CHAPTER - II

NON-LINEAR DYNAMIC RESPONSE OF MODERATELY THICK PLATES PLACED ON ELASTIC FOUNDATION

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ABSTRACT

In this chapter the non-linear dynamic response of thick plates of different shapes placed on elastic foundation of the Winkler-type is investigated by using the approximate method offered by Berger. Conformal mapping technique has been utilised in the investigation. The cases of square plates, rounded cornered plates and circular plates have been studied in detail. The ratios of the non-linear time periods including shear deformation and the linear time period of the classical plate theory have been computed for these plates for different values of the foundation modulus $\frac{k_1}{D}a^4$ and discussed.

(a) Vibrations of square plates and square plates with rounded corners.*

Let us consider the free vibrations of thick plates of thickness h. The deflections are considered to be of the same order of magnitude as the thickness of the plate. Berger's equations given by 20(a) and 20(b) in chapter I are rewritten

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in the following form for thick plates placed on elastic foundation of the Winkler-type (Ariman [34])

$$\left[1+\frac{\bar{\alpha}^2 h^2}{10(1-\nu^2)}\cdot\frac{E}{Gc}\right]\nabla^4 w-\bar{\alpha}^2\nabla^2 w-\frac{6}{5}\cdot\frac{\rho}{Gc}\cdot\frac{\partial^2}{\partial t^2}(\nabla^2 w)$$

$$+\frac{12}{h^2 C_p^2}\cdot\frac{\partial^2 w}{\partial t^2}-\frac{h^2}{10}\cdot\frac{k_1}{D}\cdot\frac{2-\nu}{1-\nu}\nabla^2 w+\frac{k_1}{D}w=0 \qquad \text{.... (21a)}$$

where k_1 = foundation modulus

D = flexural rigidity

and the coupling parameter $\bar{\alpha}^2$ is given by

$$\tau^2(t)\frac{\bar{\alpha}^2 h^2}{12}=\frac{\partial u_o}{\partial x}+\frac{\partial v_o}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \text{.... (21b)}$$

To solve the governing equations let us assume the deflection in the following form

$$w(x,y,t)=w_1(x,y)\,\tau(t) \qquad \text{.... (22)}$$

Substituting (22) in 21(a) we get

$$\left[\frac{\nabla^4 w_1}{w_1}-\frac{h^2}{10}\cdot\frac{k_1}{D}\cdot\frac{2-\nu}{1-\nu}\cdot\frac{\nabla^2 w_1}{w_1}+\frac{k_1}{D}\right]\tau(t)$$

$$+\left[\frac{12}{h^2 C_p^2}-\frac{6}{5}\cdot\frac{\rho}{Gc}\frac{\nabla^2 w_1}{w_1}\right]\ddot{\tau}(t)$$

$$+\left[\frac{\bar{\alpha}^2 h^2}{10(1-\nu^2)}\cdot\frac{E}{Gc}\cdot\frac{\nabla^4 w_1}{w_1}-\bar{\alpha}^2\frac{\nabla^2 w_1}{w_1}\right]\tau^3(t)=0 \qquad \text{.... (23)}$$

A solution of equation (23) is possible if

$$\frac{\nabla^4 w_1}{w_1} = \lambda_1^4 \qquad \ldots\ldots 24(a)$$

$$\frac{\nabla^2 w_1}{w_1} = -\lambda_1^2 \qquad \ldots\ldots 24(b)$$

From 24(a) we have

$$(\nabla^2 - \lambda_1^2)(\nabla^2 + \lambda_1^2) w_1 = 0 \qquad \ldots\ldots 25(a)$$

and from 24(b) we have

$$(\nabla^2 + \lambda_1^2) w_1 = 0 \qquad \ldots\ldots 25(b)$$

It is evident that to get a complete solution it is sufficient to solve

$$(\nabla^2 + \lambda_1^2) w_1 = 0 \qquad \ldots\ldots 25(c)$$

In a complex co-ordinate system,

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy$$

The equation 25(c) changes and

let $z = f(\xi)$, $\bar{z} = f(\bar{\xi})$ ....(26)

be the analytic function which maps the given shape in the Z - plane on to a unit circle in the ξ - plane. After transforming equation 25(c) into the complex co-ordinates (z , $\bar{z}$) and using relation (26) we obtain the following differential equation in (ξ , $\bar{\xi}$) co-ordinates for the deflection function w_1

$$\frac{\partial^2 w_1}{\partial \xi \, \partial \bar{\xi}} + \frac{\lambda_1^2}{4} \cdot \frac{dz}{d\xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} \cdot w_1 = 0 \qquad \ldots\ldots 27(a)$$

