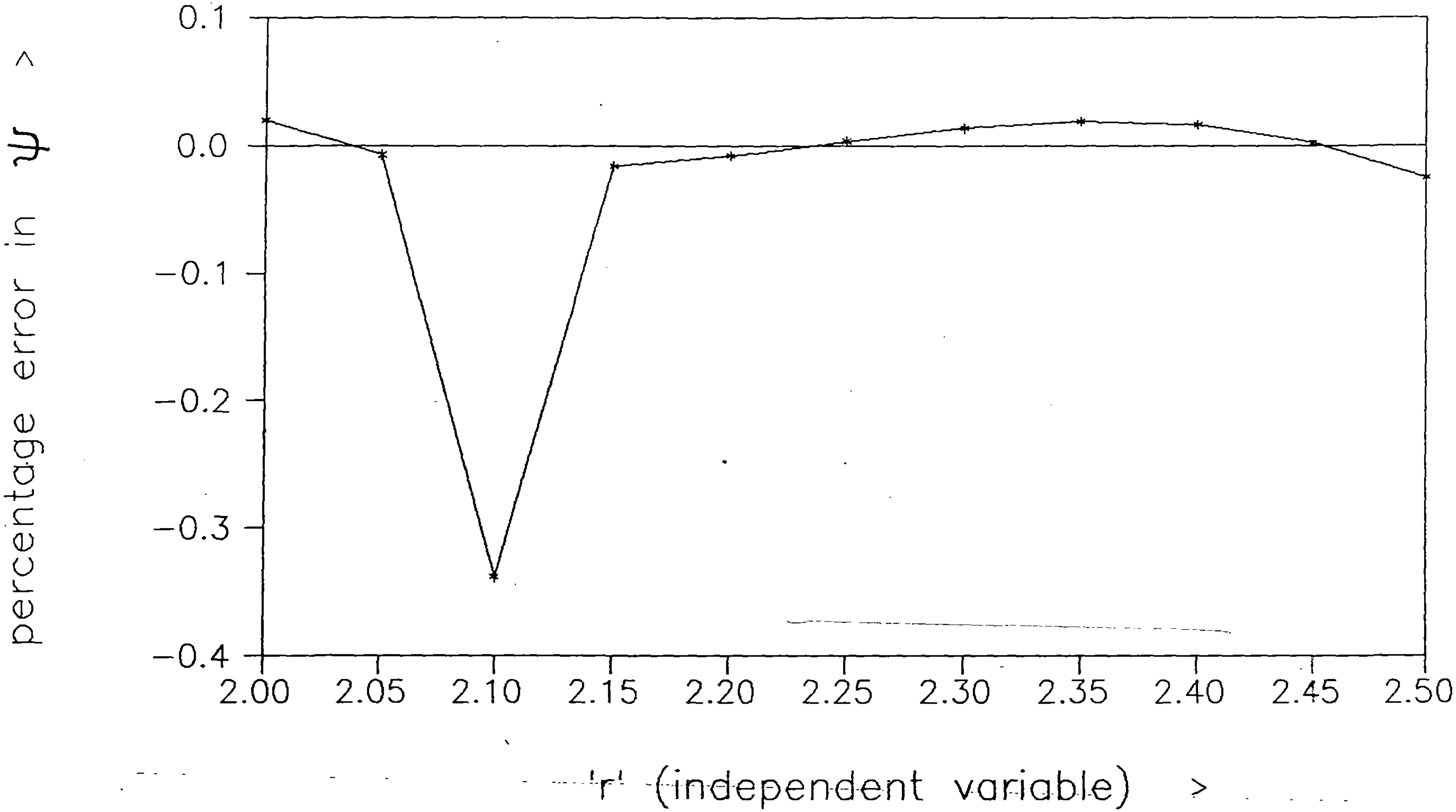
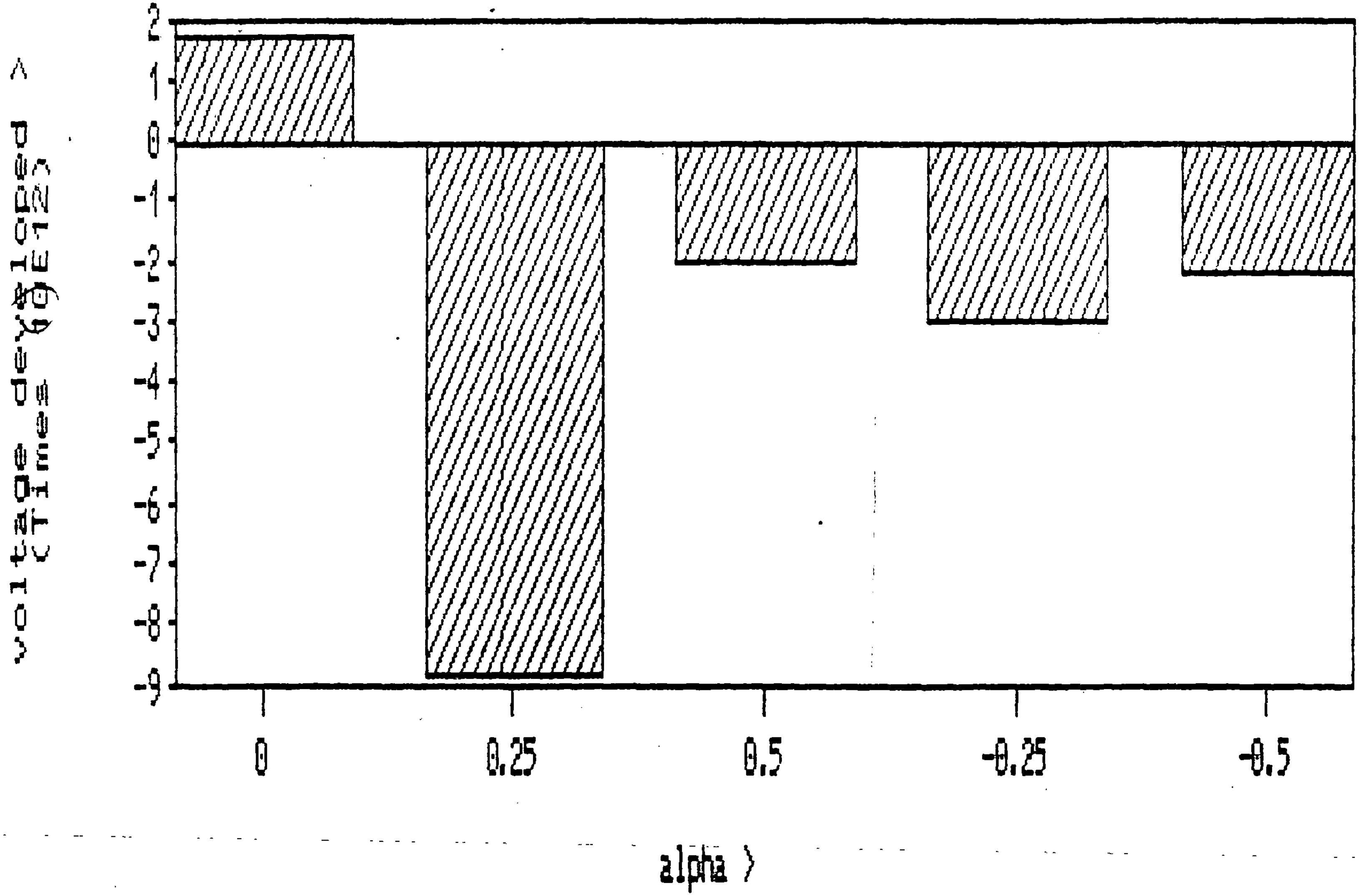

