NON-LINEAR ANALYSIS OF MODERATELY THICK PLATES—A NEW APPROACH

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P R E F A C E

Structural members commonly known as plates are used in machine parts, in aircraft design and also in modern structural design. The study of bending properties of such members is imperative to a design engineer. The bending properties of a plate depend greatly on its thickness as compared with its other dimensions. To study these properties we shall have to distinguish between two kinds of plates - (A) Thin plates and (B) Thick plates.

A. Thin plates :

Structural members whose one dimension is small in comparison with other two dimensions are commonly known as thin plates Within the elastic limit, the static, the thermal and the dynamic behaviours/responses of thin plates are influenced by the following factors :

(1) Material properties defined by Young's modulus E and Poisson's ratio ν. E and ν may be variable.

(2) Geometry of plate —

Geometry may be simple such as circular or complicated. Thickness of the plate may be variable.

(3) Types of loading and

(4) Nature of supports i.e. edge conditions.

It is well-known that if deflections w of a thin plate are small in comparison with its thickness h , a very satisfactory approximate theory of bending of the plate under lateral loads can be developed by making the following assumptions :

(a) There is no deformation in the middle plane of the plate and this plane remains neutral during bending.

(b) Points initially lying on a normal to the middle plane of the plate remain on the normal to the middle surface of the plate after bending and

(c) The normal stresses in the direction transverse to the plate can be disregarded.

The above assumptions constitute the simplest and most widely used classical small deflection theory developed by Lagrange [1].

The first assumption is completely satisfied only if a plate is bent into a developable surface. In other cases bending of a plate is accompanied by strain in the middle plane, but calculations show that the corresponding stresses in the middle plane are negligible if the deflections of the plate are small in comparison with its thickness. If the deflections are not small, these supplementary stresses must be taken into account in deriving the differential equations governing the deflections of the plates. In this way, we obtain non-linear equations and the solution of the problem becomes much more complicated.

With the advent of modern plate and shell constructions subjected to severe operational conditions, the classical linear theory for small deflections is no longer applicable in many cases. Methods of analysis dealing with large deflections, therefore, are of increasingly practical importance. It is well-known that the classical plate equations for studying the nonlinear behaviours of thin plates are due to Von Karman [2]. Von Karman's equations are in the coupled form and hence difficult to solve. Different numerical methods have been offered by several authors to solve them. Outstanding research workers who worked on Von Karman's equations are Chu and Herrman [3], Yamaki [4], Nowiniski [5] and Baur [6]. Other note worthy works in this field are due to Dutta [7] and Chowdhury [8], [9].

Berger [10] offered a simplified approach to study the nonlinear behaviours of thin plates. According to Berger's hypothesis the elastic energy due to the second invariant of the membrane strain may be disregarded as compared to the square of the first invariant without appreciably impairing the accuracy of the results. The Euler-Lagrange equations so derived from the variational equations turn out to be much simpler than those of Von Karman. Hence, this method gains popularity due to its simplicity, but its application is limited to the case of immovable edge conditions only [11]. Successful research workers who carried out useful investigations on this method are Nash and Modeer [12], Wah [13], Nowiniski [14], Banerjee [15]. Other interesting works on Berger's equations are due to Kamaiya [16], Karmakar [17] who carried out their investigations on sandwich plates. Later Banerjee [18] offered a modified strain-energy expression for the investigation of the nonlinear behaviours of thin plates. Banerjee's hypothesis is based on introducing directly the expression for the membrane stress into the total potential energy of the system. As a consequence, a new set of differential equations has been obtained in an uncoupled form. This hypothesis states that the radial stretching is proportional to $\left(\frac{dw}{dr}\right)^2$. This is reasonable as because the contribution of the term $\left(\frac{dw}{dr}\right)^2$ in the expression for the radial term is greater than that of $\frac{du}{dr}$ in bending. The author has carried out investigations on the nonlinear analysis of different elastic plates [19], [20] and obtained satisfactory results. Later Banerjee with Sinha Roy extended his line of thought to the large deflection of shallow shells [21] and obtained excellent results.

Another useful method to carry out the non-linear behaviours of thin plates is the finite element method. Eminent research workers in this field are Striz, Jang and Bert [22] and Chi-lung Huang [23].

B. Thick plates :

The approximate theories of thin plates discussed above, become unreliable in the case of plates of considerable thickness. In such a case, the thick plate theory should be applied. This theory considers the problem as a three dimensional problem of elasticity where the effects of transverse shear deformations and rotatory inertia are to be considered.

In recent years, a number of plate theories has been developed in an effort to extend the range of applicability of classical plate theory to that of thicker plates by including the effects of transverse shear deformation and transverse normal stress. It has been shown by Reissner [24], [25] that the inclusion of transverse shear deformation permits a return to Navier's three dimensional boundary conditions. Later on, Reissner [26] proposed a variational principle for the development of both the governing equations and the boundary conditions. Donnell [27] has given a three dimensional solution in the form of an infinite series in the loading functions for plates. Fredrick [28] investigated stresses on thick plates on elastic foundation. Donnell and Lee [29] have studied the problem of thick plates under tangential loads applied on the faces. Rectangular plates under different edge conditions have been studied in detail by many authors among which the works of Salerno and Goldberg [30], Volterra [31], Essenburg [32] and Volterra [33] need special mention. All these authors used either

Reissner's theory in their investigation or equations very simil to those obtained by Reissner, Starting with the assumptions concerning the components of displacements. Ariman [34] quite successfully investigated stresses of thick plates on elastic foundation. Lee [35] has given a three dimensional solution fo simply supported thick rectangular plates by applying the method followed by Donnell. Goldenviezer [36] has given an approxima theory of bending of a plate by the method of asymptotic integration of the governing equations. A three dimensional elasticity solution for rectangular plates has been developed by Srinivas [37]. This paper is also interesting.

The study of the nonlinear behaviours of moderately thick plates is gaining momentum day by day due to its wide application in modern structure and design. An attractive work in this field is due to Wu and Vinson [38]. The authors have used an improve Reissner's variational theorem along with Berger's hypothesis to propose a set of governing equations including the effects of transverse shear deformation and rotatory inertia for large amplitude free vibrations of plates composed of transversely isotropic material. Another important work is due to Iyenger, Chandrashekhara and Sebastian [39] who carried out the analysis of thick rectangular plates by using a higher order theory which is an extension of Reissner's shear deformation theory. Kanaka Raju and Venkateswara Rao [40] have studied the axisymmetric vibrations of circular plates including the effects of geometric non-linearity, shear deformation and rotatory inertia by employing the finite element method to obtain their solution. Another paper can be located by Kanaka Raju, [41] where the nonlinear vibrations of beams considering shear deformation and rotatory inertia have been studied in detail. Stresses in a thick plate with a circular hole under axisymmetric loading have been quite successfully investigated by Chandrashekhara and Muthanna [42]. The authors have obtained an exact theoretical solution in terms of Fourier -Bessel series and integrals. Kanaka Raju, Venkateswara Rao and I. S. Raju [43] further studied the geometric nonlinearity on the free flexural vibrations of moderately thick rectangular plates. The authors employed finite element formulation to obtain the non-linear to linear period ratios for rectangular plates. A conformal finite element of rectangular shape, wherein the effects of shear deformation and rotatory inertia are included is developed and used for the analysis. Another paper by Kanaka Raju and Hinton, [44] needs special mention in which they quite satisfactorily analysed the non-linear vibrations of thick plates of different shapes having different boundary conditions by using Mindlin plate elements.

A discussion on various non-linear theories applicable for moderately thick plates can be found in papers by Sathyamoorthy and Chia [45] and Sathyamoorthy [46] where it has been shown that the effects of transverse shear and rotatory inertia play a significant role in the large amplitude vibrations of moderatel thick plates of various geometries. Reddy and Chao [47] have studied the finite element analysis of the equations governing the large amplitude free, flexural oscillations of laminated anisotropic rectangular plates.

Very recently Reissner [48] has generalised some formulas of the theory of moderately thick plates. The author restates formulas for stresses and stress couples for a theory of isotropic moderately thick plates (in the classical tests of Love and of Timoshenko) in a simplified form. Fuh-Gwo Yuan and Miller [49] have presented the development of a straight forward displacement type rectangular finite element for bending a flat plate with the inclusion of transverse (or lateral) shear effects. A simple higher order non-linear shear deformation plate theory has been proposed by Lee, Senthilnathan, Lim and Chow [50]. The Von Karman extension of the theory is found to be remarkably simple for obtaining the approximate solution for the non-linear bending and vibration of thick, isotropic and transversely isotropic plates.

To sum up

(i) Thick plate theory is an extension of the classical thin plate theory, where the effects of transverse shear deformation and rotatory inertia are to be included.

(ii) The analytical works so far carried out for investigation of the non-linear behaviours of thick plates are based mainly on single mode approximation and have often been done with the aid of either Von Karman type non-linear equations or Berger type approximation, along with Reissner's variational principle.

(iii) Finite element formulation has recently been used by some authors.

It is to be noted that Berger's equation is a purely approximate method. It is meaningful only for immovable edge conditions. Von Karman equations are in the coupled form and thus difficult to solve, whereas finite element method needs much computational labour and lacks in the essence of formulation of the classical plate equations.

Aim of the present project :

The aim of the present thesis is to offer a simplified approach for the non-linear analysis of thick plates by using Reissner's variational theorem along with Banerjee's hypothesis. A set of uncoupled differential equations have been formed to study the non-linear behaviours of different elastic plates showing the effects of shear deformation and rotatory inertia. Accuracy of the results obtained from these equations has been tested for different plates and compared with other known results. The present study seems to be more advantageous than the previous investigations, because,

(a) The results can be obtained from a single differential equation both for movable as well as immovable edge conditions.

(b) The results are sufficiently accurate from the practical point of view.

(c) The proposed differential equations are in the uncoupled form and hence easy to solve. Computational labour is minimum for its simple form.

The thesis has been divided into three chapters. The first chapter is devoted to deducing the proposed differential equations governing the vibrations of thick plates with shear deformation and rotatory inertia effect. Banerjee's hypothesis suggesting a modified strain energy expression along with Reissner's variational principle has been utilised.

The second chapter deals with the application of Berger's equation on thick plate theory. Non-linear responses of thick plates having different shapes placed on elastic foundation have been studied in detail. Numerical results showing the effects of shear modulus and rotatory inertia for different values of the foundation modulus have been given in tables and compared with other known results. The study shows that Berger's approximate theory can be conveniently applied to the analysis of the thick plates. But it has been shown that for movable edge conditions Berger's theory fails miserably.

The third and the concluding chapter is devoted to the application of the new set of differential equations proposed in the present thesis. The non-linear dynamic behaviours of thick plates of square, circular and regular polygonal shapes have been studied in detail. The static behaviours of elliptical and right angled isosceles triangular plates have also been studied. Different edge conditions have been considered. For regular polygonal plates conformal mapping technique has been employed. Numerical results showing the ratio of the non-linear time periods to linear time periods for different values of transverse shear deformations have been plotted graphically in few cases and given in tabular form for other cases. It has been observed that the results obtained from the present study are in very good agreement with other known results. So, the proposed differential equations of the present project, showing the effects of shear deformation and rotatory inertia, seem to predict the non-linear behaviours of different thick elastic plates of both movable as well as immovable edges, with ease and accuracy.

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Glossary of Symbols :

Following symbols have been used in this thesis :

E	=	Young's modulus.
ν	=	Poisson's ratio.
w	=	Deflection normal to the middle plane of the plate.
h	=	Thickness of the plate.
G, G_c	=	Shear module.
ρ	=	Mass density.
λ	=	Material constant.
A_o	=	Amplitude of oscillations.
D	=	$\frac{Eh^3}{12(1-\nu^2)}$ = Flexural rigidity.
F_i	=	Body force components.
T_i	=	Surface force components.
u, v, w	=	Displacements in x, y and z - directions respectively.
$\tau(t)$	=	Time dependent function.
T	=	Kinetic energy.
$W(\sigma_{ij})$	=	Strain-energy.

M_x, M_y, M_{xy}	=	Stress couples.
N_x, N_y, N_{xy}	=	In-plane stress resultants.
Q_x, Q_y	=	Transverse shear resultants.
α, β	=	Rotational displacements in x and y - directions respectively.
ϵ_{ij}, σ_{ij}	=	Strains and stresses respectively.
C_p	=	$\left[\frac{E}{\rho(1-\nu^2)}\right]^{1/2}$ = speed of wave propagation along the surface of the plate.
$\bar{\alpha}$	=	coupling parameter.
$\bar{\beta}$	=	Dimensionless amplitude.
k	=	Tracing constant.
T^*, T	=	Non-linear and linear time period of oscillation.
q_0	=	Load function.
a	=	Dimension of a plate.
a, b	=	Semi-major and semi-minor axes of the elliptic plate.

CHAPTER - I.

A NEW APPROACH TO THE NON-LINEAR ANALYSIS OF MODERATELY THICK ELASTIC PLATES.

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A NEW APPROACH TO THE NON-LINEAR ANALYSIS OF MODERATELY THICK ELASTIC PLATES

ABSTRACT

In this chapter a set of uncoupled differential equations in cartesian as well as in polar co-ordinates have been formulated to study the non-linear behaviours of thick plates showing the effects of shear deformation and rotatory inertia. Banerjee's hypothesis [18] along with Reissner's variational theorem [26] has been utilised in the formulation.

FORMULATION OF THE DIFFERENTIAL EQUATIONS

We consider the free vibration of thick elastic plates of thickness h. The material is transversely isotropic (such as pyrolytic graphite, for example). The origin of the co-ordinates is located at the centre of the plate. The deflections are considered to be of the same order of magnitude as the plate thickness.

So far as Reissner's variational theorem is applied, the stresses are taken in the form [26]

$$(\sigma_x, \sigma_y, \sigma_z) = \frac{1}{h}(N_x, N_y, N_{xy}) + \frac{12z}{h^3}(M_x, M_y, M_{xy}) \qquad \ldots. \ (1)$$

$$(\sigma_{xz}, \sigma_{yz}) = \frac{3}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right](Q_x, Q_y) \qquad \ldots. \ (2)$$

$$\sigma_z = 0 \qquad \ldots. \ (3)$$

Note that the expressions of σ_{xz}, σ_{yz} are assumed to be the same form as those for small deflection case. Since free vibrations are concerned, σ_z is assumed equal to zero. The membrane stresses N_x/h, N_y/h and N_{xy}/h, involved in the expressions of σ_x, σ_y and σ_{xy} respectively, which are neglected in the linear theory as outlined in [26], can no longer be disregarded in the analysis of large deflection problems. The foregoing equations also satisfy all the stress boundary conditions.

In order to account for transverse shear deformation and rotatory inertia effects in the plate theory, where the lateral deflection is comparable with the thickness, the displacement components are assumed to be of the following expressions [38]

$$u(x, y, z, t) = u_o(x, y, t) + z\alpha(x, y, t) \qquad \ldots. \ (4)$$

$$v(x, y, z, t) = v_o(x, y, t) + z\beta(x, y, t) \qquad \ldots. \ (5)$$

$$w(x, y, z, t) = w(x, y, t) \qquad \ldots. \ (6)$$

The subscript O is used to associate with the middle surface. It should be noted that the relations involve the combined action of bending and stretching which characterizes the behaviour of the problem. However, the thickness is assumed to be unchanged during the deformation procedure, and the elements normal to the middle surface before deformation are not required to remain perpendicular to the deformed middle plane.

In view of equations (4) - (6), the strain-displacement relations for large deflection of plates are of the form :

$$\begin{aligned}
\epsilon_x &= \frac{\partial u_o}{\partial x} + z\frac{\partial \alpha}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\epsilon_y &= \frac{\partial u_o}{\partial y} + z\frac{\partial \beta}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + z\frac{\partial \alpha}{\partial y} + z\frac{\partial \beta}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \\
\epsilon_{yz} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \beta\right) \\
\epsilon_{xz} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \alpha\right) \\
\epsilon_z &= 0
\end{aligned} \qquad \ldots. \ (7)$$

The membrane stress resultants in terms of strains are given by :

$$N_x = \frac{Eh}{1-\nu^2}\left(\epsilon_{x_o} + \nu \epsilon_{y_o}\right)$$

$$N_y = \frac{Eh}{1-\nu^2}\left(\epsilon_{y_o} + \nu \epsilon_{x_o}\right)$$

$$N_{xy} = \frac{Eh}{1+\nu}\epsilon_{x_o y_o} \qquad \text{.... (8)}$$

where ϵ_{x_o} , ϵ_{y_o} are the normal strains of the middle surface in the x and y-directions, respectively; $\epsilon_{x_o y_o}$ is the middle surface shearing strain. From (7) it is seen that

$$\epsilon_{x_o} = \frac{\partial u_o}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_{y_o} = \frac{\partial v_o}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\epsilon_{x_o y_o} = \frac{1}{2}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \qquad \text{.... (9)}$$

Recalling that the normal stress in the transverse direction is assumed to be zero in equation (3), the strain-stress relations for a transverse isotropic material, such as a pyrolytic graphite material, are found to be [38]

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right); \quad \epsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right);$$

$$\epsilon_z = 0$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy}; \quad \epsilon_{yz} = \frac{1}{2G_c}\sigma_{yz}$$

$$\epsilon_{xz} = \frac{1}{2G_c}\sigma_{xz} \qquad \text{.... (10)}$$

The Reissner's functional as outlined in [26] is of the form

$$\Psi = \int_V [\sigma_{ij}\epsilon_{ij} - W(\sigma_{ij})]dv - \int_V F_i u_i dv - \int_{s_i} T_i u_i ds \qquad \ldots. (11)$$

It is to be noted that the last two integrations, concerned with body forces and surface tractions, are eliminated in this problem. Now, the Reissner's functional Ψ becomes a strain-energy expression as the first term on the right-hand side of equation (11).