Similarly equation 21(b) changes into

$$\tau^2(t)\cdot\frac{\bar{\alpha}^2 h^2}{12}\left(\frac{dz}{d\xi}\cdot\frac{d\bar{z}}{d\bar{\xi}}\right)^2 = \frac{\partial u_0}{\partial\xi}\cdot\frac{d\bar{z}}{d\bar{\xi}} + \frac{\partial u_0}{\partial\bar{\xi}}\cdot\frac{dz}{d\xi}$$

$$+\left(\frac{\partial v_0}{\partial\xi}\cdot\frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial v_0}{\partial\bar{\xi}}\cdot\frac{dz}{d\xi}\right)$$

$$+2\frac{\partial w_1}{\partial\xi}\cdot\frac{\partial w_1}{\partial\bar{\xi}}\cdot\frac{dz}{d\xi}\cdot\frac{d\bar{z}}{d\bar{\xi}} \qquad \text{.... 27(b)}$$

Here $\xi = re^{i\theta}$ and $\bar{\xi} = r.e^{-i\theta}$, r being the radius of the circle. For transverse vibrations the inplane displacements u_0 and v_0 are of no interest and they have been eliminated finally through integrations by choosing suitable expressions for the displacements compatible with their boundary conditions i.e. $u_0 = 0$, $v_0 = 0$ on the boundary.

To solve equation 27(a) let us choose the deflection function $W_1(\xi, \bar{\xi})$ in the following form

$$W_1 = A_0\left[1-\xi\bar{\xi}\right]\left[1-\frac{1}{3}\xi\bar{\xi}+\frac{1}{2}\left(\xi^2+\bar{\xi}^2\right)\left(1-\xi\bar{\xi}\right)^2\right] \qquad \text{.... (28)}$$

Clearly W_1 is θ dependent and satisfies the simply supported edge conditions, namely,

$$W_1 = 0 \quad \text{at} \quad r = 1$$

$$\frac{\partial^2 W_1}{\partial\xi\,\partial\bar{\xi}} = 0 \quad \text{at} \quad r = 1$$

Substituting (28) in 27(a) and inserting the values of $\frac{dz}{d\xi}$, $\frac{d\bar{z}}{d\bar{\xi}}$ from the given mapping functions $z = f(\xi)$ we get the error function $\epsilon(\xi, \bar{\xi})$. Galerkin's technique requires that

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \epsilon(\xi, \bar{\xi}) \, w_1(\xi, \bar{\xi}) \, r \, dr \, d\theta = 0 \qquad \ldots. \ (29)$$

The values of $\bar{\alpha}^2$ have been determined by substituting (28) in 27(b), remembering the values of $\frac{dz}{d\xi}$, $\frac{d\bar{z}}{d\bar{\xi}}$ from the given mapping functions and finally integrating over the area of the plate. After evaluating the integrals in (29) we get the necessary values of λ_1^2. Thus the values of $\frac{\nabla^4 w_1}{w_1}$, and $\frac{\nabla^2 w_1}{w_1}$ are determined. Inserting all these values in (23) we get the following cubic equation determining the unknown time function $\tau(t)$ in the form

$$\ddot{\tau}(t) + \alpha_1 \tau(t) + \beta_1 \tau^3(t) = 0 \qquad \ldots. \ (30)$$

The solution of the above equation subject to the boundary conditions

$$\tau(0) = 1$$

$$\dot{\tau}(0) = 0$$

is well-known and is obtained in terms of Jacobi's elliptic function.

The ratio of the non-linear time periods to the linear time periods of the classical plate (thin plate) is

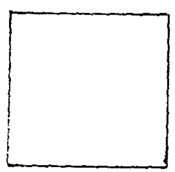
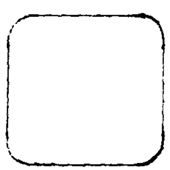
$$\frac{T^*}{T} = \frac{2K/\pi}{\left[1 + \frac{\beta_1}{\alpha_1} \bar{\beta}^2\right]^{1/2}} \qquad \ldots. \ (31)$$

where $\bar{\beta} = \frac{A_c}{h}$ is the ratio of the static deflection to the thickness of the plate.

