CHAPTER II

LARGE DEFLECTION ANALYSES OF SKEW PLATES OF VERIABLE THICKNESS

PAPER I

NON-LINEAR ANALYSIS OF RHOMBIC PLATES OF VARIABLE THICKNESS

ABSTRACT

This paper deals with the non-linear static and dynamic behaviours of a simply-supported rhombic plate (skew plate of aspect ratio 1) of linearly varying thickness. Banerjee's hypothesis has been followed to form a set of decoupled differential equations and then the Galerkin's procedure has been utilised to solve the equations. Various numerical results for a rhombic plate of isotropic material, under both static and dynamic loadings have been computed and compared with the other results known from literature. It is seen that the present approach offers sufficiently accurate results for both movable and immovable edge conditions.

GOVERNING EQUATIONS

Let us consider a rhombic plate of elastic isotropic material having thickness varying linearly, the central thickness being 'h_o' and thickness-variation parameter being 'p'. Let the size of each side of the plate be '2a' which is sufficiently large compared to 'h_o'. The plate is considered to be simply-supported along its edges and its faces respond to the bending and membrane actions.

We now posit a rectangular cartesian co-ordinate system (x,y,z,) at the centre of the plate, (x,y) being in the middle plane and z the thickness direction positive downwards. Also let us set an oblique co-ordinate system (x_1, y_1, θ) at the same origin, (x_1, y_1) being parallel to the sides of the plate, and θ the skew angle of the plate (Vide Fig.3 in paper II Chapter I). Obviously

$$x = x_1 \cos\theta \quad \text{and} \quad y = y_1 + x_1 \sin\theta$$

are the co-ordinate transformation equations.

Now following Banerjee's hypothesis, the differential equations in rectangular cartesian co-ordinate system governing the deflections and vibrations of plates of linearly varying thickness will be

$$\begin{aligned} & h^3 \nabla^4 W + 6h^2\left(\frac{dh}{dx}\right)\nabla^2\left(\frac{\partial W}{\partial x}\right) + 6h\left(\frac{dh}{dx}\right)^2\left(\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\right) \\ & - A_1\left(\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\right) - 6\lambda h\left[\left\{\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right\}\left\{\nabla^2 W\right.\right. \\ & \left. + \frac{1}{h}\left(\frac{dh}{dx}\right)\left(\frac{\partial W}{\partial x}\right)\right\} + 2\left\{\frac{\partial^2 W}{\partial x^2}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{\partial^2 W}{\partial y^2}\left(\frac{\partial W}{\partial y}\right)^2\right. \\ & \left.\left. + 2\left(\frac{\partial W}{\partial x}\right)\left(\frac{\partial W}{\partial y}\right)\left(\frac{\partial^2 W}{\partial x \partial y}\right)\right\}\right] = \frac{q}{16L} \quad . \end{aligned} \tag{1}$$

for non-linear static deflections under uniform loading, where A_1 is a constant given by

$$\frac{A_1}{12h} = \frac{1}{2}\left\{\left(\frac{\partial W}{\partial x}\right)^2 + \nu\left(\frac{\partial W}{\partial y}\right)^2\right\} + \frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} \quad . \tag{2}$$

and

$$h^3 \nabla^4 W + 6h^2\left(\frac{dh}{dx}\right)\nabla^2\left(\frac{\partial W}{\partial x}\right) + 6h\left(\frac{dh}{dx}\right)^2\left(\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\right)$$
$$-A_2\left(\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\right) - 6\lambda h\left[\left\{\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right\}\left\{\nabla^2 W\right.\right.$$
$$\left.+\frac{1}{h}\left(\frac{dh}{dx}\right)\left(\frac{\partial W}{\partial x}\right)\right\} + 2\left\{\frac{\partial^2 W}{\partial x^2}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{\partial^2 W}{\partial y^2}\left(\frac{\partial W}{\partial y}\right)^2\right.$$
$$\left.\left.+2\left(\frac{\partial W}{\partial x}\right)\left(\frac{\partial W}{\partial y}\right)\left(\frac{\partial^2 W}{\partial x \partial y}\right)\right\}\right] + \frac{\rho h}{L}\frac{\partial^2 W}{\partial t^2} = 0 \quad . \qquad (3)$$

for non-linear free elastic vibrations, where A_2 is a time - dependent constant given by

$$\frac{A_2}{12h} = \frac{1}{2}\left\{\left(\frac{\partial W}{\partial x}\right)^2 + \nu\left(\frac{\partial W}{\partial y}\right)^2\right\} + \frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} \quad . \qquad (4)$$