The substitution of equations (7) and (10) into equation (11) gives

$$\begin{aligned}
\Psi = \iiint_{-h/2}^{+h/2} \Bigg\{ & \sigma_x \left[\frac{\partial u_o}{\partial x} + z\frac{\partial \alpha}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \\
& + \sigma_y \left[\frac{\partial v_o}{\partial y} + z\frac{\partial \beta}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] \\
& + \sigma_{xy}\left[\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + z\left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right) + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] \\
& + \sigma_{yz}\left(\frac{\partial w}{\partial y} + \beta\right) + \sigma_{xz}\left(\frac{\partial w}{\partial x} + \alpha\right) \\
& - \frac{1}{2}\Bigg[\frac{1}{E}\left(\sigma_x^2 + \sigma_y^2\right) - \frac{2\nu}{E}\sigma_x\sigma_y + \frac{2(1+\nu)}{E}\sigma_{xy}^2 \\
& + \frac{1}{G_c}\left(\sigma_{xz}^2 + \sigma_{yz}^2\right)\Bigg]\Bigg\}\, dx\, dy\, dz \qquad \ldots. (12)
\end{aligned}$$

Integrating equation (12) with respect to z and using equations (1), (2), (8) and (9) in the preceding, we obtain [38]

$$\Psi = \iint \Big\{ \frac{Eh}{2(1-\nu^2)} \left[\bar{I}_e^2 - 2(1-\nu)\,\bar{II}_e \right] + M_x \frac{\partial \alpha}{\partial x} + M_y \frac{\partial \beta}{\partial y} + M_{xy}\left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \right) + Q_x\left(\frac{\partial w}{\partial x} + \alpha \right) + Q_y\left(\frac{\partial w}{\partial y} + \beta \right) - \frac{1}{2E}\left[\frac{12}{h^3}\left(M_x^2 + M_y^2 \right) - \frac{24\, M_x M_y \nu}{h^3} \right] - \frac{3}{5G_c h}\left(Q_x^2 + Q_y^2 \right) \Big\}\, dx\, dy \qquad \text{.... (13)}$$

where $\bar{I}_e$, $\bar{II}_e$ are the first and second invariants of the middle surface strains. These are

$$\bar{I}_e = \epsilon_{x_0} + \epsilon_{y_0} \; ; \; \bar{II}_e = \epsilon_{x_0}\epsilon_{y_0} - \epsilon_{x_0 y_0}^2 \qquad \text{.... (14)}$$

Let us now apply Banerjee's hypothesis and rewrite the expression for Ψ in the following form :

$$\Psi' = \iint \Big\{ \frac{Eh}{2(1-\nu^2)} \left(\bar{\bar{I}}_e^2 + \frac{\lambda}{4}\left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\}^2 \right) + M_x \frac{\partial \alpha}{\partial x} + M_y \frac{\partial \beta}{\partial y} + M_{xy}\left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \right) + Q_x\left(\frac{\partial w}{\partial x} + \alpha \right) + Q_y\left(\frac{\partial w}{\partial y} + \beta \right) - \frac{1}{2E}\left[\frac{12}{h^3}\left(M_x^2 + M_y^2 \right) - \frac{24\, M_x M_y \nu}{h^3} \right] - \frac{3}{5G_c h}\left(Q_x^2 + Q_y^2 \right) \Big\}\, dx\, dy \qquad \text{.... 15(a)}$$

where

$$\bar{\bar{I}}_e = \frac{\partial u_o}{\partial x} + \nu\frac{\partial v_o}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \text{.... 15(b)}$$

and λ is a constant depending on Poisson's ratio of the plate material [18].

The Kinetic energy equation after integrating through the thickness is

$$T = \iiint \left\{ \frac{\rho h}{2}\left[\left(\frac{\partial u_o}{\partial t}\right)^2 + \left(\frac{\partial v_o}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] + \frac{\rho h^3}{24}\left[\left(\frac{\partial \alpha}{\partial t}\right)^2 + \left(\frac{\partial \beta}{\partial t}\right)^2\right]\right\} dx\, dy \qquad \text{.... (16)}$$

In order to derive the equation of motion we now apply Hamilton's principle in conjunction with the strain-energy equation given by (15) as well as the kinetic energy given by (16). Therefore we have to minimise the integral,

$$\Phi = \int_{t_1}^{t_2} (\Psi - T)\, dt \qquad \text{.... (17)}$$

Taking the variation of Φ , equating it to zero and eliminating M_x, M_y, M_{xy}, etc. we get the following set of decoupled differential equations governing the vibrations of the plates : *

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$$\nabla^4 W + \frac{6}{5(1-\nu^2)} \cdot k\left(\frac{E}{Gc}\right) \frac{\bar{\alpha}^2 h^2}{12} \tau^2(t) \nabla^2\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$+ \frac{3\lambda}{5(1-\nu^2)} k\left(\frac{E}{Gc}\right) \nabla^2 \left[\nabla^2 w \left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}\right.$$

$$\left. + 2\left\{\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2\right\} + 4\,\frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}\right]$$

$$- \frac{6}{5} \cdot \frac{\rho}{Gc} \cdot \frac{\partial^2}{\partial t^2}\left(\nabla^2 w\right) - \bar{\alpha}^2 \tau^2(t)\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right]$$

$$- \frac{6\lambda}{h^2}\left[\nabla^2 w \left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}\right.$$

$$\left. + 2\left\{\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2\right\} + 4\,\frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}\right]$$

$$+ \frac{12}{h^2 C_p^2} \cdot \frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{.... 18(a)}$$

The coupling parameter $\bar{\alpha}^2$ is given by

$$\frac{\bar{\alpha}^2 h^2}{12} \tau^2(t) = \frac{\partial u_o}{\partial x} + \nu \frac{\partial v_o}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \text{.... 18(b)}$$

where $\tau(t)$ is a function of time and $C_p = \left[\frac{E}{\rho(1-\nu^2)}\right]^{1/2}$ is the speed of wave propagation along the surface of the plates.

For movable edge condition $\bar{\alpha} = 0$ and for immovable edge condition $u_0 = v_0 = 0$ at the boundary.

In polar co-ordinates the above set of equations are transformed into the following form : *

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right]$$

$$+\frac{6}{5(1-\nu^2)}k\left(\frac{E}{G_c}\right)\frac{\bar{\alpha}^2 h^2}{12}\cdot r^{\nu-1}\left[\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\frac{\partial w}{\partial r}\right]\tau^2(t)$$

$$+\frac{3\lambda}{5(1-\nu^2)}k\left(\frac{E}{G_c}\right)\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)\left(\frac{\partial w}{\partial r}\right)^2+\right.$$

$$\left.2\frac{\partial^2 w}{\partial r^2}\left(\frac{\partial w}{\partial r}\right)^2\right]-\frac{6}{5}\frac{\rho}{G_c}\cdot\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right]$$

$$-\bar{\alpha}^2\tau^2(t)r^{\nu-1}\left[\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\cdot\frac{\partial w}{\partial r}\right]$$

$$-\frac{6\lambda}{h^2}\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial w}{\partial r}\right)\left(\frac{\partial w}{\partial r}\right)^2+2\frac{\partial^2 w}{\partial r^2}\left(\frac{\partial w}{\partial r}\right)^2\right]$$

$$+\frac{12}{h^2 c_p^2}\cdot\frac{\partial^2 w}{\partial t^2}=0 \qquad \ldots\ 19(a$$

where

$$r^{\nu-1}\frac{\bar{\alpha}^2 h^2}{12}\tau^2(t)=\frac{\partial u_0}{\partial r}+\nu\frac{u_0}{r}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \qquad \ldots\ 19(b$$

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* Published in the Journal of Sound and Vibration (U.K.), 133(1), pp. 185 - 188, 1989

For movable edge condition $\bar{\alpha} = 0$ and for immovable edge condition $u_0 = 0$ at the boundary. Equations 18(a) - 19(b) will be utilised to study the nonlinear behaviours of different elastic plates.

BERGER'S EQUATIONS :

To derive Berger's equations, as outlined by Wu and Vinson [38], let us now recall our attention to the strain energy expression given by equation (13). If Π_e is neglected in the expression (13) we shall arrive at the following set of decoupled equations by using the same procedure as adopted in the previous case.

$$\left[1 + \frac{\bar{\alpha}^2 h^2}{10(1-\nu^2)} \left(\frac{E}{G_c}\right) \tau^2(t) \right] \nabla^4 w - \bar{\alpha}^2 \tau^2(t)(\nabla^2 w)$$

$$- \frac{6}{5} \cdot \frac{\rho}{G_c} \frac{\partial^2}{\partial t^2} (\nabla^2 w) + \frac{12}{h^2 C_p^2} \cdot \frac{\partial^2 w}{\partial t^2} = 0$$

.... 20(a)

where

$$\bar{I}_e = \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\bar{\alpha}^2 h^2}{12} \tau^2(t)$$

.... 20(b)

Equations 20(a) and 20(b) are well-known Berger's equations on the thick plate theory and will now be utilised to investigate the nonlinear behaviours of different elastic plates.

CHAPTER - II

NON-LINEAR DYNAMIC RESPONSE OF MODERATELY THICK PLATES PLACED ON ELASTIC FOUNDATION

CHAPTER - II

NON-LINEAR DYNAMIC RESPONSE OF MODERATELY THICK PLATES PLACED ON ELASTIC FOUNDATION

ABSTRACT

In this chapter the non-linear dynamic response of thick plates of different shapes placed on elastic foundation of the Winkler-type is investigated by using the approximate method offered by Berger. Conformal mapping technique has been utilised in the investigation. The cases of square plates, rounded cornered plates and circular plates have been studied in detail. The ratios of the non-linear time periods including shear deformation and the linear time period of the classical plate theory have been computed for these plates for different values of the foundation modulus $\frac{k_1}{D}a^4$ and discussed.

(a) <u>Vibrations of square plates and square plates with rounded corners</u> .*

Let us consider the free vibrations of thick plates of thickness h . The deflections are considered to be of the same order of magnitude as the thickness of the plate. Berger's equations given by 20(a) and 20(b) in chapter I are rewritten

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* Accepted for publication in the Journal of Sound and Vibration U.K., July 1990.

in the following form for thick plates placed on elastic foundation of the Winkler-type (Ariman [34])

$$\left[1+\frac{\bar{\alpha}^2 h^2}{10(1-\nu^2)}\cdot\frac{E}{Gc}\right]\nabla^4 w-\bar{\alpha}^2\nabla^2 w-\frac{6}{5}\cdot\frac{\rho}{Gc}\cdot\frac{\partial^2}{\partial t^2}(\nabla^2 w)$$

$$+\frac{12}{h^2 C_p^2}\cdot\frac{\partial^2 w}{\partial t^2}-\frac{h^2}{10}\cdot\frac{k_1}{D}\cdot\frac{2-\nu}{1-\nu}\nabla^2 w+\frac{k_1}{D}w=0 \qquad \text{.... (21a)}$$

where k_1 = foundation modulus

D = flexural rigidity

and the coupling parameter $\bar{\alpha}^2$ is given by

$$\tau^2(t)\frac{\bar{\alpha}^2 h^2}{12}=\frac{\partial u_o}{\partial x}+\frac{\partial v_o}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \text{.... (21b)}$$

To solve the governing equations let us assume the deflection in the following form

$$w(x,y,t)=w_1(x,y)\,\tau(t) \qquad \text{.... (22)}$$

Substituting (22) in 21(a) we get

$$\left[\frac{\nabla^4 w_1}{w_1}-\frac{h^2}{10}\cdot\frac{k_1}{D}\cdot\frac{2-\nu}{1-\nu}\cdot\frac{\nabla^2 w_1}{w_1}+\frac{k_1}{D}\right]\tau(t)$$

$$+\left[\frac{12}{h^2 C_p^2}-\frac{6}{5}\cdot\frac{\rho}{Gc}\frac{\nabla^2 w_1}{w_1}\right]\ddot{\tau}(t)$$

$$+\left[\frac{\bar{\alpha}^2 h^2}{10(1-\nu^2)}\cdot\frac{E}{Gc}\cdot\frac{\nabla^4 w_1}{w_1}-\bar{\alpha}^2\frac{\nabla^2 w_1}{w_1}\right]\tau^3(t)=0 \qquad \text{.... (23)}$$

A solution of equation (23) is possible if

$$\frac{\nabla^4 w_1}{w_1} = \lambda_1^4 \qquad \text{.... 24(a)}$$

$$\frac{\nabla^2 w_1}{w_1} = -\lambda_1^2 \qquad \text{.... 24(b)}$$

From 24(a) we have

$$(\nabla^2 - \lambda_1^2)(\nabla^2 + \lambda_1^2)\, w_1 = 0 \qquad \text{.... 25(a)}$$

and from 24(b) we have

$$(\nabla^2 + \lambda_1^2)\, w_1 = 0 \qquad \text{.... 25(b)}$$

It is evident that to get a complete solution it is sufficient to solve

$$(\nabla^2 + \lambda_1^2)\, w_1 = 0 \qquad \text{.... 25(c)}$$

In a complex co-ordinate system,

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy$$

The equation 25(c) changes and

let $z = f(\xi)$, $\bar{z} = f(\bar{\xi})$ ....(26)

be the analytic function which maps the given shape in the Z - plane on to a unit circle in the ξ - plane. After transforming equation 25(c) into the complex co-ordinates (z , $\bar{z}$) and using relation (26) we obtain the following differential equation in (ξ , $\bar{\xi}$) co-ordinates for the deflection function w_1

$$\frac{\partial^2 w_1}{\partial \xi\, \partial \bar{\xi}} + \frac{\lambda_1^2}{4} \cdot \frac{dz}{d\xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} \cdot w_1 = 0 \qquad \text{.... 27(a)}$$

Similarly equation 21(b) changes into

$$\tau^2(t)\cdot\frac{\bar{\alpha}^2 h^2}{12}\left(\frac{dz}{d\xi}\cdot\frac{d\bar{z}}{d\bar{\xi}}\right)^2 = \frac{\partial u_0}{\partial \xi}\cdot\frac{d\bar{z}}{d\bar{\xi}} + \frac{\partial u_0}{\partial \bar{\xi}}\cdot\frac{dz}{d\xi}$$

$$+\left(\frac{\partial v_0}{\partial \xi}\cdot\frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial v_0}{\partial \bar{\xi}}\cdot\frac{dz}{d\xi}\right)$$

$$+2\frac{\partial w_1}{\partial \xi}\cdot\frac{\partial w_1}{\partial \bar{\xi}}\cdot\frac{dz}{d\xi}\cdot\frac{d\bar{z}}{d\bar{\xi}} \qquad \text{.... 27(b)}$$

Here $\xi = re^{i\theta}$ and $\bar{\xi} = r.e^{-i\theta}$, r being the radius of the circle. For transverse vibrations the inplane displacements u_0 and v_0 are of no interest and they have been eliminated finally through integrations by choosing suitable expressions for the displacements compatible with their boundary conditions i.e. $u_0 = 0$, $v_0 = 0$ on the boundary.

To solve equation 27(a) let us choose the deflection function w_1 ($\xi, \bar{\xi}$) in the following form

$$w_1 = A_0\left[1 - \xi\bar{\xi}\right]\left[1 - \frac{1}{3}\xi\bar{\xi} + \frac{1}{2}\left(\xi^2 + \bar{\xi}^2\right)\left(1 - \xi\bar{\xi}\right)^2\right] \qquad \text{.... (28)}$$

Clearly w_1 is θ dependent and satisfies the simply supported edge conditions, namely,

$$w_1 = 0 \quad \text{at} \quad r = 1$$

$$\frac{\partial^2 w_1}{\partial\xi\partial\bar{\xi}} = 0 \quad \text{at} \quad r = 1$$

Substituting (28) in 27(a) and inserting the values of $\frac{dz}{d\xi}$, $\frac{d\bar{z}}{d\bar{\xi}}$ from the given mapping functions $z = f(\xi)$ we get the error function $\epsilon(\xi, \bar{\xi})$. Galerkin's technique requires that

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \epsilon(\xi, \bar{\xi}) w_1(\xi, \bar{\xi}) r \, dr \, d\theta = 0 \qquad \ldots. \ (29)$$

The values of $\bar{\alpha}^2$ have been determined by substituting (28) in 27(b), remembering the values of $\frac{dz}{d\xi}$, $\frac{d\bar{z}}{d\bar{\xi}}$ from the given mapping functions and finally integrating over the area of the plate. After evaluating the integrals in (29) we get the necessary values of λ_1^2 . Thus the values of $\frac{\nabla^4 w_1}{w_1}$, and $\frac{\nabla^2 w_1}{w_1}$ are determined. Inserting all these values in (23) we get the following cubic equation determining the unknown time function $\tau(t)$ in the form

$$\ddot{\tau}(t) + \alpha_1 \tau(t) + \beta_1 \tau^3(t) = 0 \qquad \ldots. \ (30)$$

The solution of the above equation subject to the boundary conditions

$$\tau(0) = 1$$

$$\dot{\tau}(0) = 0$$

is well-known and is obtained in terms of Jacobi's elliptic function.

The ratio of the non-linear time periods to the linear time periods of the classical plate (thin plate) is

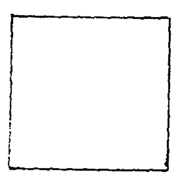
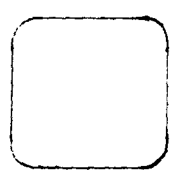
$$\frac{T^*}{T} = \frac{2K/\pi}{\left[1 + \frac{\beta_1}{\alpha_1} \bar{\beta}^2\right]^{1/2}} \qquad \ldots. \ (31)$$

where $\bar{\beta} = \frac{A_c}{h}$ is the ratio of the static deflection to the thickness of the plate.

Table I shows different values of $\frac{T^*}{T}$ Vs. $\frac{k_1}{D} a^4$ for different $\bar{\beta}$ for the simply supported square and rounded cornered square plates.