Table I shows different values of $\frac{T^*}{T}$ Vs. $\frac{k_1}{D} a^4$ for different $\bar{\beta}$ for the simply supported square and rounded cornered square plates.

Table - 1

Ratio of the non-linear to linear periods for the free vibrations of simply supported square plates and square plates with rounded corners. Immovable edge conditions have been considered.

PLATE SHAPE	MAPPING FUNCTION	$\bar{\beta} = \frac{A_0}{h}$	$\frac{T^*}{T}$ FOR $\nu = 0.3$, $\frac{h}{a} = 0.2$, $\frac{E}{G_c} = 2.5$.				
			WITHOUT ELASTIC FOUNDATION $k_1 . \frac{a^4}{D} = 0$		WITH ELASTIC FOUNDATION.		
			PRESENT STUDY	REF.[38]	$k . \frac{a^4}{D} = 10$	$k . \frac{a^4}{D} = 15$	$k . \frac{a^4}{D} = 20$
SIMPLY SUPPORTED SQUARE PLATE OF SIDE 2a 	$Z = 1.08a\xi - 0.11a\xi^5$	0.2	0.9772	1.0037	0.9836	0.9856	0.9872
		0.4	0.9175	0.9418	0.9391	0.9461	0.9517
		0.6	0.8353	0.8606	0.8768	0.8878	0.9003
		0.8	0.7564	0.7758	0.8076	0.8258	0.8408
		1.0	0.6793	0.6976	0.7387	0.7608	0.7793
SIMPLY SUPPORTED SQUARE PLATE WITH ROUNDED CORNERS. 	$Z = \frac{25}{48}a\xi - \frac{1}{48}a\xi^5$	0.2	0.9704	–	0.9711	0.9714	0.9718
		0.4	0.8957	–	0.8981	0.8990	0.9000
		0.6	0.8025	–	0.8060	0.8077	0.8094
		0.8	0.7104	–	0.7151	0.7169	0.7190
		1.0	0.6283	–	0.6330	0.6353	0.6376

$\frac{T^*}{T}$ VS $\bar{\beta}$ FOR

SIMPLY SUPPORTED SQUARE PLATE WITH SIDE $2a$.
AND
ROUNDED CORNERED SQUARE PLATE.

(IMMOVABLE EDGES)