In both the equations (1) and (3)

$$L = \frac{E}{12(1-\nu^2)} \; , \quad D = \frac{Eh_o^3}{12(1-\nu^2)} \; .$$

The thickness variation is expressed by

$$h = h_o(1 + px/a) \; ,$$

where $p < 1$.

ANALYSIS

(A) Non-linear static behaviours of skew plates of variable thickness -

We consider, here, the bending of simply-supported rhombic plate of variable thickness with constrained in-plane displacements at the boundaries. We now transform equation (1) and (2) in oblique co-ordinates by the transformation operators as given in Paper I, Chapter I.

We choose , as usual, the deflection function W in the following form for simply-supported edge conditions

$$W = W_o \cos\frac{\pi X_1}{2a} \cos\frac{\pi Y_1}{2a} \qquad (5)$$

Now integrating the transformed equation (2) over the entire area of the plate we get

$$A_1 = 6h_o p\cos\theta\left\{\frac{\pi^2 W_o^2}{8a^2}\left(1+\nu+2\tan^2\theta\right)\right\}\Big/\log_e\frac{(1+p\cos\theta)}{(1-p\cos\theta)} . \qquad (6)$$

Again introducing (5) and (6) in the transformed form of equation (1) and then applying the Galerkin's procedure, we arrive at the following cubic equation determining the non-dimensional central deflection $\beta = W_o/h_o$ of the simply-supported rhombic plate of variable thickness

$$\left[(1+2\tan^2\theta)\left\{\sec^2\theta+\left(1-\frac{6}{\pi^2}\right)p^2\right\}+\frac{6p^2}{\pi^2}(1-\nu+2\tan^2\theta)\right]\beta$$
$$+\frac{3}{8}\left[\lambda(5+11\tan^2\theta+6\tan^4\theta)\right.$$
$$\left.+2p\cos\theta(1+\nu+2\tan^2\theta)^2\Big/\log_e\frac{(1+p\cos\theta)}{(1-p\cos\theta)}\right]\beta^3 = \frac{4}{\pi^6}Q . \qquad (7)$$

where $Q = qa^4/Dh_o$ is the load function parameter, q being the load per unit area of the plate .

NUMERICAL RESULTS

Table 1 shows different numerical results of the central deflections of a rhombic plate of variable thickness having $\nu = 0.3$; load parameters are taken the same as in Paper I, Chapter I. It is to be noted that the results for thickness variation parameter p = 0 (i.e. for a plate of constant thickness), agree exactly with those in Paper I, Chapter I, both for movable and immovable edge conditions, which have been experimentally verified by the author. The results for other values of p are new. (Note - For movable edge conditions $A_1 = 0$).

TABLE 1

Static Deflections of Rhombic Plates.

		W_o/h_o					
Skew Angle θ	Load Parameter qa^4/Dh_o	Immovable Edge			Movable Edge		
		p = 0.1	p = 0.2	p = 0.3	p = 0.1	p = 0.2	p = 0.3
15^o	111.72	0.3434	0.3374	0.3277	0.3675	0.3595	0.3472
	335.16	0.7682	0.7618	0.7513	0.9775	0.8935	0.9370
	558.60	1.0255	1.0204	1.0121	1.4210	1.4040	1.3765
30^o	111.72	0.2009	0.1974	0.1916	0.2061	0.2022	0.1958
	335.16	0.5094	0.5083	0.4991	0.5840	0.5824	0.5663
	558.60	0.7301	0.7239	0.7153	0.9250	0.9120	0.8900