0.1
0.0
−0.1
−0.2
−0.3
−0.4
percentage error in ψ >
2.00
2.05
2.10
2.15
2.20
2.25
2.30
2.35
2.40
2.45
2.50
'r' (independent variable) >

TABLE 5.5.7

COMPARISON OF VOLTAGES DEVELOPED IN HOMOGENEOUS & INHOMOGENEOUS STUDY.

VOLTAGE DEVELOPED IN HOMOG-ENEOUS QUARTZ BAR (V_H)	VOLTAGE DEVELOPED IN INHUMOGENEOUS QUARTZ BAR (V_{NH})	$\frac{V_{NH}}{V_H}$	REMARK
1.701×10^{12} at $\alpha=0$	-8.853×10^{12} at $\alpha = 0.25$	5.204	It is obse observed that the average voltage in homogeneous case is 2.4 times then that of Homogeneous
	$- 2.032 \times 10^{12}$ at $\alpha = 0.50$	1.194	
	$- 2.97 \times 10^{12}$ at $\alpha = -0.25$	1.74	
	$- 2.20 \times 10^{12}$ at $\alpha = -0.50$	1.29	

FIG 5.5.7 Comparision of voltages
in Homogeneous & Inhomogeneous study



materials. The magnitude of the output voltages in Inhomogencous material is about 2.5 times than that of Homogeneous materials.This magnitude can be further increased by the suitable choice of the values of r_1, r_2, c_{11}, c_{12} and α.

Another important aspect of the present analysis is that when the α is varying in parabolic variations (i.e. α=0.25) the magnitude of the voltage is much greater that in other three variations. In some cases it also happens that the polarity of the output voltage is also reversed. Bar chart for variations of voltages are also shown in Fig 5.5.7 for different Inhomogenous and Homogeneous cases under investigation.