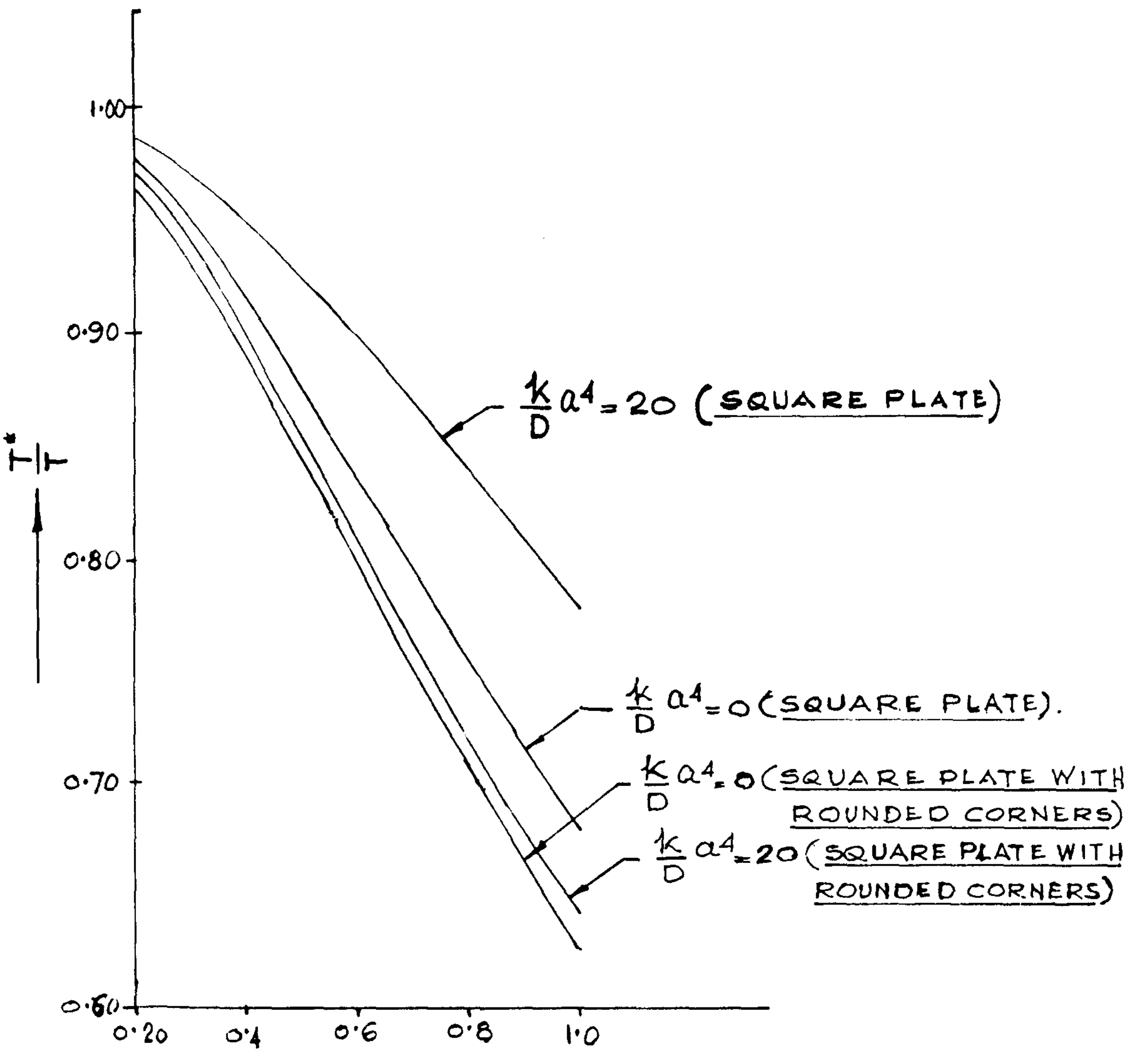


FIG. - 1.

(b) Vibrations of circular plates.

Let us consider the free vibrations of thick circular plates of radius a. The deflections are considered to be of the same order of magnitude as the plate thickness.

For circular plate of radius a the mapping function is $z = a\xi$

Thus $$\frac{dz}{d\xi} = \frac{d\bar{z}}{d\bar{\xi}} = a = \text{constant.}$$

On this assumption the equations 25(a) and 25(b) namely

$$(\nabla^2 - \lambda_1^2)(\nabla^2 + \lambda_1^2) \cdot w_1 = 0 \qquad \ldots.\ 32(a)$$

$$(\nabla^2 + \lambda_1^2)\, w_1 = 0 \qquad \ldots.\ 32(b)$$

offer an interesting closed form solution. Changing equation 32(a) into complex co-ordinates as shown in the previous case we have the following two differential equations

$$\left.\begin{aligned} \frac{\partial^2 w_1'}{\partial\xi\,\partial\bar{\xi}} + \frac{\lambda_1^2}{4} a^2 w_1' &= 0 \\ \frac{\partial^2 w_1''}{\partial\xi\,\partial\bar{\xi}} - \frac{\lambda_1^2}{4} a^2 w_1'' &= 0 \end{aligned}\right\} \qquad \ldots.\ 32(c)$$

where $w_1 = w_1' + w_1''$

The equation determining $\bar{\alpha}^2$ given by the equation 27(b) remains same as in the previous case. The solution of the differential equations 32(c) is obtained in terms of Bessel function in the form

$$W_1 = W_1' + W_1''$$

$$= A J_0 \left(2P\sqrt{\xi\bar{\xi}}\right) + B I_0 \left(2P\sqrt{\xi\bar{\xi}}\right)$$

$$= A J_0(2Pr) + B I_0(2Pr) \qquad \ldots. \ (33)$$

where $p = \dfrac{a\lambda_1}{2}$

For clamped edge boundary conditions

$$W_1 = 0 \quad \text{at} \quad r = 1$$

$$\frac{\partial W_1}{\partial r} = 0 \quad \text{at} \quad r = 1.$$

Thus the frequency equation is obtained in the form

$$\begin{vmatrix} J_0(2p) & I_0(2p) \\ -J_1(2p) & I_1(2p) \end{vmatrix} = 0 \qquad \ldots. \ (34)$$

Solving (34) the value of $2p = 3{\cdot}20$ is obtained from the table [54].