(B) Non-linear dynamic behaviours of skew plates of variable thickness -

Let us now consider free vibrations of variable thickness rhombic plates. Neglecting in-plane inertia, transforming equation (4) in oblique co-ordinates, choosing

$$W = W_o F(t) \cos \frac{\pi X_1}{2a} \cos \frac{\pi Y_1}{2a} \tag{8}$$

for fundamental mode of vibration and then integrating the transformed equation over the whole domain of the plate we get

$$A_2 = 6h_o p \cos\theta \frac{\pi^2 W_o^2 F^2}{8a^2}(1+\nu+2\tan^2\theta) \Big/ \log_e \frac{(1+p\cos\theta)}{(1-p\cos\theta)} . \tag{9}$$

Here W_o is the initial amplitude of vibration and F(t) is some unspecified function of time. It is to be noted that,we are interested in the normal displacement W only and so the in-plane displacements u and v are eliminated here also by considering suitable expressions for them compatible with the boundary conditions of the plate.

Now transforming equation (3) in oblique co-ordinates, inserting (8) and (9) in the transformed equation and then applying the Galerkin's procedure we get the following differential equation for time function

$$\frac{d^2F}{d\tau^2} + \Big[(1+2\tan^2\theta)\frac{\pi^4}{4}\Big\{\sec^2\theta + \Big(1-\frac{6}{\pi^2}\Big)p^2\Big\} + \frac{3}{2}\pi^2 p^2(1-\nu$$
$$+2\tan^2\theta)\Big]F + \frac{3\pi^4}{32}\Big[\lambda(5+11\tan^2\theta+6\tan^4\theta)$$
$$+2(1+\nu+2\tan^2\theta)^2 \cdot p\cos\theta \Big/ \log_e \frac{(1+p\cos\theta)}{(1-p\cos\theta)}\Big]\beta^2 F^3 = 0 . \tag{10}$$

where $\beta = W_o/h_o$, the non-dimensional amplitude and

$$\tau = (Lh_o^2/\rho a^4)^{\frac{1}{2}} t$$

some time function.

The equation (10) is in the form $\ddot{F} + AF + BF^3 = 0$, the familiar Duffing's Equation. With the initial conditions $F(0) = 1$ and $\dot{F}(0) = 0$, the solution of equation (10) is the well-known elliptic integral $F(t) = C_n(\omega^*, t, k)$. Then the ratio of the non-linear frequency ω^* to the linear frequency ω is given by

$$\frac{\omega^*}{\omega} = \sqrt{1 + B/A}$$

where

$$A = \Big[(1 + 2\tan^2\theta)\frac{\pi^4}{4}\Big\{\sec^2\theta + \Big(1 - \frac{6}{\pi^2}\Big)p^2$$

$$+ \frac{3}{2}\pi^2 p^2 (1 - \nu + 2\tan^2\theta)\Big] .$$

and

$$B = \frac{3\pi^4}{32}\Big[\lambda(5 + 11\tan^2\theta + 6\tan^4\theta)$$

$$+ 2(1 + \nu + 2\tan^2\theta)^2 \cdot p\cos\theta \Big/ \log_e\frac{(1 + p\cos\theta)}{(1 - p\cos\theta)} .$$

NUMERICAL RESULTS

Numerical results of the ratio ω^*/ω are shown in Tables 2 and 3. Table 2 shows the results for a square plate ($\theta = 0^\circ$) compared with those obtainable from ref.174 , after converting the shell equations into plate equations. It is seen that the results agree perfectly. Table 3 shows the results for rhombic plates with skew angles $\theta = 15^\circ$, 22.5° and 30° and thickness variation parameters $p = 0, 0.1, 0.2$ and 0.3. These results are new to the author's sincere belief. Here the results for skew angles higher than 30° are not considered, because for greater values of θ, the effect of non-liearity does not play important role in design. (Note - For movable edge conditions $A_2 = 0$).