Table - 1

Ratio of the non-linear to linear periods for the free vibrations of simply supported square plates and square plates with rounded corners. Immovable edge conditions have been considered.

PLATE SHAPE	MAPPING FUNCTION	$\bar{\beta} = \frac{A_0}{h}$	$\frac{T^*}{T}$ FOR $\nu = 0.3$, $\frac{h}{a} = 0.2$, $\frac{E}{G_c} = 2.5$.				
			WITHOUT ELASTIC FOUNDATION		WITH ELASTIC FOUNDATION.		
			$k_1 . \frac{a^4}{D} = 0$		$k . \frac{a^4}{D} = 10$	$k . \frac{a^4}{D} = 15$	$k . \frac{a^4}{D} = 20$
			PRESENT STUDY	REF.[38]			
SIMPLY SUPPORTED SQUARE PLATE OF SIDE 2a 	$z = 1.08a\xi - 0.11a\xi^5$	0·2	0·9772	1·0037	0·9836	0·9856	0·9872
		0·4	0·9175	0·9418	0·9391	0·9461	0·9517
		0·6	0·8353	0·8606	0·8768	0·8878	0·9003
		0·8	0·7564	0·7758	0·8076	0·8258	0·8408
		1·0	0·6793	0·6976	0·7387	0·7608	0·7793
SIMPLY SUPPORTED SQUARE PLATE WITH ROUNDED CORNERS. 	$z = \frac{25}{48}a\xi - \frac{1}{48}a\xi^5$	0·2	0·9704	–	0·9711	0·9714	0·9718
		0·4	0·8957	–	0·8981	0·8990	0·9000
		0·6	0·8025	–	0·8060	0·8077	0·8094
		0·8	0·7104	–	0·7151	0·7169	0·7190
		1·0	0·6283	–	0·6330	0·6353	0·6376

$\frac{T^*}{T}$ VS $\bar{\beta}$ FOR

SIMPLY SUPPORTED SQUARE PLATE WITH SIDE $2a$.
AND
ROUNDED CORNERED SQUARE PLATE.

(IMMOVABLE EDGES)

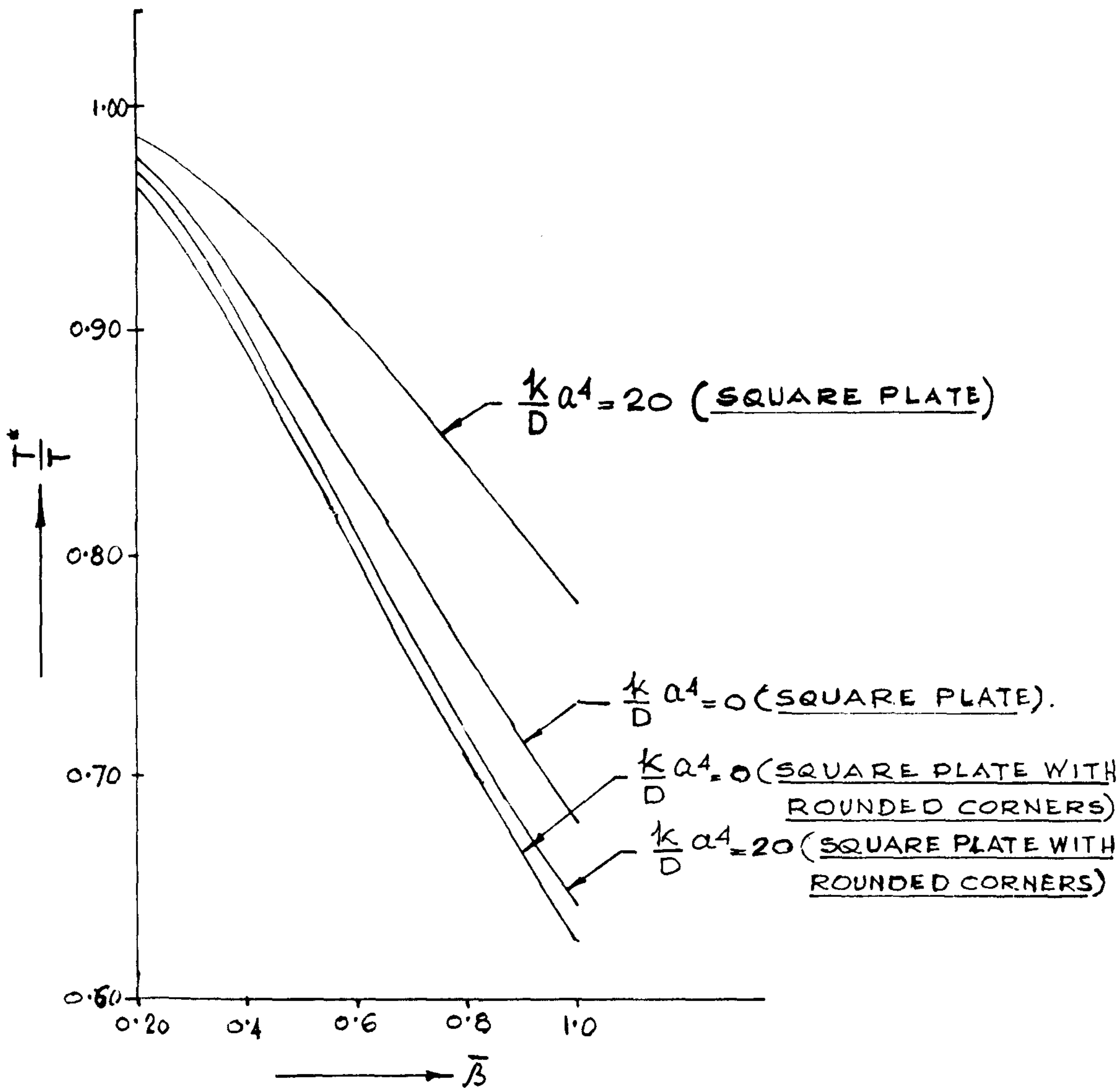


FIG. - 1.

(b) Vibrations of circular plates.

Let us consider the free vibrations of thick circular plates of radius a. The deflections are considered to be of the same order of magnitude as the plate thickness.

For circular plate of radius a the mapping function is $z = a\xi$

Thus $$\frac{dz}{d\xi} = \frac{d\bar{z}}{d\bar{\xi}} = a = \text{constant.}$$

On this assumption the equations 25(a) and 25(b) namely

$$(\nabla^2 - \lambda_1^2)(\nabla^2 + \lambda_1^2)\cdot w_1 = 0 \qquad \text{.... 32(a)}$$

$$(\nabla^2 + \lambda_1^2)\, w_1 = 0 \qquad \text{.... 32(b)}$$

offer an interesting closed form solution. Changing equation 32(a) into complex co-ordinates as shown in the previous case we have the following two differential equations

$$\left.\begin{aligned} \frac{\partial^2 w_1'}{\partial\xi\,\partial\bar{\xi}} + \frac{\lambda_1^2}{4} a^2 w_1' &= 0 \\ \frac{\partial^2 w_1''}{\partial\xi\,\partial\bar{\xi}} - \frac{\lambda_1^2}{4} a^2 w_1'' &= 0 \end{aligned}\right\} \qquad \text{.... 32(c)}$$

where $w_1 = w_1' + w_1''$

The equation determining $\bar{\alpha}^2$ given by the equation 27(l) remains same as in the previous case. The solution of the differential equations 32(c) is obtained in terms of Bessel function in the form

$$W_1 = W_1' + W_1''$$

$$= A J_0 \left(2p\sqrt{\xi\bar{\xi}}\right) + B I_0 \left(2p\sqrt{\xi\bar{\xi}}\right)$$

$$= A J_0 (2pr) + B I_0 (2pr) \qquad \ldots. \ (33)$$

where $p = \dfrac{a\lambda_1}{2}$

For clamped edge boundary conditions

$$W_1 = 0 \quad \text{at} \quad r = 1$$

$$\frac{\partial W_1}{\partial r} = 0 \quad \text{at} \quad r = 1.$$

Thus the frequency equation is obtained in the form

$$\begin{vmatrix} J_0(2p) & I_0(2p) \\ -J_1(2p) & I_1(2p) \end{vmatrix} = 0 \qquad \ldots. \ (34)$$

Solving (34) the value of $2p = 3{\cdot}20$ is obtained from the table [54].

For simply supported edge conditions

$$W_1 = 0 \quad \text{at} \quad r = 1$$

$$\frac{\partial^2 W_1}{\partial r^2} + \frac{\nu}{r}\cdot\frac{\partial W_1}{\partial r} = 0 \quad \text{at} \quad r = 1$$

Thus the frequency equation in this case is

$$\begin{vmatrix} J_0(2p) & I_0(2p) \\ (1-\nu)J_1(2p) - 2pJ_0(2p) & -(1-\nu)I_1(2p) + 2pI_0(2p) \end{vmatrix} = 0 \qquad \ldots. \ (35)$$

Solving (35) we get $2p = 2\cdot 22$ from table [54].

The values of $\bar{\alpha}^2$ have been determined as in the previous case by putting (33) in 27(b) and integrating over the entire area of the plate. Now knowing the values of $p = a\lambda_{1/2}$, the values of $\frac{\nabla^4 W_1}{W_1}$ and $\frac{\nabla^2 W_1}{W_1}$ are determined in each case. Inserting all these values in equation (23) we get a similar type of cubic equation as equation (30) for determining the time function $T(t)$. The solution of this cubic equation is obtained in terms of Jacobi's elliptic function as in the previous case.

The following table shows $\frac{T^*}{T}$ Vs $\frac{k_1}{D} a^4$ for different values of $\bar{\beta}$

Table - 2.

Ratio of non-linear to linear periods for the vibrations of circular plates.

$$\left[\frac{h}{a} = 0.2\ ,\quad \nu = 0.3\right]$$

PLATE SHAPE WITH EDGE CONDITION	$\bar{\beta} = \frac{A_o}{h}$	$\frac{T^*}{T}$ WITHOUT ELASTIC FOUNDATION ($K_1=0$): PRESENT STUDY	REF. [52]	REF. [40]	$\frac{T^*}{T}$ WITH ELASTIC FOUNDATION: $\frac{k_1}{D}.a^4=10$	$\frac{k_1}{D}.a^4=15$	$\frac{k_1}{D}.a^4=20$
CLAMPED CIRCULAR PLATE $\frac{E}{G_c} = 8.1971$	0·2	0·9926	0·9821	0·9921	0·9760	0·9771	0·9780
	0·6	0·9395	0·9138	0·9366	0·8403	0·8492	0·8572
	1·0	0·8551	0·8029	0·8533	0·7032	0·7238	0·7416

PLATE SHAPE WITH EDGE CONDITION	$\bar{\beta} = \frac{A_o}{h}$	PRESENT STUDY	REF. [40]	$\frac{k_1}{D}.a^4=10$	$\frac{k_1}{D}.a^4=15$	$\frac{k_1}{D}.a^4=20$
SIMPLY SUPPORTED CIRCULAR PLATE $\frac{E}{G_c} = 2.9374$	0·2	0·9755	0·9745	0·9828	0·9850	0·9867
	0·6	0·8294	0·8265	0·8721	0·8861	0·8972
	1·0	0·6657	0·6688	0·7304	0·7542	0·7737

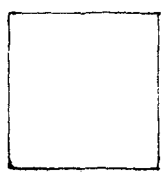
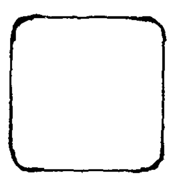
Observations :

The tables 1 and 2 clearly show the response of the foundation modulus $k_1 \frac{a^4}{D}$ on the ratio of the time periods $\frac{T^*}{T}$ for different $\bar{\beta}$. As $k_1 \frac{a^4}{D}$ increases $\frac{T^*}{T}$ increases. This is quite expected from the practical point of view. When $k_1 \frac{a^4}{D} = 0$ i.e. when there is no elastic foundation, the results obtained in the present study for square plates and for circular plates with simply supported as well as clamped edges are found to be in very good agreement with other known results. The results of the rounded cornered square plates are completely new.

(C) A comparative study on the time period ratio $\frac{T^*}{T}$ Vs. $k_1\frac{a^4}{D}$ for different $\bar{\beta}$ of the simply supported square plates and square plates with rounded corners on the basis of the area of the plates :

Table - 3.

$$\left[\ \bar{\beta}=1,\ \nu=0{\cdot}3,\ \frac{E}{G_c}=2{\cdot}5,\ \frac{h}{a}=0{\cdot}2\ \right]$$

PLATE SHAPE [SIMPLY SUPPORTED EDGE CONDITION]	$\frac{T^*}{T}$	
	WITHOUT ELASTIC FOUNDATION. ($K_1=0$)	WITH ELASTIC FOUNDATION, $k_1\frac{a^4}{D}=10$
SQUARE PLATE OF SIDE 2a. 	0·6793	0·7387
SQUARE PLATE WITH ROUNDED CORNERS OF SIDE 2a 	0·6283	0·6330

Observation :

An interesting observation on Table - 3 is variation of the time period ratio $\frac{T^*}{T}$ with the area of the respective plates. It is observed that as the area of the plate increases this ratio increases. This is true irrespective of the response of the foundation modulus. Obviously, the area of the square plate is greater than that of the rounded cornered square plate. $\frac{T^*}{T}$ increases accordingly. This is noteworthy from the practical point of view.

(d) A useful observation on Berger's equations :

Let us now examine Berger's equations on the movable edge conditions.

A clamped circular plate of radius a is considered here. The deflection function for the circular plate is assumed as

$$W_1 = A_0 \tau(t)\left[1 - \frac{r^2}{a^2}\right]^2 \qquad \ldots . \ (36)$$

This clearly satisfies the required boundary conditions of the clamped edges.

Before recalling our attention to the original equation (23) let us turn to the equation 21(b) which reduces (in the present case) to the following form.

$$\tau^2(t)\frac{\bar{\alpha}^2 h^2}{12} = \frac{du_0}{dr} + \frac{u_0}{r} + \frac{1}{2}\left(\frac{dW_1}{dr}\right)^2 \qquad \ldots . \ (37)$$

$$\text{Let } u_0 = u_1(r)\,\tau^2(t) \qquad \ldots . \ (37a)$$

Putting (36) in (37) multiplying the equation by r we obtain the following equation after integration

$$u_1 r + c = \frac{\bar{\alpha}^2 h^2}{24}\cdot r^2 - \frac{8A_0^2}{a^4}\left[\frac{r^8}{8a^4} + \frac{r^4}{4} - \frac{1}{3}\cdot\frac{r^6}{a^2}\right] \qquad \ldots . \ 38(a)$$

At $r = 0$, $u_1 = 0$. Therefore $c = 0$

Hence u_1 is obtained as

$$u_1 = \frac{\bar{\alpha}^2 h^2}{24}\cdot r - \frac{8A_0^2}{a^4}\left[\frac{r^3}{4} - \frac{r^5}{3a^2} + \frac{r^7}{8a^4}\right] \qquad \ldots . \ 38(b)$$

For movable edge condition at $r = a$

$$\frac{du_1}{dr} + \nu \frac{u_1}{r} + \frac{1}{2}\left(\frac{dw_1}{dr}\right)^2 = 0$$

Using this condition we obtain the value for $\bar{\alpha}^2$ as

$$\bar{\alpha}^2 = \frac{8A_0^2}{h^2a^2} \cdot \frac{\nu-1}{\nu+1} \qquad \text{.... (39)}$$

Inserting now (36) in equation (23), remembering $\bar{\alpha}^2$ thus evaluated from (39) and applying Galerkin's technique we get as usual a cubic equation determining the time function $\tau(t)$ in the form

$$\ddot{\tau}(t) + \alpha_1 \tau(t) - \beta_1 \tau^3(t) = 0 \qquad \text{.... (40)}$$

Which leads to meaningless results because the coefficient of $\tau^3(t)$ is negative.

Thus we arrive at the following conclusion :

Although Berger's equation can be conveniently applied to the nonlinear theory of thick plates, its application is limited to the immovable edge conditions only.

CHAPTER - III.

INFLUENCES OF LARGE AMPLITUDES, TRANSVERSE SHEAR DEFORMATION AND ROTATORY INERTIA ON FREE VIBRATIONS OF TRANSVERSELY ISOTROPIC PLATES - A NEW APPROACH.

CHAPTER - III.

INFLUENCES OF LARGE AMPLITUDES, TRANSVERSE SHEAR DEFORMATION AND ROTATORY INERTIA ON FREE VIBRATIONS OF TRANSVERSELY ISOTROPIC PLATES - A NEW APPROACH.

ABSTRACT

In this Chapter the non-linear static and dynamic behaviours of moderately thick plates of different shapes have been analysed with the help of a new set of uncoupled differential equations proposed in the chapter I. Numerical results for different plates with different edge conditions have been computed and compared with other known results.

A. Non-linear analysis of square, elliptical and isosceles right angled triangular plates.

Analysis : -

Let us consider the free vibrations of thick plates of thickness h in cartesian co-ordinate system. The material is transversely isotropic (such as pyrolytic graphite, for example). The origin of co-ordinates is located at the centre of the square plate of side 2a and at the centre of the elliptic plate with semi-axes a and b. For isosceles right angled triangular plate of equal side a, it is located at one corner. The deflections are considered to be of the same order of magnitude as the plate thickness.

*(i) Square plate :

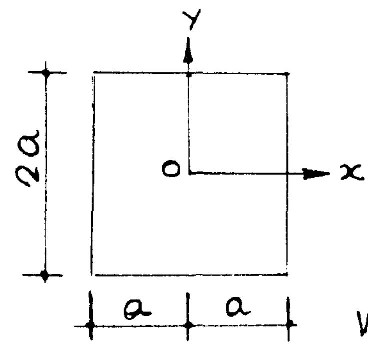


FIG - 2.