For simply supported edge conditions

$$W_1 = 0 \quad \text{at} \quad r = 1$$

$$\frac{\partial^2 W_1}{\partial r^2} + \frac{\nu}{r}\cdot\frac{\partial W_1}{\partial r} = 0 \quad \text{at} \quad r = 1$$

Thus the frequency equation in this case is

$$\begin{vmatrix} J_0(2p) & I_0(2p) \\ (1-\nu)J_1(2p) - 2pJ_0(2p) & -(1-\nu)I_1(2p) + 2pI_0(2p) \end{vmatrix} = 0 \qquad \ldots. \ (35)$$

Solving (35) we get $2p = 2{\cdot}22$ from table [54].

The values of $\bar{\alpha}^2$ have been determined as in the previous case by putting (33) in 27(b) and integrating over the entire area of the plate. Now knowing the values of $p = a\lambda_{1/2}$, the values of $\frac{\nabla^4 W_1}{W_1}$ and $\frac{\nabla^2 W_1}{W_1}$ are determined in each case. Inserting all these values in equation (23) we get a similar type of cubic equation as equation (30) for determining the time function $T(t)$. The solution of this cubic equation is obtained in terms of Jacobi's elliptic function as in the previous case.

The following table shows $\frac{T^*}{T}$ Vs $\frac{k_1}{D}a^4$ for different values of $\bar{\beta}$

Table - 2.

Ratio of non-linear to linear periods for the vibrations of circular plates.

$$\left[\frac{h}{a} = 0{\cdot}2\ ,\ \nu = 0{\cdot}3\right]$$

PLATE SHAPE WITH EDGE CONDITION	$\bar{\beta} = \frac{A_0}{h}$	$\frac{T^*}{T}$ WITHOUT ELASTIC FOUNDATION ($K_1=0$): PRESENT STUDY	REF. [52]	REF. [40]	WITH ELASTIC FOUNDATION: $\frac{k_1}{D}.a^4=10$	$\frac{k_1}{D}.a^4=15$	$\frac{k_1}{D}.a^4=20$
CLAMPED CIRCULAR PLATE $\frac{E}{G_c} = 8{\cdot}1971$	0·2	0·9926	0·9821	0·9921	0·9760	0·9771	0·9780
	0·6	0·9395	0·9138	0·9366	0·8403	0·8492	0·8572
	1·0	0·8551	0·8029	0·8533	0·7032	0·7238	0·7416

PLATE SHAPE WITH EDGE CONDITION	$\bar{\beta} = \frac{A_0}{h}$	PRESENT STUDY	REF. [40]	$\frac{k_1}{D}.a^4=10$	$\frac{k_1}{D}.a^4=15$	$\frac{k_1}{D}.a^4=20$
SIMPLY SUPPORTED CIRCULAR PLATE $\frac{E}{G_c} = 2{\cdot}9374$	0·2	0·9755	0·9745	0·9828	0·9850	0·9867
	0·6	0·8294	0·8265	0·8721	0·8861	0·8972
	1·0	0·6657	0·6688	0·7304	0·7542	0·7737

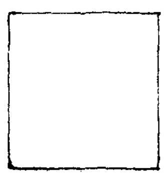
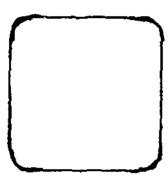
Observations :

The tables 1 and 2 clearly show the response of the foundation modulus $k_1 \frac{a^4}{D}$ on the ratio of the time periods $\frac{T^*}{T}$ for different $\bar{\beta}$. As $k_1 \frac{a^4}{D}$ increases $\frac{T^*}{T}$ increases. This is quite expected from the practical point of view. When $k_1 \frac{a^4}{D} = 0$ i.e. when there is no elastic foundation, the results obtained in the present study for square plates and for circular plates with simply supported as well as clamped edges are found to be in very good agreement with other known results. The results of the rounded cornered square plates are completely new.