TABLE 2

Showing Dynamic Characteristics of Square Plates $\theta = 0^{\circ}$

		ω^*/ω							
Edge condition	W_o/h_o	p = 0		p = 0.1		p = 0.2		p = 0.3	
		Present Method	Sinha-Banerjee Method	Present Method	Sinha-Banerjee Method	Present Method	Sinha-Banerjee Method	Present Method	Sinha-Banerjee Method
Immovable Edge	0.25	1.02477	1.02477	1.02450	1.02447	1.02374	1.02350	1.02253	1.0220
	0.50	1.09573	1.09573	1.09473	1.09520	1.09187	1.09099	1.08734	1.0853
	0.75	1.20474	1.20474	1.20270	1.2025	1.19683	1.19600	1.18451	1.18330
	1.00	1.34257	1.34257	1.33932	1.33890	1.32992	1.32700	1.31500	1.3082
Movable Edge	0.25	1.00526	1.00526	1.00522	1.00526	1.00509	1.00514	1.00490	1.00493
	0.50	1.02088	1.02088	1.02071	1.02088	1.02022	1.02040	1.01946	1.01950
	0.75	1.04640	1.04640	1.04600	1.04640	1.04495	1.04530	1.04327	1.04340
	1.00	1.08110	1.08110	1.08040	1.08110	1.07860	1.07930	1.07572	1.07600

TABLE 3

Showing Dynamic Characteristics of Rhombic Plates, $\theta = 15^{\circ}$, $\theta = 22.5^{\circ}$, and $\theta = 30^{\circ}$

		ω^*/ω							
Skew Angle θ	w_o/h_o	Immovable Edge				Movable Edge			
		p = 0	p = 0.1	p = 0.2	p = 0.3	p = 0	p = 0.1	p = 0.2	p = 0.3
15°	0.25	1.02463	1.02438	1.02365	1.02250	1.00500	1.00496	1.00484	1.00467
	0.50	1.09520	1.09425	1.09152	1.08721	1.01984	1.01968	1.01924	1.01854
	0.75	1.20366	1.20172	1.19612	1.18724	1.04410	1.04377	1.04279	1.04125
	1.00	1.34084	1.33774	1.32880	1.31462	1.07716	1.07658	1.07489	1.07224
22.5°	0.25	1.02454	1.02430	1.02362	1.02254	1.00472	1.0046	1.00439	1.00433
	0.50	1.09486	1.09388	1.09142	1.08737	1.01876	1.01863	1.01823	1.01760
	0.75	1.20297	1.20115	1.19591	1.18758	1.04174	1.04144	1.04057	1.03920
	1.00	1.33974	1.33684	1.32847	1.31511	1.07308	1.07256	1.07106	1.06868
30°	0.25	1.02452	1.02431	1.02370	1.02272	1.00442	1.00439	1.00430	1.00417
	0.50	1.09481	1.09401	1.09700	1.08802	1.01756	1.01745	1.01711	1.01658
	0.75	1.20286	1.20120	1.19650	1.19189	1.03910	1.03885	1.03810	1.03692
	1.00	1.33957	1.33695	1.32940	1.31726	1.06853	1.06810	1.06680	1.06476

OBSERVATIONS

For static behaviour of a rhombic plate of varying thickness, it is observed that

(i) with the increase of skew angle,central deflection decredases for the same loading whether the edge conditions of the plate are movable or immovable,

(ii) for any assumed skew angle,the central deflection is greater for movable edge conditions than that for immovable edge conditions,the load reamining same in both the cases,

(iii) increasing thickness parameter decreases the central delfection.

All the above observations are quite expected from practical point of view.

As regards dynamic behaviour of a variable thickness rhombic plate, the following observations are made:

(i) The frequency ratio decreases with increasing thickness variation parameter irrespective of the edge conditions.

(ii) The frequency ratio increases with the non-dimensional amplitude.

(iii) The frequency ratio gradually decreases with the increase of p, the edge conditions being movable or immovable. This is an expected result, because the thickness is minimum at the centre and maximum at the edge of the tapered plate,

(iv) The frequency ratio decreases with the increase of skew angle for lower values of p. For comparatively higher values of p, the vibration character tends to change in the case of immovable edge conditions. But for movable edge conditions the vibration character does not show such irregularity. This situation demands further investigation.