For square plate of side 2a, let us choose the deflection function in the following form

$$W = A_0 \Upsilon(t) \cos\frac{\pi x}{a} . \cos\frac{\pi y}{a} \quad \ldots . (41)$$

for fundamental mode of vibrations. Clearly this form of W satisfies the following simply supported edge conditions

$$W = 0 \quad \text{at} \quad x = \pm a$$

$$W = 0 \quad \text{at} \quad y = \pm a$$

$$\frac{\partial^2 W}{\partial x^2} = 0 \quad \text{at} \quad x = \pm a$$

$$\frac{\partial^2 W}{\partial y^2} = 0 \quad \text{at} \quad y = \pm a.$$

Putting (41) in (18b) of the first chapter and integrating over the area of the plate one gets

$$\bar{\alpha}^2 = \frac{3}{8} . \frac{A_0^2 \pi^2 (1+\nu)}{a^2 h^2} \quad \ldots . (42)$$

For transverse vibration the normal displacement is our primary interest, So the inplane displacements have been eliminated through integration by choosing suitable expression for them compatible with their boundary conditions i.e. $u_0 = 0$, $v_0 = 0$ on the boundary.

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*Published in the International Journal of Nonlinear Mechanics, (U.S.A.) Vol. 24. No. 3 PP. 159 - 164 , 1989.

Now inserting (41) in (18a) of the 1st chapter. Considering (42) and applying Galerkin's error minimising technique one gets the following differential equation for the time function $\tau(t)$

$$\left[\frac{12}{h^2 C_p^2} + \frac{3}{5}\frac{\pi^2 \rho}{a^2 G_c}\right]\ddot{\tau}(t) + \frac{\pi^4}{4a^4}\tau(t) + \left[\frac{15}{32}\frac{\pi^4}{a^4}\lambda\left(\frac{A_o}{h}\right)^2 + \frac{3}{32}\cdot\frac{\pi^4}{a^4}(1+\nu)^2\left(\frac{A_o}{h}\right)^2 + \frac{3}{640}\cdot\frac{\pi^6}{a^6}\,\frac{h^2(1+\nu)^2}{(1-\nu^2)}\,k\left(\frac{E}{G_c}\right)\left(\frac{A_o}{h}\right)^2 + \frac{3}{128}\lambda\cdot\frac{\pi^6}{a^6}\cdot\frac{h^2}{(1-\nu^2)}\cdot k\left(\frac{E}{G_c}\right)\left(\frac{A_o}{h}\right)^2\right]\tau^3(t) = 0 \quad \ldots. \ (43)$$

The above equation is of the form

$$\ddot{\tau}(t) + \alpha_1 \tau(t) + \beta_1 \tau^3(t) = 0 \quad \ldots. \ (44)$$

The solution of equation (44) subject to the initial conditions

$$\tau(0) = 1$$

$$\dot{\tau}(0) = 0$$

is well-known and is obtained in terms of Jacobi's elliptic function. The ratio of the nonlinear and linear time periods is

$$\frac{T^*}{T} = \frac{2K}{\pi}\left[\frac{1 + \dfrac{\pi^2}{20(1-\nu^2)}\cdot\dfrac{E}{G_c}\cdot\dfrac{h^2}{a^2}}{1 + \frac{15}{8}\lambda\bar{\beta}^2 + \frac{3}{8}(1+\nu)^2\bar{\beta}^2 + \frac{3}{32}\frac{\lambda}{(1-\nu^2)}k\left(\frac{E}{G_c}\right)\frac{\pi^2 h^2}{a^2}\bar{\beta}^2 + \frac{3}{160}k\left(\frac{E}{G_c}\right)\frac{\pi^2 h^2 (1+\nu)^2}{a^2(1-\nu^2)}\bar{\beta}^2}\right]^{1/2} \quad \ldots. \ (45)$$

where $\bar{\beta} = \frac{A_o}{h}$ is the ratio of the static deflection to the thickness of the plate.

RATIO OF NON-LINEAR TO LINEAR PERIOD FOR THE FUNDAMENTAL MODE OF VIBRATION OF A SIMPLY SUPPORTED SQUARE PLATE.

($\nu = 0.3$, $\lambda = \nu^2$ [18])

IMMOVABLE EDGES. (TABLE 4 TO 7)

PRESENT FINDINGS | REF. [38]

TABLE-4

	$\bar{\beta} = \frac{A_o}{h}$	$\frac{T^*}{T}$							
		$k\left(\frac{E}{G_c}\right)$				$k\left(\frac{E}{G_c}\right)$			
		2·5	20	30	50	2·5	20	30	50
$\frac{h}{2a} = \frac{1}{10}$	0	1·0268	1·1976	1·2850	1·4440	1·0268	1·1976	1·2850	1·4440
	0·2	1·0140	1·1774	1·2602	1·4092	1·0037	1·1606	1·2397	1·3806
	0·4	0·9785	1·1228	1·1940	1·3187	0·9418	1·0683	1·1290	1·2290
	0·6	0·9270	1·0469	1·1066	1·2012	0·8606	0·9577	0·9978	1·0656
	0·8	0·8624	0·9636	1·0113	1·0819	0·7758	0·8422	0·8710	0·9159
	1·0	0·8055	0·8809	0·9123	0·9678	0·6976	0·7449	0·7648	0·7948

PRESENT FINDINGS | REF. [38].

TABLE-5

	$\bar{\beta} = \frac{A_0}{h}$	$\frac{T^*}{T}$							
		$k\left(\frac{E}{G_c}\right)$				$k\left(\frac{E}{G_c}\right)$			
		2.5	20	30	50	2.5	20	30	50
$\frac{h}{2a} = \frac{1}{20}$	0	1.0067	1.0529	1.0785	1.1274	1.0067	1.0529	1.0785	1.1274
	0.2	0.9947	1.0391	1.0635	1.1100	0.9846	1.0173	1.0511	1.0966
	0.4	0.9610	1.0009	1.0227	1.0644	0.9270	0.9617	0.9810	1.0176
	0.6	0.9121	0.9460	0.9643	0.9905	0.8487	0.8757	0.8963	0.9175
	0.8	0.8548	0.8825	0.8968	0.9251	0.7670	0.7869	0.7973	0.8166
	1.0	0.7948	0.8170	0.8273	0.8503	0.6900	0.7049	0.7119	0.7255

PRESENT FINDINGS | REF. [38]

TABLE-6

		$\frac{T^*}{T}$							
	$\bar{\beta} = \frac{A_c}{h}$	$k\left(\frac{E}{G_c}\right)$				$k\left(\frac{E}{G_c}\right)$			
		2·5	20	30	50	2·5	20	30	50
$\frac{h}{2a} = \frac{1}{30}$	0	1·0030	1·0239	1·0355	1·0585	1·0030	1·0239	1·0355	1·0585
	0·2	0·9912	1·0111	1·0225	1·0445	0·9811	1·0005	1·0113	1·0221
	0·4	0·9578	0·9759	0·9860	1·0058	0·9172	0·9393	0·9482	0·9656
	0·6	0·9093	0·9247	0·9334	0·9501	0·8464	0·8586	0·8653	0·8784
	0·8	0·8525	0·8641	0·8722	0·8855	0·7630	0·7742	0·7791	0·7885
	1·0	0·7930	0·8031	0·8087	0·8197	0·6889	0·6952	0·6986	0·7052

PRESENT STUDY | REF [38] — CLASSICAL THIN PLATE THEORY.

TABLE - 7.

$\bar{\beta} = \frac{A_0}{h}$	$\frac{T^*}{T}$ $\frac{E}{G_c} = 0$	$\frac{T^*}{T}$ $\frac{E}{G_c} = 0$
0	1	1
0·2	0·9882	0·9782
0·4	0·9552	0·9210
0·6	0·9072	0·8446
0·8	0·8507	0·7640
1·0	0·7917	0·6878

PRESENT FINDINGS FOR MOVABLE EDGES (TABLE 8 TO 11)

TABLE - 8

	$\bar{\beta} = \frac{A_o}{h}$	$\frac{T^*}{T}$ $(\nu = 0.3, \lambda = \nu^2$ [18]$)$			
		$k\left(\frac{E}{G_c}\right)$			
		2·5	20	30	50
$\frac{h}{2a} = \frac{1}{10}$	0	1·02680	1·19760	1·2850	1·4440
	0·2	1·02406	1·19325	1·2796	1·4366
	0·4	1·01601	1·18056	1·2641	1·4147
	0·6	1·00298	1·1604	1·2394	1·3802
	0·8	0·9857	1·13376	1·2071	1·3362
	1·0	0·9647	1·1022	1·1693	1·2852

TABLE - 9

	$\bar{\beta} = \frac{A_o}{h}$	$\frac{T^*}{T}$ $(\nu = 0.3 \;\; \lambda = \nu^2$ [18]$)$			
		$k\left(\frac{E}{G_c}\right)$			
		2·5	20	30	50
$\frac{h}{2a} = \frac{1}{20}$	0	1·0067	1·0529	1·07850	1·12754
	0·2	1·0042	1·0499	1·0752	1·1239
	0·4	0·9966	1·0412	1·0659	1·1134
	0·6	0·9844	1·0291	1·0509	1·0963
	0·8	0·9678	1·0089	1·0310	1·0738
	1·0	0·9480	0·9859	1·0070	1·0468

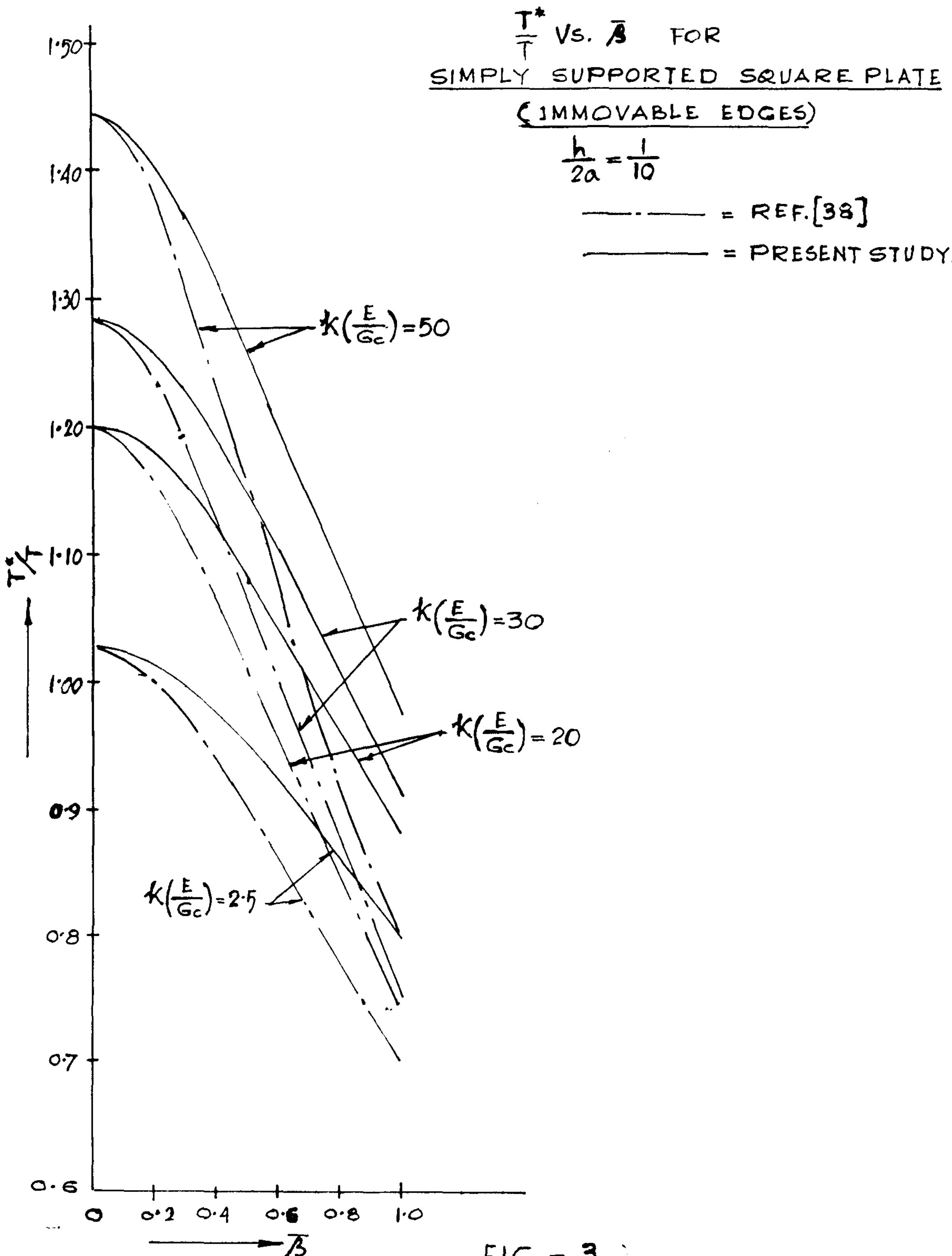
Table 10

		$\frac{T^*}{T}$			
		$k\left(\frac{E}{G_c}\right)$			
	$\bar{\beta} = \frac{A_o}{h}$	2·5	20	30	50
$\frac{h}{2a} = \frac{1}{30}$	0	1·0030	1·0239	1·0355	1·0585
	0·2	1·0005	1·0213	1·0327	1·0555
	0·4	0·9930	1·0132	1·0245	1·0467
	0·6	0·9808	1·0003	1·0110	1·0326
	0·8	0·9647	0·9831	0·9934	1·0139
	1·0	0·9449	0·9622	0·9720	0·9910

Table 11

		$\frac{T^*}{T}$
$k\left(\frac{E}{G_c}\right) = 0$	$\bar{\beta} = \frac{A_o}{h}$	
	0	1
	0·2	0·9975
	0·4	0·9900
	0·6	0·9779
	0·8	0·9616
	1·0	0·9416

Note that absurd results are obtained by Berger's method for movable edge conditions.



$\frac{T^*}{T}$ Vs. $\bar{\beta}$ FOR
SIMPLY SUPPORTED SQUARE PLATE
(IMMOVABLE EDGES)
$\frac{h}{2a} = \frac{1}{10}$
= REF.[38]
= PRESENT STUDY.
$k(\frac{E}{G_c}) = 50$
$k(\frac{E}{G_c}) = 30$
$k(\frac{E}{G_c}) = 20$
$k(\frac{E}{G_c}) = 2.5$
T^*/T
1.50
1.40
1.30
1.20
1.10
1.00
0.9
0.8
0.7
0.6
0
0.2
0.4
0.6
0.8
1.0
$\bar{\beta}$

FIG.- 3

$\frac{T^*}{T}$ VS β FOR

SIMPLY SUPPORTED SQUARE PLATE
(IMMOVABLE EDGES)

—·— = REF [38]

—— = PRESENT STUDY.

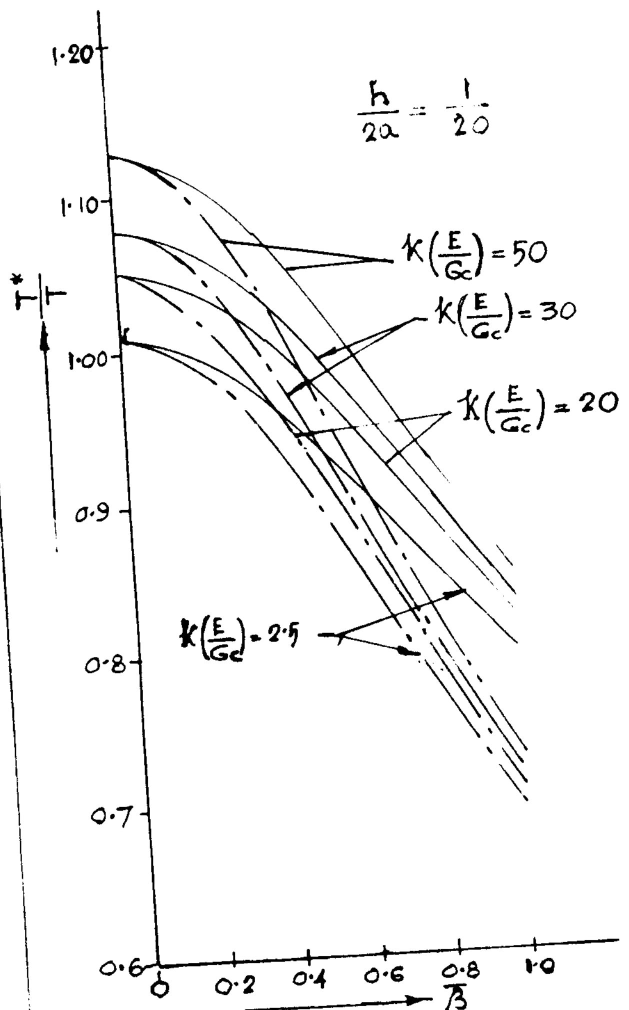


FIG.- 4

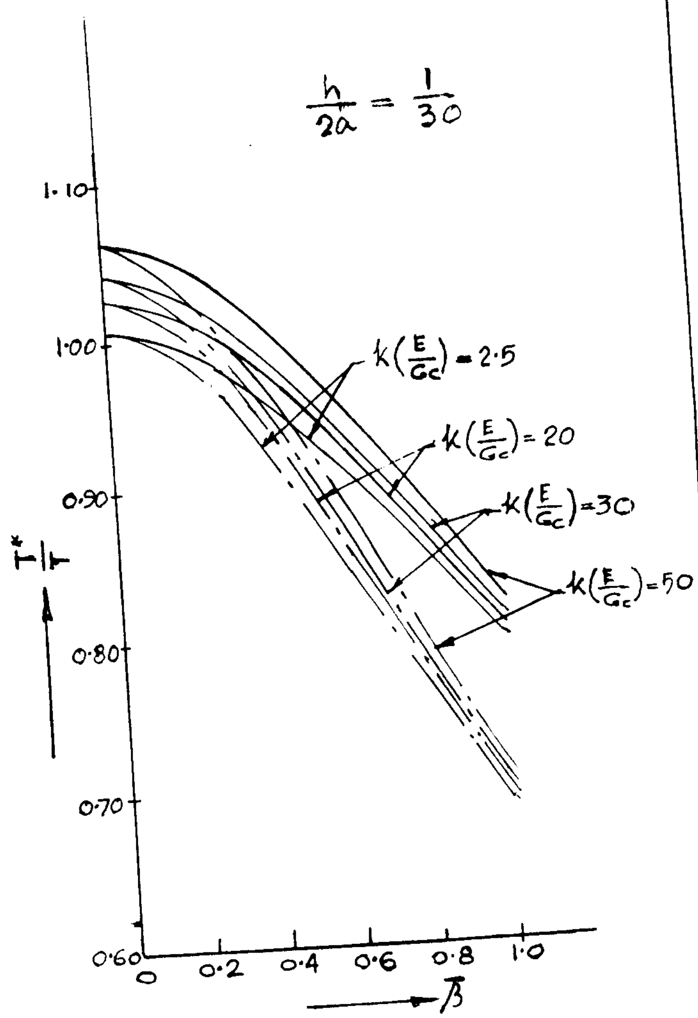


FIG.- 5

$\frac{T^*}{T}$ VS $\bar{\beta}$ FOR

SIMPLY SUPPORTED SQUARE PLATE.