(C) <u>A comparative study on the time period ratio $\frac{T^*}{T}$ Vs. $k_1 \frac{a^4}{D}$ for different $\bar{\beta}$ of the simply supported square plates and square plates with rounded corners on the basis of the area of the plates :</u>

Table - 3.

$$\left[\ \bar{\beta} = 1,\ \nu = 0{\cdot}3,\ \frac{E}{G_c} = 2{\cdot}5,\ \frac{h}{a} = 0{\cdot}2\ \right]$$

PLATE SHAPE [SIMPLY SUPPORTED EDGE CONDITION]	$\frac{T^*}{T}$	
	WITHOUT ELASTIC FOUNDATION. $(K_1 = 0)$	WITH ELASTIC FOUNDATION, $k_1 \frac{a^4}{D} = 10$
SQUARE PLATE OF SIDE $2a$. 	0·6793	0·7387
SQUARE PLATE WITH ROUNDED CORNERS OF SIDE $2a$ 	0·6283	0·6330

Observation :

An interesting observation on Table - 3 is variation of the time period ratio $\frac{T^*}{T}$ with the area of the respective plates. It is observed that as the area of the plate increases this ratio increases. This is true irrespective of the response of the foundation modulus. Obviously, the area of the square plate is greater than that of the rounded cornered square plate. $\frac{T^*}{T}$ increases accordingly. This is noteworthy from the practical point of view.

(d) A useful observation on Berger's equations :

Let us now examine Berger's equations on the movable edge conditions.

A clamped circular plate of radius a is considered here. The deflection function for the circular plate is assumed as

$$W_1 = A_0 \tau(t)\left[1 - \frac{r^2}{a^2}\right]^2 \qquad \ldots . \quad (36)$$

This clearly satisfies the required boundary conditions of the clamped edges.

Before recalling our attention to the original equation (23) let us turn to the equation 21(b) which reduces (in the present case) to the following form.

$$\tau^2(t)\frac{\bar{\alpha}^2 h^2}{12} = \frac{du_0}{dr} + \frac{u_0}{r} + \frac{1}{2}\left(\frac{dw_1}{dr}\right)^2 \qquad \ldots . \quad (37)$$

$$\text{Let } u_0 = u_1(r)\,\tau^2(t) \qquad \ldots . \quad (37a)$$

Putting (36) in (37) multiplying the equation by r we obtain the following equation after integration

$$u_1 r + c = \frac{\bar{\alpha}^2 h^2}{24} \cdot r^2 - \frac{8A_0^2}{a^4}\left[\frac{r^8}{8a^4} + \frac{r^4}{4} - \frac{1}{3}\cdot\frac{r^6}{a^2}\right] \qquad \ldots . \quad 38(a)$$

At $r = 0$, $u_1 = 0$. Therefore $c = 0$

Hence u_1 is obtained as

$$u_1 = \frac{\bar{\alpha}^2 h^2}{24} \cdot r - \frac{8A_0^2}{a^4}\left[\frac{r^3}{4} - \frac{r^5}{3a^2} + \frac{r^7}{8a^4}\right] \qquad \ldots . \quad 38(b)$$

For movable edge condition at $r = a$

$$\frac{du_1}{dr} + \nu \frac{u_1}{r} + \frac{1}{2}\left(\frac{dw_1}{dr}\right)^2 = 0$$

Using this condition we obtain the value for $\bar{\alpha}^2$ as

$$\bar{\alpha}^2 = \frac{8A_0^2}{h^2a^2} \cdot \frac{\nu-1}{\nu+1} \qquad \text{.... (39)}$$

Inserting now (36) in equation (23), remembering $\bar{\alpha}^2$ thus evaluated from (39) and applying Galerkin's technique we get as usual a cubic equation determining the time function $\tau(t)$ in the form

$$\ddot{\tau}(t) + \alpha_1 \tau(t) - \beta_1 \tau^3(t) = 0 \qquad \text{.... (40)}$$

Which leads to meaningless results because the coefficient of $\tau^3(t)$ is negative.

Thus we arrive at the following conclusion :

Although Berger's equation can be conveniently applied to the nonlinear theory of thick plates, its application is limited to the immovable edge conditions only.