(MOVABLE EDGES.)

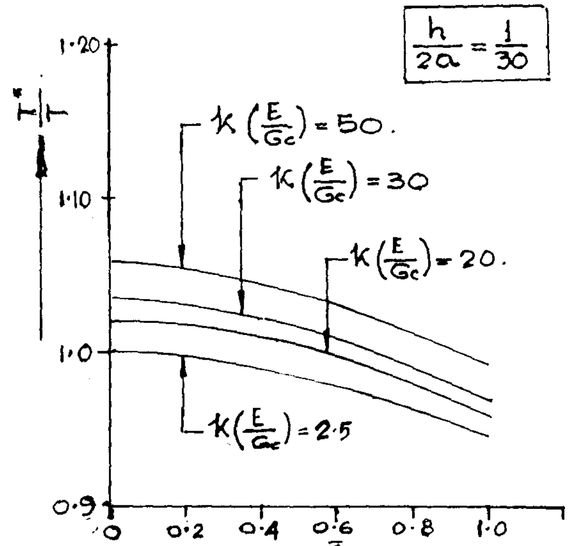


FIG.-8.

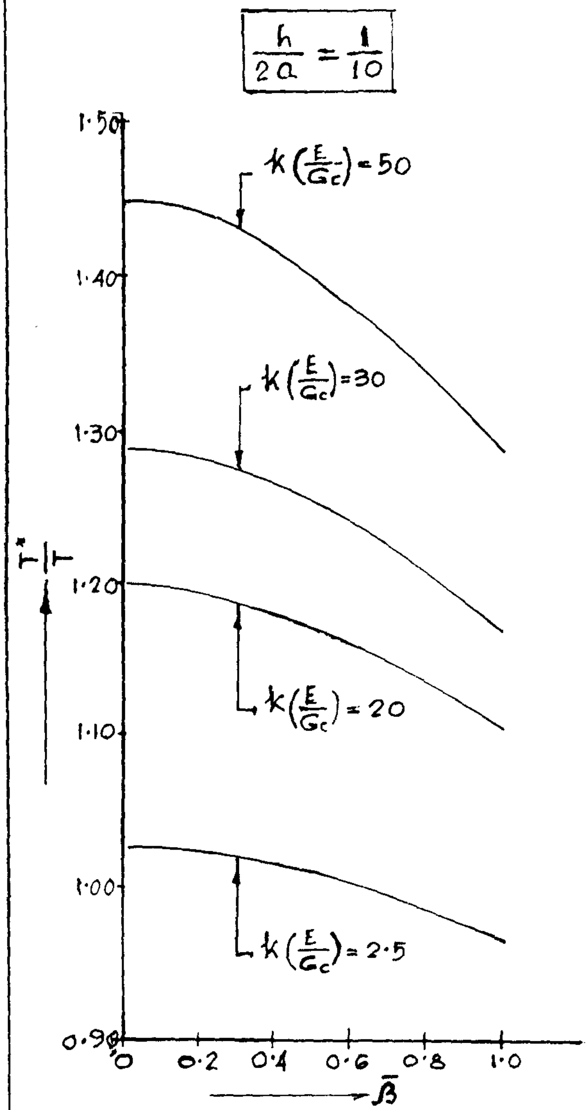


FIG.-6

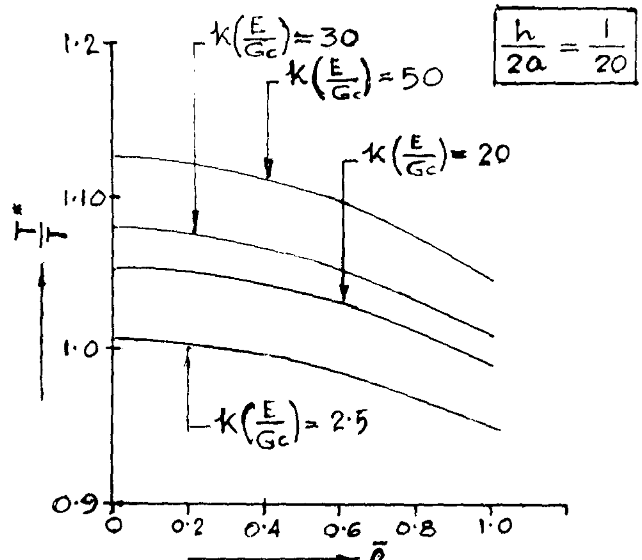


FIG.-7.

(ii) <u>Large deflections of uniformly loaded elliptical plate with clamped edges</u> :

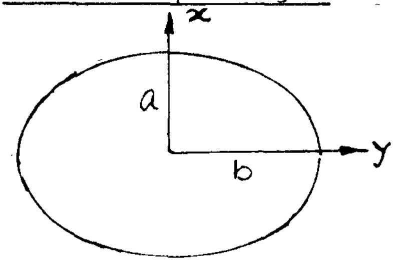


FIG.- 9

The plate geometry and co-ordinate system are shown in the fig.9

The elliptical plate with semi-axes a and b is clamped along the boundary whose equation is given by

$$1-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \qquad \ldots . \; (46)$$

For static deflection let us rewrite the differential equation 18(a) and 18(b) in the following forms by replacing the inertial term by the corresponding term due to mechanical loading.

$$\nabla^4 w+\frac{6}{5(1-\nu^2)}\cdot K\left(\frac{E}{G_c}\right)\cdot\frac{\bar{\alpha}^2 h^2}{12}\nabla^2\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$+\frac{3\lambda}{5(1-\nu^2)}\cdot K\cdot\left(\frac{E}{G_c}\right)\nabla^2\left[\nabla^2 w\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}\right.$$

$$\left.+2\left\{\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2\right\}+4\cdot\frac{\partial^2 w}{\partial x\,\partial y}\cdot\frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial x}\right]$$

$$-\bar{\alpha}^2\left[\frac{\partial^2 w}{\partial x^2}+\nu\cdot\frac{\partial^2 w}{\partial y^2}\right]-\frac{6\lambda}{h^2}\left[\nabla^2 w\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}+\right.$$

$$\left.2\left\{\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2\right\}+4\frac{\partial^2 w}{\partial x.\partial y}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right]=\frac{q_0}{D} \qquad \ldots . \; (47)$$

where q_0 is the intensity of continuously distributed load,

and

$$\frac{\bar{\alpha}^2 h^2}{12} = \frac{\partial u_0}{\partial x} + \nu \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

.... (48)

For movable edge condition

$$\bar{\alpha} = 0$$

Let us assume the deflection function in the following form

$$W = W_0\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]^2 \qquad \text{.... (49)}$$

Clearly this form of W satisfies the clamped edge conditions of the plate.

Now putting (49) in (48) and integrating over the area of the plate we get,

$$\bar{\alpha}^2 = \frac{4 W_0^2}{h^2}\left(\frac{1}{a^2} + \frac{\nu}{b^2}\right) \qquad \text{.... (50)}$$

Inserting (49) in (47), remembering (50) and applying Galerking's technique, as before, we get the cubic equation determining the deflection function $\frac{W_0}{h}$

$$\frac{W_0}{h} + \left[\frac{W_0}{h}\right]^3 \cdot \frac{1}{\frac{1}{a^4} + \frac{1}{b^4} + \frac{2}{3a^2 b^2}} \left[\frac{2h^2}{5(1-\nu^2)} \cdot k\left(\frac{E}{G_c}\right)\left(\frac{1}{a^2} + \frac{\nu}{b^2}\right)\right.$$

$$\left. \cdot \left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1+\nu}{3a^2 b^2}\right) + \frac{18\lambda h^2}{25(1-\nu^2)} \cdot k\left(\frac{E}{G_c}\right)\left(\frac{1}{a^6} + \frac{1}{b^6} + \frac{7}{9a^4 b^2} + \frac{7}{9a^2 b^4}\right)\right.$$

$$+\frac{1}{3}\left(\frac{1}{a^2}+\frac{\nu}{b^2}\right)^2+\frac{24\lambda}{35}\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{2}{3a^2b^2}\right)\Bigg]$$

$$=\frac{q_0}{2}\cdot\frac{(1-\nu^2)}{Eh^4}\cdot\frac{1}{\frac{1}{a^4}+\frac{1}{b^4}+\frac{2}{3a^2b^2}} \qquad \ldots. (51)$$

(iii) Large deflections of simply supported isosceles right angled triangular plate :

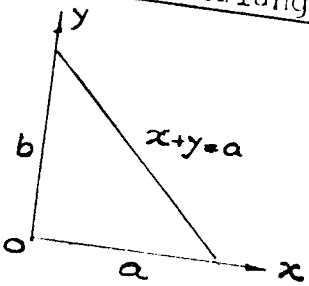


FIG-10

The plate geometry and co-ordinate system are shown in the fig.10

Here the deflection function is chosen in the following form

$$W = W_0\left[\sin\frac{\pi x}{a}\cdot\sin\frac{2\pi y}{a}+\sin\frac{\pi y}{a}\cdot\sin\frac{2\pi x}{a}\right] \qquad \ldots. (52)$$

This form of W satisfies the following simply supported edge conditions, namely -

$$\left.\begin{aligned}
W &= 0 \quad \text{at} \quad x = 0, a\\
W &= 0 \quad \text{at} \quad y = 0, a\\
\frac{\partial^2 W}{\partial x^2} &= 0 \quad \text{at} \quad x = 0, a\\
\frac{\partial^2 W}{\partial y^2} &= 0 \quad \text{at} \quad y = 0, a\\
W &= 0 \quad \text{at} \quad x + y = a\\
\text{and} \quad \frac{\partial^2 W}{\partial \nu^2} &= 0 \quad \text{at} \quad x + y = a\\
\text{where} \quad \frac{\partial}{\partial \nu} &\equiv \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)
\end{aligned}\right\} \qquad \ldots. (53)$$

Putting (52) in (48) and using the same method as in the case of elliptic plate we get

$$\bar{\alpha}^2 = \frac{15}{2} \cdot \frac{W_o^2 \pi^2}{a^2 h^2} \cdot (1+\nu) \qquad \ldots. (54)$$

Now inserting (52) in (48), remembering (54) and applying Galerkin's technique as before we get the following cubic equation determining $\left(\frac{W_o}{h}\right)$

$$\frac{W_o}{h} + \left(\frac{W_o}{h}\right)^3 \Big[6{\cdot}8665\, k \left(\frac{E}{G_c}\right) \cdot \frac{h^2}{a^2} + 4{\cdot}875\lambda + 1{\cdot}2675 + 18{\cdot}6086\, \lambda k \left(\frac{E}{G_c}\right) \cdot \frac{h^2}{a^2} \Big]$$

$$= \frac{48(1-\nu^2)}{25\pi^6} \cdot \frac{q_o a^4}{E h^4} \qquad \ldots. (55)$$

Numerical Results :

Numerical results are presented here in tabular forms both for movable as well as immovable edges for different moderately thick isotropic plates and compared with other known results. The results of the isosceles right angled triangular plates are new.

For free vibrations the ratios of the non-linear period T* of vibrations including the effects of transverse shear deformation to the corresponding linear period T of the classical plate (not including transverse shear and rotatory inertia) are computed for various thickness parameter and material constants at different nondimensional amplitudes of vibration. It is to be noted that the effects of rotatory inertia have been neglected in each case because these are considered to be small compared with effects due to transverse shear deformation as the plate is undergoing flexural vibrations.

To study the non-linear static behaviours of the plates the nondimensional deflection functions at the centre $\frac{W_o}{h}$ have been obtained for different values of the nondimensional load parameter $\frac{q.a^4}{Dh}$.

It is observed that for moderately thick plates, the non-linear periods are dependent on the thickness parameter whereas they are independent of the same for thin plates.

STATIC DEFLECTIONS OF CLAMPED ELLIPTICAL PLATE.

TABLE - 12. $\lambda = 0{\cdot}18\,[18],\ k\left(\frac{E}{G_c}\right) = 1.$

$\frac{a}{b}=1$ $\frac{q_0 a^4}{Eh^4}$ REF [46]	$\frac{W_0}{h}$ REF. [46]	$\frac{W_0}{h}$ IMMOVABLE EDGES. PRESENT STUDY	$\frac{W_0}{h}$ MOVABLE EDGES PRESENT STUDY.	$\frac{a}{b}=1{\cdot}5$ $\frac{q_0 a^4}{Eh^4}$ REF. [46]	$\frac{W_0}{h}$ REF. [46]	$\frac{W_0}{h}$ IMMOVABLE EDGES PRESENT STUDY	$\frac{W_0}{h}$ MOVABLE EDGES PRESENT STUDY.	$\frac{a}{b}=2{\cdot}0$ $\frac{q_0 a^4}{Eh^4}$ REF. [46]	$\frac{W_0}{h}$ REF. [46]	$\frac{W_0}{h}$ IMMOVABLE EDGES PRESENT STUDY.	$\frac{W_0}{h}$ MOVABLE EDGES PRESENT STUDY.
3·2756	0·5	0·5123	0·5392	9·2855	0·5	0·5220	0·5385	24·1379	0·5	0·5263	0·5382
8·6227	1·0	1·0619	1·2350	24·4213	1·0	1·1138	1·2320	63·4326	1·0	1·1404	1·2281
18·1126	1·5	1·6235	2·0346	51·2577	1·5	1·7365	2·0270	133·0412	1·5	1·7975	2·0170
33·8169	2·0	2·1885	2·8548	95·6450	2·0	2·3622	2·8420	248·1205	2·0	2·4595	2·8253.

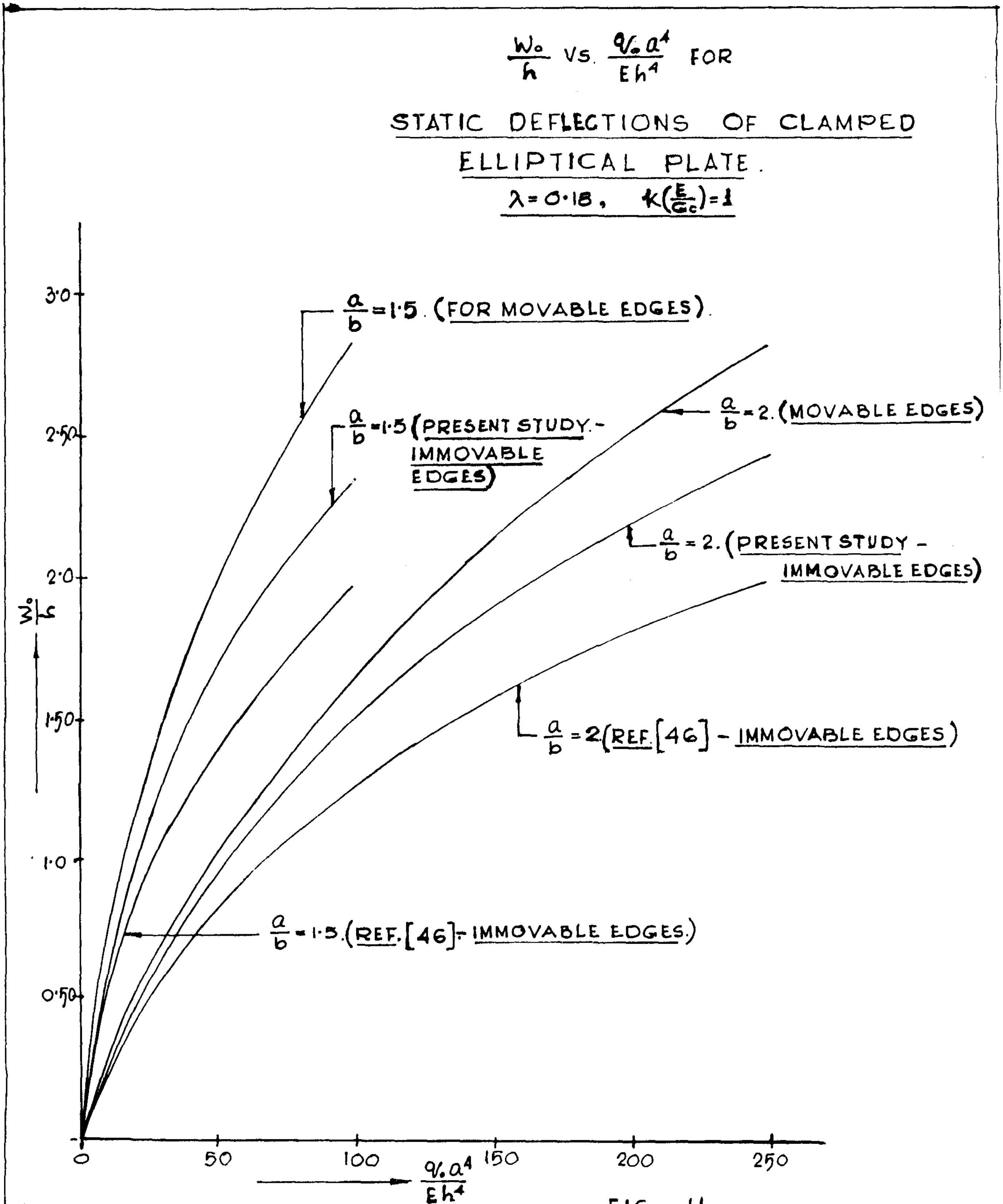

$\frac{W_o}{h}$ VS. $\frac{q_o a^4}{E h^4}$ FOR
STATIC DEFLECTIONS OF CLAMPED
ELLIPTICAL PLATE.
$\lambda = 0.18$, $k(\frac{E}{G_c}) = 1$
$\frac{a}{b} = 1.5$. (FOR MOVABLE EDGES).
$\frac{a}{b} = 1.5$ (PRESENT STUDY - IMMOVABLE EDGES)
$\frac{a}{b} = 2$. (MOVABLE EDGES)
$\frac{a}{b} = 2$. (PRESENT STUDY - IMMOVABLE EDGES)
$\frac{a}{b} = 2$ (REF. [46] - IMMOVABLE EDGES)
$\frac{a}{b} = 1.5$. (REF. [46] - IMMOVABLE EDGES.)
3.0
2.50
2.0
1.50
1.0
0.50
$\frac{W_o}{h}$
0
50
100
150
200
250
$\frac{q_o a^4}{E h^4}$

FIG. - 11

STATIC DEFLECTIONS OF SIMPLY SUPPORTED ISOSCELES RIGHT ANGLED TRIANGULAR PLATE.

TABLE-13 $\lambda = 0{\cdot}09$ [18], $K\left(\frac{E}{G_c}\right) = 1$.

$\frac{q_0 a^4}{E h^4}$	$\frac{W_0}{h}$	
	IMMOVABLE EDGES.	MOVABLE EDGES.
500	0·5735	0·7322
1000	0·8232	1·1428
1500	0·9909	1·4225
2000	1·1210	1·6393

$\frac{W_o}{h}$ VS $\frac{q_o a^4}{E h^4}$ FOR

STATIC DEFLECTIONS OF SIMPLY SUPPORTED RIGHT ANGLED ISOSCELES TRIANGULAR PLATE.

$\lambda = 0.18$, $k\left(\frac{E}{G_c}\right) = 1$.

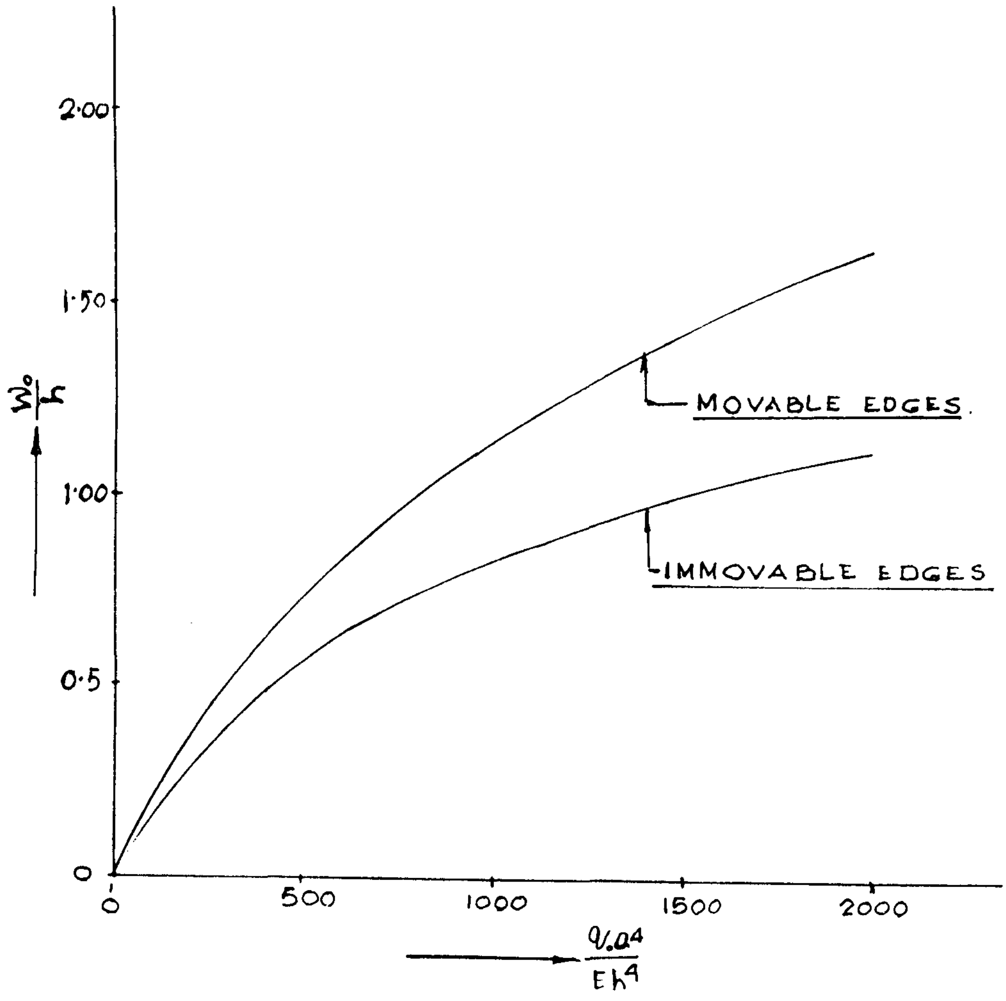


FIG.-12.

B. *Vibrations of clamped circular plates.

Let us consider a thick circular plate of radius a. The origin is located at the centre of the plates. The polar co-ordinates are chosen in the analysis. The deflection of the plate is of the same order of magnitude as the thickness of the plate.

For circular plate of radius a, let us choose the deflection function in the following form

$$W = A_0 \tau(t)\left[1-\frac{r^2}{a^2}\right]^2 \qquad \ldots\ldots (56)$$

clearly this form of W satisfies the following clamped edge conditions,

$$(W)_{r=a} = 0$$

and

$$\left(\frac{\partial w}{\partial r}\right)_{r=a} = 0$$

To evaluate the coupling parameter $\bar{\alpha}^2$, let us now recall our attention to equation 19(b) of Chapter I. Multiplying this equation by the integrating factor r^{ν}, putting (56) in this exact equation and finally integrating the equation between the limits 0 and a, the constant $\bar{\alpha}^2$ is obtained in the following form

$$\bar{\alpha}^2 = \frac{1536\nu}{a^{1+\nu}(3+\nu)(5+\nu)(7+\nu)} \cdot \frac{A_0^2}{h^2} \qquad \ldots\ldots (57)$$

Putting (56) in 19(a) of 1st Chapter, considering (57) and applying Galerkin's error minimising technique one gets the following differential equation for the time function $\tau(t)$

---

*Published in the Journal of Sound and Vibration (U.K.) 133(1), PP. 185 - 188, 1989.

$$\left[\frac{6}{5}\cdot\frac{a^2}{h^2C_p^2}+\frac{4}{5}\frac{\rho}{Gc}\right]\ddot{\tau}(t)+\frac{32}{3a^2}\cdot\tau(t)$$

$$+\left[864\cdot79872\,k\left(\frac{E}{Gc}\right)\frac{\nu}{(1-\nu^2)(\nu+5)(\nu+7)}\cdot\frac{A_o^2}{h^2}\cdot\frac{h^2}{a^4}\right.$$

$$+1541\cdot2224\cdot\frac{A_o^2}{a^2h^2}\cdot\frac{\nu}{(\nu+3)(\nu+5)(\nu+7)}$$

$$\left.+10\cdot24\lambda k\left(\frac{E}{Gc}\right)\cdot\frac{1}{(1-\nu^2)}\cdot\frac{A_o^2}{h^2}\cdot\frac{h^2}{a^4}+7\cdot3142\,\frac{\lambda}{a^2}\cdot\frac{A_o^2}{h^2}\right]\tau^3(t)=0$$

.... (58)

The ratio of the non-linear and linear time period is obtained as before in the form

$$\frac{T^*}{T}=\frac{2K}{\pi}\left[1\Big/\left\{1+81\cdot1255k\left(\frac{E}{Gc}\right)\frac{h^2}{a^2}\cdot\frac{\nu}{(1-\nu^2)(\nu+5)(\nu+7)}\bar{\beta}^2\right.\right.$$

$$+144\cdot4986\cdot\frac{\nu}{(\nu+3)(\nu+5)(\nu+7)}\bar{\beta}^2$$

$$\left.\left.+0\cdot9606\,\frac{\lambda}{(1-\nu^2)}\cdot k\left(\frac{E}{Gc}\right)\frac{h^2}{a^2}\bar{\beta}^2+0\cdot6857062\lambda\bar{\beta}^2\right\}\right]^{1/2}$$

.... (59)

Numerical results : -

Numerical results have been computed here in tabular form both for movable as well as immovable edge conditions as in the previous case.

RATIO OF NON-LINEAR TO LINEAR PERIOD FOR THE FUNDAMENTAL MODE OF VIBRATION OF A CLAMPED CIRCULAR PLATE.

$\nu = 0.3$, $\lambda = 0.18$

IMMOVABLE EDGES. (TABLE 14 TO 19)

PRESENT STUDY | REF. [40]

TABLE - 14

T^*/T					T^*/T				
$\bar{\beta} = \frac{A_0}{h}$	$\frac{h}{a} = 0.2$ $k\left(\frac{E}{G_c}\right) = 8.1971$	$\frac{h}{a} = 0.15$ $k\left(\frac{E}{G_c}\right) = 8.8133$	$\frac{h}{a} = 0.10$ $k\left(\frac{E}{G_c}\right) = 10.4869$	$\frac{h}{a} = 0.05$ $k\left(\frac{E}{G_c}\right) = 19.3165$	$\bar{\beta} = \frac{A_0}{h}$	$\frac{h}{a} = 0.20$ $k\left(\frac{E}{G_c}\right) = 8.1971$	$\frac{h}{a} = 0.15$ $k\left(\frac{E}{G_c}\right) = 8.8133$	$\frac{h}{a} = 0.10$ $k\left(\frac{E}{G_c}\right) = 10.4869$	$\frac{h}{a} = 0.05$ $k\left(\frac{E}{G_c}\right) = 19.3165$
0	1.0000	1.0000	1.0000	1.0000	0	1.0000	1.0000	1.0000	1.0000
0.20	0.9891	0.9907	0.9919	0.9926	0.2	0.9921	0.9924	0.9927	0.9928
0.40	0.9584	0.9661	0.9688	0.9716	0.4	0.9699	0.9710	0.9718	0.9722
0.60	0.9138	0.9251	0.9339	0.9394	0.6	0.9366	0.9388	0.9402	0.9410
0.80	0.8576	0.8774	0.8908	0.8993	0.8	0.8965	0.8995	0.9015	0.9026
1.00	0.8029	0.8257	0.8435	0.8547	1.0	0.8533	0.8568	0.8591	0.8603

TABLE - 15

THIN PLATE.

$\bar{\beta} = \frac{A_o}{h}$	PRESENT STUDY $\frac{T^*}{T}$, $k\left(\frac{E}{Gc}\right) = 0$	REF. [40] $\frac{T^*}{T}$
0	1·0000	1·0000
0·20	0·9933	0·9928
0·40	0·9739	0·9724
0·60	0·9441	0·9413
0·80	0·9068	0·9029
1·00	0·8648	0·8607

TABLE-16

$\bar{\beta} = \frac{A_o}{h}$	$k\left(\frac{E}{G_c}\right) = 8.1971$			
	$\frac{h}{a} = 0.2$ $\frac{T^*}{T}$	$\frac{h}{a} = 0.15$ $\frac{T^*}{T}$	$\frac{h}{a} = 0.10$ $\frac{T^*}{T}$	$\frac{h}{a} = 0.05$ $\frac{T^*}{T}$
0	1·0000	1·0000	1·0000	1·0000
0·20	0·9890	0·9909	0·9922	0·9930
0·40	0·9584	0·9651	0·9699	0·9729
0·60	0·9138	0·9263	0·9361	0·9421
0·80	0·8576	0·8793	0·8943	0·9036
1·00	0·8029	0·8283	0·8480	0·8605

TABLE-17

$\bar{\beta} = \frac{A_o}{h}$	$k\left(\frac{E}{G_c}\right) = 10.4869$			
	$\frac{h}{a} = 0.2$ $\frac{T^*}{T}$	$\frac{h}{a} = 0.15$ $\frac{T^*}{T}$	$\frac{h}{a} = 0.10$ $\frac{T^*}{T}$	$\frac{h}{a} = 0.05$ $\frac{T^*}{T}$
0	1·0000	1·0000	1·0000	1·0000
0·20	0·9879	0·9902	0·9919	0·9929
0·40	0·9542	0·9626	0·9688	0·9726
0·60	0·9051	0·9216	0·9339	0·9415
0·80	0·8477	0·8721	0·8908	0·9027
1·00	0·7879	0·8189	0·8435	0·8593

TABLE - 18

$\bar{\beta} = \frac{A_o}{h}$	$\frac{T^*}{T}$ FOR $\frac{h}{a} = 0.20$			
	$k\left(\frac{E}{G_c}\right) = 8.1971$	$k\left(\frac{E}{G_c}\right) = 8.8133$	$k\left(\frac{E}{G_c}\right) = 10.4869$	$k\left(\frac{E}{G_c}\right) = 19.3165$
0	1.0000	1.0000	1.0000	1.0000
0.20	0.9890	0.9887	0.9879	0.9834
0.40	0.9584	0.9572	0.9542	0.9385
0.60	0.9138	0.9110	0.9051	0.8758
0.80	0.8576	0.8564	0.8477	0.8061
1.00	0.8029	0.7988	0.7879	0.7369

TABLE - 19

$\bar{\beta} = \frac{A_o}{h}$	$\frac{T^*}{T}$ FOR $\frac{h}{a} = 0.10$			
	$k\left(\frac{E}{G_c}\right) = 8.1971$	$k\left(\frac{E}{G_c}\right) = 8.8133$	$k\left(\frac{E}{G_c}\right) = 10.4869$	$k\left(\frac{E}{G_c}\right) = 19.3165$
0	1.0000	1.0000	1.0000	1.0000
0.20	0.9922	0.9921	0.9919	0.9907
0.40	0.9699	0.9696	0.9688	0.9647
0.60	0.9361	0.9355	0.9339	0.9255
0.80	0.8943	0.8933	0.8908	0.8781
1.00	0.8480	0.8467	0.8435	0.8267

MOVABLE EDGES (TABLE 20 TO 24)

TABLE - 20.

$\bar{\beta} = \frac{A_o}{h}$	$k\left(\frac{E}{G_c}\right) = 8.1971$			
	$\frac{h}{a} = 0.20$	$\frac{h}{a} = 0.15$	$\frac{h}{a} = 0.10$	$\frac{h}{a} = 0.05$
0	1.0000	1.0000	1.0000	1.0000
0.20	0.9972	0.9976	0.9979	0.9981
0.40	0.9891	0.9914	0.9918	0.9924
0.60	0.9759	0.9793	0.9817	0.9833
0.80	0.9583	0.9461	0.9683	0.9708
1.00	0.9371	0.9456	0.9518	0.9555

TABLE - 21.

$\bar{\beta} = \frac{A_o}{h}$	$k\left(\frac{E}{G_c}\right) = 8.8133$			
	$\frac{h}{a} = 0.20$	$\frac{h}{a} = 0.15$	$\frac{h}{a} = 0.10$	$\frac{h}{a} = 0.05$
0	1.0000	1.0000	1.0000	1.0000
0.20	0.9972	0.9976	0.9979	0.9981
0.40	0.9888	0.9905	0.9917	0.9924
0.60	0.9753	0.9790	0.9816	0.9832
0.80	0.9573	0.9635	0.9680	0.9707
1.00	0.9357	0.9447	0.9514	0.9554

TABLE – 22

$\bar{\beta} = \frac{A_o}{h}$	$\frac{T^*}{T}$ FOR $\frac{h}{a} = 0.20$			
	$k\left(\frac{E}{G_c}\right) = 8.1971$	$k\left(\frac{E}{G_c}\right) = 8.8133$	$k\left(\frac{E}{G_c}\right) = 10.4869$	$k\left(\frac{E}{G_c}\right) = 19.3165$
0	1.0000	1.0000	1.0000	1.0000
0.20	0.9972	0.9971	0.9969	0.9959
0.40	0.9890	0.9887	0.9880	0.9842
0.60	0.9759	0.9753	0.9737	0.9655
0.80	0.9583	0.9573	0.9547	0.9410
1.00	0.9371	0.9357	0.9318	0.9123

TABLE – 23

$\bar{\beta} = \frac{A_o}{h}$	$\frac{T^*}{T}$ FOR $\frac{h}{a} = 0.10$			
	$k\left(\frac{E}{G_c}\right) = 8.1971$	$k\left(\frac{E}{G_c}\right) = 8.8133$	$k\left(\frac{E}{G_c}\right) = 10.4869$	$k\left(\frac{E}{G_c}\right) = 19.3165$
0	1.0000	1.0000	1.0000	1.0000
0.20	0.9979	0.9979	0.9978	0.9976
0.40	0.9918	0.9917	0.9915	0.9905
0.60	0.9817	0.9816	0.9812	0.9791
0.80	0.9683	0.9680	0.9673	0.9542
1.00	0.9518	0.9514	0.9504	0.9450

TABLE - 24

$\bar{\beta} = \frac{A_0}{h}$	$\frac{T^*}{T}$ FOR $\nu = 0.3$, $\lambda = 0.18$			
	$\frac{h}{a} = 0.20$ $k\left(\frac{E}{G_c}\right) = 8.1971$	$\frac{h}{a} = 0.15$ $k\left(\frac{E}{G_c}\right) = 8.8133$	$\frac{h}{a} = 0.10$ $k\left(\frac{E}{G_c}\right) = 10.4869$	$\frac{h}{a} = 0.05$ $k\left(\frac{E}{G_c}\right) = 19.3165$
0	1.0000	1.0000	1.0000	1.0000
0.20	0.9972	0.9976	0.9978	0.9980
0.40	0.9890	0.9904	0.9915	0.9921
0.60	0.9759	0.9789	0.9812	0.9826
0.80	0.9583	0.9635	0.9673	0.9697
1.00	0.9371	0.9447	0.9504	0.9538

CLAMPED CIRCULAR PLATE. (IMMOVABLE EDGES)

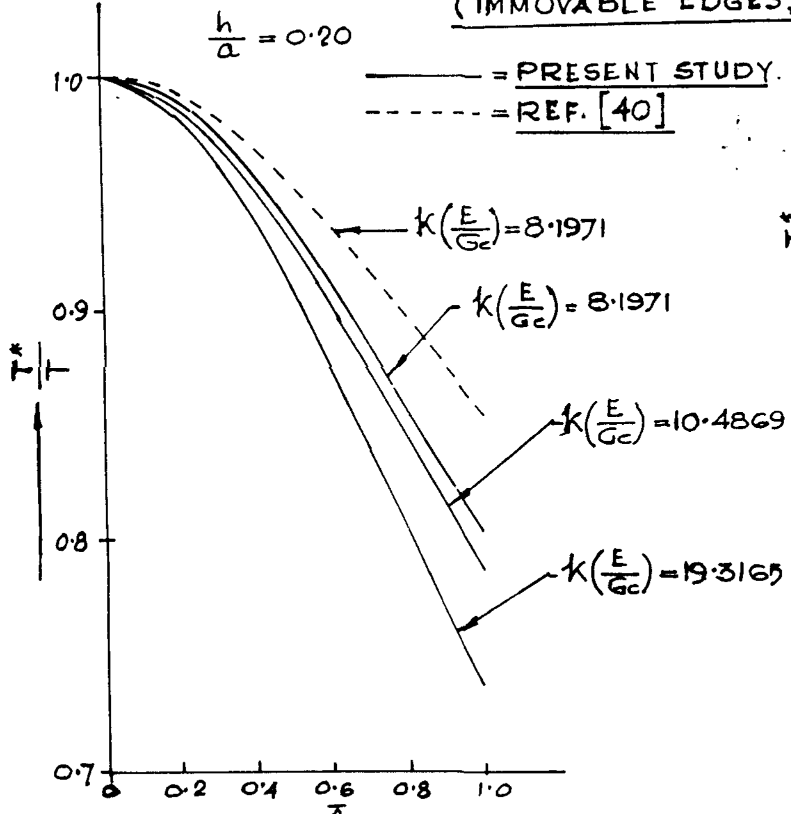


FIG.-13

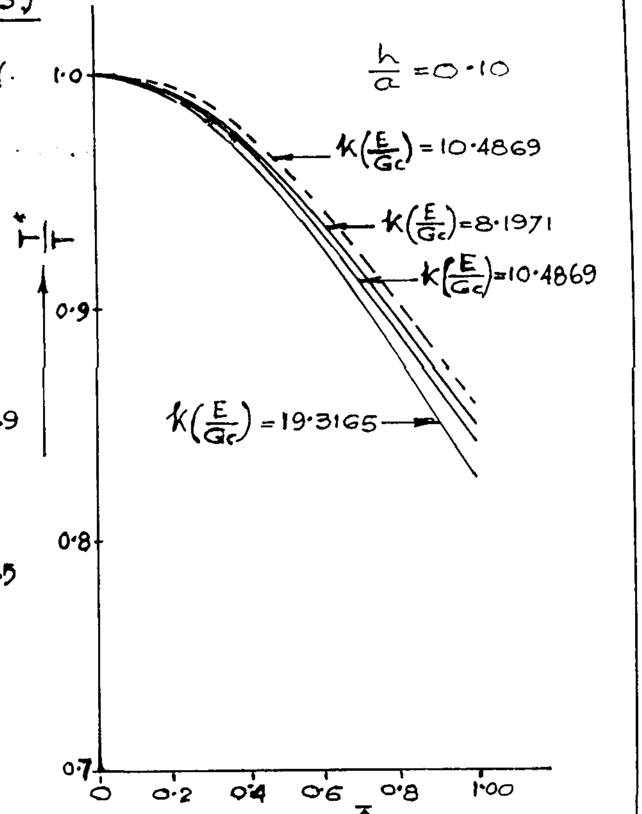


FIG.-14

CLAMPED CIRCULAR PLATE (MOVABLE EDGES)

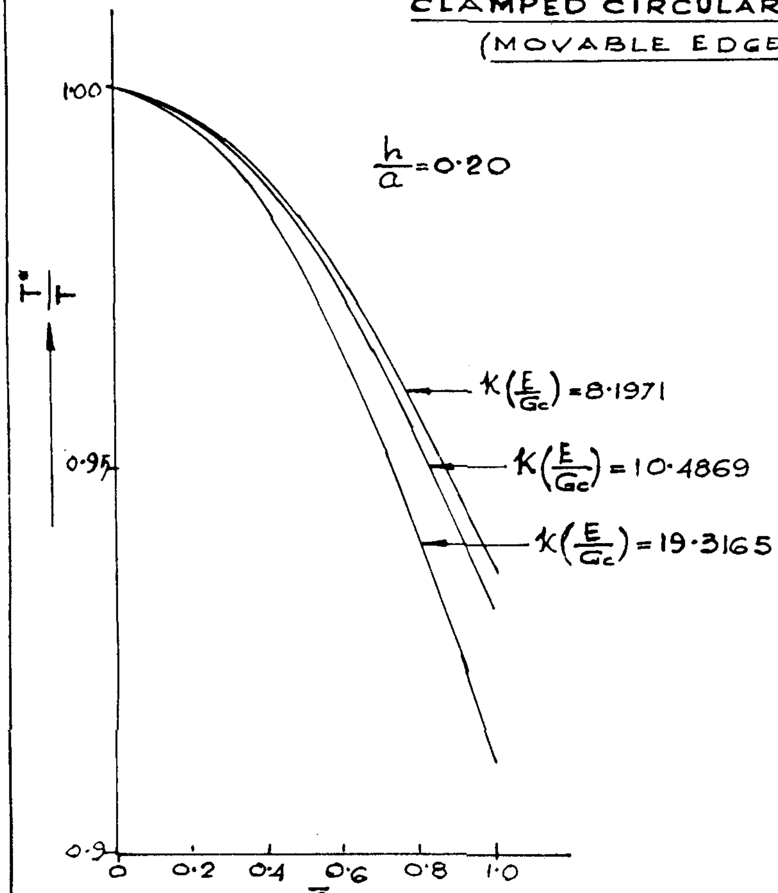


FIG.- 15

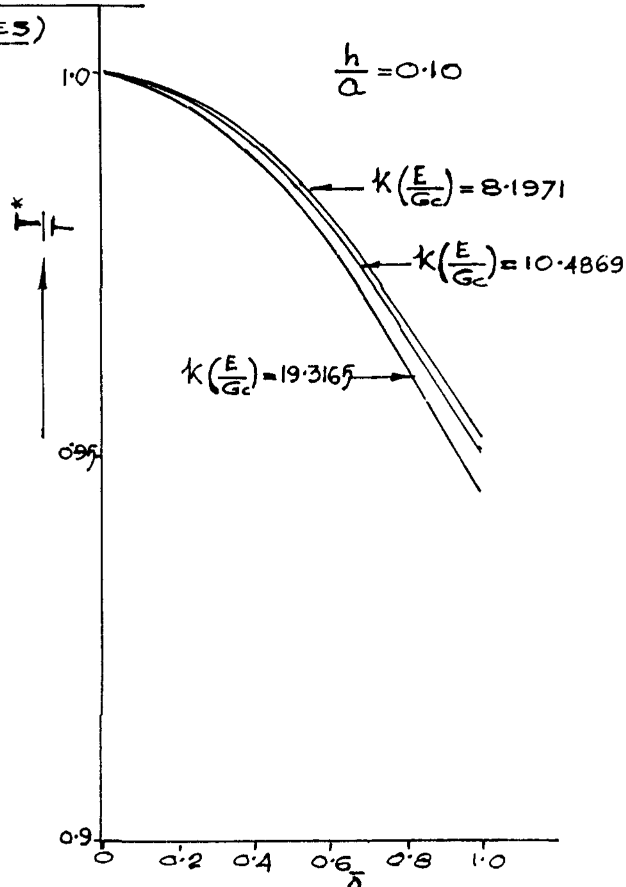


FIG.- 16

C. Large Amplitudes, Transverse shear Deformation and Rotatory Inertia on Free Vibrations of Moderately Thick Polygonal Plates*

Formulation of the differential equation :

Let us consider the free vibrations of thick polygonal plates of thickness h .

In a complex co-ordinate system $z = x+iy$, $\bar{z} = x-iy$ the equations 18(a) and 18(b) of the 1st chapter change. Let

$$z = f(\xi) \qquad \ldots \qquad \ldots \qquad \ldots . \quad (60)$$

be the analytic function which maps the given shape in the z-plane on to a unit circle in the ξ -plane. Substituting the relation (60) into the transformed equations in $(z, \bar{z})$ the following set of differential equations in (ξ , $\bar{\xi}$) co-ordinates have been obtained:

$$16\left[\frac{\partial^4 w}{\partial\xi^2\partial\bar{\xi}^2}\cdot\left(\frac{dz}{d\xi}\right)^3\cdot\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3-\frac{\partial^3 w}{\partial\xi^2\partial\bar{\xi}}\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\cdot\left(\frac{dz}{d\xi}\right)^3\right.$$
$$\left.-\frac{\partial^3 w}{\partial\bar{\xi}^2\partial\xi}\cdot\frac{d^2 z}{d\xi^2}\cdot\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3+\frac{\partial^2 w}{\partial\xi\partial\bar{\xi}}\cdot\frac{d^2 z}{d\xi^2}\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\cdot\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\right]$$
$$+\frac{2}{5(1-\nu^2)}k\left(\frac{E}{Gc}\right)\bar{\alpha}^2 h^2\tau^2(t)\left[(1-\nu)\left\{\frac{\partial^4 w}{\partial\xi^3\partial\bar{\xi}}\cdot\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^4\cdot\left(\frac{dz}{d\xi}\right)^2\right.\right.$$
$$-3\frac{\partial^3 w}{\partial\xi^2\partial\bar{\xi}}\cdot\frac{d^2 z}{d\xi^2}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^4\cdot\frac{dz}{d\xi}+3\frac{\partial^2 w}{\partial\xi\partial\bar{\xi}}\left(\frac{d^2 z}{d\xi^2}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^4$$
$$\left.-\frac{\partial^2 w}{\partial\xi\partial\bar{\xi}}\cdot\frac{d^3 z}{d\xi^3}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^4\cdot\frac{dz}{d\xi}\right\}+(1-\nu)\left\{\frac{\partial^4 w}{\partial\bar{\xi}^3\partial\xi}\cdot\left(\frac{dz}{d\xi}\right)^4\cdot\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\right.$$

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* Accepted for publication in the Journal of Applied Mechanics (ASME) - U.S.A, June 1990.

$$-3\frac{\partial^3 w}{\partial \xi^2 \partial \bar{\xi}}\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\left(\frac{dz}{d\xi}\right)^4\frac{d\bar{z}}{d\bar{\xi}}+3\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\left(\frac{d^2\bar{z}}{d\bar{\xi}^2}\right)^2\left(\frac{dz}{d\xi}\right)^4$$

$$-\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\cdot\frac{d^3\bar{z}}{d\bar{\xi}^3}\cdot\left(\frac{dz}{d\xi}\right)^4\cdot\frac{d\bar{z}}{d\bar{\xi}}\Bigg\}+2(1+\nu)\Bigg\{\frac{\partial^4 w}{\partial \xi^2 \partial \bar{\xi}^2}\left(\frac{dz}{d\xi}\right)^3\cdot\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3$$

$$-\frac{\partial^3 w}{\partial \xi^2 \partial \bar{\xi}}\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\cdot\left(\frac{dz}{d\xi}\right)^3-\frac{\partial^3 w}{\partial \bar{\xi}^2 \partial \xi}\cdot\frac{d^2 z}{d\xi^2}\cdot\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3$$

$$+\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\cdot\frac{d^2 z}{d\xi^2}\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\cdot\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\Bigg]$$

$$+\frac{96\lambda}{5(1-\nu^2)}k\left(\frac{E}{Gc}\right)\Bigg[4\Bigg\{\frac{\partial^4 w}{\partial \xi^2 \partial \bar{\xi}^2}\cdot\frac{\partial w}{\partial \xi}\cdot\frac{\partial w}{\partial \bar{\xi}}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\left(\frac{dz}{d\xi}\right)^2$$

$$-\frac{\partial^3 w}{\partial \xi^2 \partial \bar{\xi}}\cdot\frac{\partial w}{\partial \xi}\cdot\frac{\partial w}{\partial \bar{\xi}}\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\cdot\frac{d\bar{z}}{d\bar{\xi}}\cdot\left(\frac{dz}{d\xi}\right)^2$$

$$-\frac{\partial^3 w}{\partial \xi \partial \bar{\xi}^2}\cdot\frac{\partial w}{\partial \xi}\cdot\frac{\partial w}{\partial \bar{\xi}}\cdot\frac{d^2 z}{d\xi^2}\quad\frac{dz}{d\xi}\cdot\left(\frac{dz}{d\bar{\xi}}\right)^2$$

$$+\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\cdot\frac{\partial w}{\partial \xi}\cdot\frac{\partial w}{\partial \bar{\xi}}\cdot\frac{d^2 z}{d\xi^2}\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\cdot\frac{dz}{d\xi}\cdot\frac{d\bar{z}}{d\bar{\xi}}\Bigg\}$$

$$+\Bigg\{\frac{\partial^4 w}{\partial \bar{\xi}^3 \partial \xi}\left(\frac{\partial w}{\partial \xi}\right)^2\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2-3\frac{\partial^3 w}{\partial \bar{\xi}^2 \partial \xi}\left(\frac{\partial w}{\partial \xi}\right)^2\frac{d^2\bar{z}}{d\bar{\xi}^2}\left(\frac{dz}{d\xi}\right)^2\cdot\frac{d\bar{z}}{d\bar{\xi}}$$

$$+3\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\left(\frac{\partial w}{\partial \xi}\right)^2\left(\frac{d^2\bar{z}}{d\bar{\xi}^2}\right)^2\left(\frac{dz}{d\xi}\right)^2-\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\left(\frac{\partial w}{\partial \xi}\right)^2\frac{d^3\bar{z}}{d\bar{\xi}^3}\left(\frac{dz}{d\xi}\right)^2\frac{d\bar{z}}{d\bar{\xi}}\Bigg\}$$

$$+\Bigg\{\frac{\partial^4 w}{\partial \xi^3 \partial \bar{\xi}}\cdot\left(\frac{\partial w}{\partial \bar{\xi}}\right)^2\cdot\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\cdot\left(\frac{dz}{d\xi}\right)^2-3\frac{\partial^3 w}{\partial \xi^2 \partial \bar{\xi}}\left(\frac{\partial w}{\partial \bar{\xi}}\right)^2\frac{d^2 z}{d\xi^2}\cdot\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\cdot\frac{dz}{d\xi}$$

$$+3\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\left(\frac{\partial w}{\partial \bar{\xi}}\right)^2\left(\frac{d^2 z}{d\xi^2}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2-\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\cdot\left(\frac{\partial w}{\partial \bar{\xi}}\right)^2\cdot\frac{d^3 z}{d\xi^3}\cdot\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\cdot\frac{dz}{d\xi}\Bigg\}$$

$$+4\left(\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\right)^3\cdot\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2+6\Bigg\{\frac{\partial^3 w}{\partial \bar{\xi}^2 \partial \xi}\cdot\frac{\partial^2 w}{\partial \xi^2}\cdot\frac{\partial w}{\partial \bar{\xi}}\cdot\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2$$

$$- \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}} \cdot \frac{\partial^2 w}{\partial \xi^2} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^2 \bar{z}}{d \bar{\xi}^2} \left(\frac{dz}{d\xi}\right)^2 \cdot \frac{d\bar{z}}{d\bar{\xi}} - \frac{\partial^3 w}{\partial \bar{\xi}^2 \partial \xi} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^2 z}{d\xi^2}$$

$$\cdot \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 \cdot \frac{dz}{d\xi} + \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{d^2 \bar{z}}{d\bar{\xi}^2} \cdot \frac{dz}{d\xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} \Bigg\}$$

$$+ 6 \Bigg\{ \frac{\partial^3 w}{\partial \xi^2 \partial \bar{\xi}} \cdot \frac{\partial^2 w}{\partial \bar{\xi}^2} \cdot \frac{\partial w}{\partial \xi} \left(\frac{dz}{d\xi}\right)^2 \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 - \frac{\partial^2 w}{\partial \xi \cdot \partial \bar{\xi}} \cdot \frac{\partial^2 w}{\partial \bar{\xi}^2} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{dz}{d\xi} \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2$$

$$- \frac{\partial^3 w}{\partial \xi^2 \partial \bar{\xi}} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^2 \bar{z}}{d\bar{\xi}^2} \left(\frac{dz}{d\xi}\right)^2 \cdot \frac{d\bar{z}}{d\bar{\xi}} + \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{d^2 \bar{z}}{d\bar{\xi}^2} \cdot \frac{dz}{d\xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} \Bigg\}$$

$$+ 10 \Bigg\{ \frac{\partial^3 w}{\partial \xi^2 \partial \bar{\xi}} \cdot \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}} \cdot \frac{\partial w}{\partial \bar{\xi}} \left(\frac{dz}{d\xi}\right)^2 \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 - \left(\frac{\partial^2 w}{\partial \xi \partial \bar{\xi}}\right)^2 \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^2 z}{d\xi^2} \cdot \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 \cdot \frac{dz}{d\xi} \Bigg\}$$

$$+ 10 \Bigg\{ \frac{\partial^3 w}{\partial \bar{\xi}^2 \partial \xi} \cdot \frac{\partial^2 w}{\partial \xi \cdot \partial \bar{\xi}} \cdot \frac{\partial w}{\partial \xi} \left(\frac{dz}{d\xi}\right)^2 \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 - \left(\frac{\partial^2 w}{\partial \xi \cdot \partial \bar{\xi}}\right)^2 \cdot \frac{\partial w}{\partial \xi} \cdot \frac{d^2 \bar{z}}{d\bar{\xi}^2} \cdot \left(\frac{dz}{d\xi}\right)^2 \cdot \frac{d\bar{z}}{d\bar{\xi}} \Bigg\}$$

$$+ 8 \Bigg\{ \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}} \cdot \frac{\partial^2 w}{\partial \xi^2} \cdot \frac{\partial^2 w}{\partial \bar{\xi}^2} \left(\frac{dz}{d\xi}\right)^2 \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 - \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}} \cdot \frac{\partial^2 w}{\partial \bar{\xi}^2} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{dz}{d\xi} \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2$$

$$- \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}} \cdot \frac{\partial^2 w}{\partial \xi^2} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^2 \bar{z}}{d\bar{\xi}^2} \left(\frac{dz}{d\xi}\right)^2 \cdot \frac{d\bar{z}}{d\bar{\xi}} + \frac{\partial^2 w}{\partial \xi \partial \bar{\xi}} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^2 \bar{z}}{d\bar{\xi}^2} \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{dz}{d\xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} \Bigg\}$$

$$+ 2 \Bigg[\Bigg\{ \frac{\partial^3 w}{\partial \bar{\xi}^3} \cdot \frac{\partial^2 w}{\partial \xi^2} \cdot \frac{\partial w}{\partial \xi} \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 \left(\frac{dz}{d\xi}\right)^2 - 3 \frac{\partial^2 w}{\partial \xi^2} \cdot \frac{\partial^2 w}{\partial \bar{\xi}^2} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{d^2 \bar{z}}{d\bar{\xi}^2} \cdot \frac{d\bar{z}}{d\bar{\xi}} \cdot \left(\frac{dz}{d\xi}\right)^2$$

$$+ 3 \frac{\partial^2 w}{\partial \xi^2} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \left(\frac{d^2 \bar{z}}{d\bar{\xi}^2}\right)^2 \left(\frac{dz}{d\xi}\right)^2 - \frac{\partial^2 w}{\partial \xi^2} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^3 \bar{z}}{d\bar{\xi}^3} \cdot \left(\frac{dz}{d\xi}\right)^2 \cdot \frac{d\bar{z}}{d\bar{\xi}} \Bigg\}$$

$$+ \Bigg\{ - \frac{\partial^3 w}{\partial \bar{\xi}^3} \left(\frac{\partial w}{\partial \xi}\right)^2 \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{dz}{d\xi} \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 + 3 \frac{\partial^2 w}{\partial \bar{\xi}^2} \left(\frac{\partial w}{\partial \xi}\right)^2 \frac{d^2 \bar{z}}{d\bar{\xi}^2} \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{dz}{d\xi} \cdot \frac{d\bar{z}}{d\bar{\xi}}$$

$$- 3 \frac{\partial w}{\partial \bar{\xi}} \left(\frac{\partial w}{\partial \xi}\right)^2 \left(\frac{d^2 \bar{z}}{d\bar{\xi}^2}\right)^2 \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{dz}{d\xi} + \frac{\partial w}{\partial \bar{\xi}} \left(\frac{\partial w}{\partial \xi}\right)^2 \cdot \frac{d^3 \bar{z}}{d\bar{\xi}^3} \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{dz}{d\xi} \cdot \frac{d\bar{z}}{d\bar{\xi}} \Bigg\} \Bigg]$$

$$+ 2 \Bigg[\Bigg\{ \frac{\partial^3 w}{\partial \xi^3} \cdot \frac{\partial^2 w}{\partial \bar{\xi}^2} \cdot \frac{\partial w}{\partial \bar{\xi}} \left(\frac{dz}{d\xi}\right)^2 \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 - 3 \frac{\partial^2 w}{\partial \xi^2} \cdot \frac{\partial^2 w}{\partial \bar{\xi}^2} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^2 z}{d\xi^2} \cdot \frac{dz}{d\xi} \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2$$

$$+ 3 \frac{\partial^2 w}{\partial \bar{\xi}^2} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \left(\frac{d^2 z}{d\xi^2}\right)^2 \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 - \frac{\partial^2 w}{\partial \bar{\xi}^2} \cdot \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot \frac{d^3 z}{d\xi^3} \left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2 \cdot \frac{dz}{d\xi} \Bigg\}$$

$$+ \Bigg\{ - \frac{\partial^3 w}{\partial \xi^3} \left(\frac{\partial w}{\partial \bar{\xi}}\right)^2 \frac{d^2 \bar{z}}{d\bar{\xi}^2} \left(\frac{dz}{d\xi}\right)^2 \frac{d\bar{z}}{d\bar{\xi}} + 3 \frac{\partial^2 w}{\partial \xi^2} \left(\frac{\partial w}{\partial \bar{\xi}}\right)^2 \frac{d^2 z}{d\xi^2} \cdot \frac{d^2 \bar{z}}{d\bar{\xi}^2} \cdot \frac{d\bar{z}}{d\bar{\xi}} \cdot \frac{dz}{d\xi}$$

$$-3\frac{\partial w}{\partial \xi}\left(\frac{\partial w}{\partial \bar{\xi}}\right)^2\left(\frac{d^2 z}{d\xi^2}\right)^2\frac{d^2\bar{z}}{d\bar{\xi}^2}\cdot\frac{d\bar{z}}{d\bar{\xi}}+\frac{\partial w}{\partial \xi}\left(\frac{\partial w}{\partial \bar{\xi}}\right)^2\frac{d^3 z}{d\xi^3}\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\cdot\frac{d\bar{z}}{d\bar{\xi}}\cdot\frac{dz}{d\xi}\Bigg\}\Bigg]$$

$$-\frac{24}{5}\frac{\rho}{Gc}\ddot{\tau}(t)\frac{\partial^2 w}{\partial\xi\,\partial\bar{\xi}}\left(\frac{dz}{d\xi}\right)^4\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^4-\bar{\alpha}^2\tau^2(t)\Bigg[(1-\nu)\Bigg\{\frac{\partial^2 w}{\partial\xi^2}\left(\frac{dz}{d\xi}\right)^3\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^5$$

$$-\frac{\partial w}{\partial\xi}\cdot\frac{d^2 z}{d\xi^2}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^5\Bigg\}+(1-\nu)\Bigg\{\frac{\partial^2 w}{\partial\bar{\xi}^2}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3\left(\frac{dz}{d\xi}\right)^5-\frac{\partial w}{\partial\bar{\xi}}\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\left(\frac{dz}{d\xi}\right)^5\Bigg\}$$

$$+2(1+\nu)\frac{\partial^2 w}{\partial\xi\,\partial\bar{\xi}}\left(\frac{dz}{d\xi}\right)^4\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^4\Bigg]$$

$$-\frac{48\lambda}{h^2}\Bigg[4\Bigg\{\frac{\partial^2 w}{\partial\xi\,\partial\bar{\xi}}\cdot\frac{\partial w}{\partial\xi}\cdot\frac{\partial w}{\partial\bar{\xi}}\left(\frac{dz}{d\xi}\right)^3\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3\Bigg\}+\frac{\partial^2 w}{\partial\bar{\xi}^2}\left(\frac{\partial w}{\partial\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3\left(\frac{dz}{d\xi}\right)^3$$

$$+\frac{\partial^2 w}{\partial\xi^2}\left(\frac{\partial w}{\partial\bar{\xi}}\right)^2\left(\frac{dz}{d\xi}\right)^3\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3-\frac{\partial w}{\partial\bar{\xi}}\left(\frac{\partial w}{\partial\xi}\right)^2\cdot\frac{d^2\bar{z}}{d\bar{\xi}^2}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2\left(\frac{dz}{d\xi}\right)^3$$

$$-\frac{\partial w}{\partial\xi}\left(\frac{\partial w}{\partial\bar{\xi}}\right)^2\frac{d^2 z}{d\xi^2}\left(\frac{dz}{d\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^3\Bigg]+\frac{12}{h^2 c_0^2}\ddot{\tau}(t)\,w(\xi,\bar{\xi})\left(\frac{dz}{d\xi}\right)^5\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^5=0$$

.... (61)

where $\bar{\alpha}^2$ is obtained from the following equation

$$\frac{\bar{\alpha}^2 h^2}{12}\left\{\left(\frac{dz}{d\xi}\right)\left(\frac{d\bar{z}}{d\bar{\xi}}\right)\right\}^2\tau^2(t)=\frac{1}{2}(1-\nu)\left(\frac{\partial w}{\partial\xi}\right)^2\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2+\frac{1}{2}(1-\nu)\left(\frac{\partial w}{\partial\bar{\xi}}\right)^2\left(\frac{dz}{d\xi}\right)^2$$

$$+\frac{\partial u_0}{\partial\xi}\cdot\frac{dz}{d\xi}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2+\frac{\partial u_0}{\partial\bar{\xi}}\cdot\frac{d\bar{z}}{d\bar{\xi}}\left(\frac{dz}{d\xi}\right)^2$$

$$+\nu i\left\{\frac{\partial v_0}{\partial\xi}\cdot\frac{dz}{d\xi}\left(\frac{d\bar{z}}{d\bar{\xi}}\right)^2-\frac{\partial v_0}{\partial\bar{\xi}}\cdot\frac{d\bar{z}}{d\bar{\xi}}\left(\frac{dz}{d\xi}\right)^2\right\}$$

$$+(1+\nu)\frac{\partial w}{\partial\xi}\cdot\frac{\partial w}{\partial\bar{\xi}}\cdot\frac{dz}{d\xi}\cdot\frac{d\bar{z}}{d\bar{\xi}}$$

.... (62)

Here $\xi = r.e^{i\theta}$, $\bar{\xi} = r.e^{-i\theta}$, r being the radius of the circle. Values for λ have been obtained from the condition $\frac{\partial V}{\partial \lambda} = 0$, for minimum potential energy.

For regular polygons the mapping function is

$$Z = L\xi + \lambda_2 \xi^S \qquad \text{.... (63)}$$

where values L and λ_2 are given in a separate table.

Let us choose the deflection function in the following form

$$W = A_0 \tau(t)[1-\xi\bar{\xi}][1-\tfrac{1}{3}\xi\bar{\xi} + \tfrac{1}{2}(\xi^2+\bar{\xi}^2)(1-\xi\bar{\xi})^2] \qquad \text{.... (64)}$$

clearly W is θ dependent and satisfies the simply supported edge conditions, namely,

$$W = 0 \text{ at } r = 1$$

$$\frac{\partial^2 W}{\partial \xi \partial \bar{\xi}} = 0 \text{ at } r = 1$$

Substituting equation (63) and (64) in (61) the error function $\epsilon(\xi, \bar{\xi}, t)$ is obtained. Galerkin's technique requires

$$\int_0^{2\pi}\int_{r=0}^{1} \epsilon(\xi,\bar{\xi},t)\, W(\xi,\bar{\xi},t)\, r\, dr\, d\theta = 0 \qquad \text{.... (65)}$$

The constant $\bar{\alpha}$ is determined by putting (64) in (62) using (63) and integrating over the area of the plate.

It is to be noted that for transverse vibrations the normal displacement $W(\xi, \bar{\xi}, t)$ is our primary interest. So, the in-plane displacements u_0 and v_0 in equations (62) have been eliminated through integration by choosing suitable expressions for them compatible with their boundary conditions, namely, $u_0 = 0, v_0 = 0$ on the boundary for immovable edges.

For movable edges $\bar{\alpha} = 0$ .... (66)

Evaluating the integrals in (65) and considering the values of $\bar{\alpha}$ obtained from (62)(after integrating over the area of the plate) one obtains the Duffing's equation as in the previous cases in the form

$$\ddot{\tau}(t)+\alpha_1\tau(t)+\beta_1\tau^3(t)=0 \qquad \dots\dots (67)$$

Here the β_1 consists of a huge number of terms. So these terms have not been shown. Numerical results coming out from these terms have been presented in the tables.

The ratio of the nonlinear time period and linear time period in this case is

$$\frac{T^*}{T}=\frac{\frac{2K}{\pi}}{\left[1+\frac{\beta_1}{\alpha_1}.\bar{\beta}^2\right]^{1/2}} \qquad \dots\dots (68)$$

where $\bar{\beta}=\frac{A_0}{h}$

Numerical results : -

Numerical results are presented here in the tabular form for movable as well as immovable edges, for moderately thick polygonal plates. If the mapping function is known, the nonlinear behaviours of thick plates of any shape can be studied with ease and accuracy by using the proposed differential equations.

TABLE - 25

MAPPING FUNCTION COEFFICIENTS.[57]

POLYGONS	L	λ_2
SQUARE	1·08a	-0·11a
PENTAGON	1·053a	-0·07a
HEXAGON	1·038a	-0·05a
HEPTAGON	1·029a	-0·036a
OCTAGON	1·022a	-0·028a

TABLE - 26

LINEARTIME PERIOD.

$T_L^* = (\text{THICK PLATE}) = \frac{2\pi}{\sqrt{\alpha}} . (\frac{E}{G_c} \neq 0)$, $T_L(\text{THIN PLATE}) = \frac{2\pi}{\sqrt{\alpha}} . (\frac{E}{G_c} = 0)$

POLYGONS	$T_L^* (\frac{h}{a} = 0{\cdot}2, \frac{E}{G_c} = 2{\cdot}5)$	$\frac{T_L^*}{T_L}$
SQUARE	1·5613	1·0261
PENTAGON	1·2121	1·0285
HEXAGON	1·1185	1·0296
HEPTAGON	1·0722	1·0303
OCTAGON	1·0469	1·0308

RATIO OF NON-LINEAR TO LINEAR PERIOD FOR THE FUNDAMENTAL MODE OF VIBRATION OF SIMPLY SUPPORTED POLYGONAL PLATES (SQUARE OF SIDE 2a).

TABLE - 27

$\bar{\beta}=\frac{A_0}{h}$	$\frac{T^*}{T}$ FOR IMMOVABLE EDGES $[\nu=0.3\ \lambda=\nu^2[18],\ \frac{h}{2a}=\frac{1}{10})$									
	THIN PLATE									
	$k(\frac{E}{G_c})=0$	REF. [51]	$k(\frac{E}{G_c})=2.5$	REF. [51]	$k(\frac{E}{G_c})=20$	REF. [51]	$k(\frac{E}{G_c})=30$	REF. [51]	$k(\frac{E}{G_c})=50$	REF. [51]
0.60	0.9143	0.9072	0.9346	0.9270	1.0582	1.0469	1.1177	1.1066	1.2191	1.2012
0.80	0.8613	0.8507	0.8784	0.8624	0.9797	0.9636	1.0268	1.0113	1.1046	1.0819
1.00	0.8050	0.7917	0.8191	0.8055	0.9006	0.8809	0.9372	0.9123	0.9959	0.9678

$\bar{\beta}=\frac{A_0}{h}$	$\frac{T^*}{T}$ FOR MOVABLE EDGES $(\nu=0.3,\ \lambda=\nu^2[18],\ \frac{h}{2a}=\frac{1}{10})$									
	THIN PLATE									
	$k(\frac{E}{G_c})=0$	REF. [51]	$k(\frac{E}{G_c})=2.5$	REF. [51]	$k(\frac{E}{G_c})=20$	REF. [51]	$k(\frac{E}{G_c})=30$	REF. [51]	$k(\frac{E}{G_c})=50$	REF. [51]
0.60	0.9850	0.9779	1.0105	1.0029	1.1717	1.1664	1.2505	1.2334	1.3981	1.3802
0.80	0.9722	0.9616	1.0017	0.9857	1.1498	1.1337	1.2226	1.2071	1.3589	1.3362
1.00	0.9550	0.9416	0.9782	0.9647	1.1213	1.1022	1.1942	1.1693	1.3133	1.2852

RATIO OF NON-LINEAR TO LINEAR PERIOD FOR THE FUNDAMENTAL MODE OF VIBRATION OF DIFFERENT POLYGONS.

TABLE-28

$\frac{T^*}{T}$ $(\nu=0.3,\ \lambda=\nu^2[18],\ \frac{h}{2a}=\frac{1}{10})$ [2a is a dimension in length and related to the side of each polygon]

	$\bar{\beta}=\frac{A_0}{h}$	PENTAGON $k(\frac{E}{G_c})$					HEXAGON $k(\frac{E}{G_c})$				
		0	2.5	20	30	50	0	2.5	20	30	50
IMMOVABLE EDGES.	0.60	0.9341	0.9546	1.0794	1.1397	1.2391	0.9534	0.9747	1.0994	1.1607	1.2591
	0.80	0.8814	0.8995	1.0007	1.0476	1.1267	0.9016	0.9207	1.0207	1.0691	1.1467
	1.00	0.8255	0.8411	0.9209	0.9577	1.0170	0.8453	0.8621	0.9429	0.9777	1.0381
MOVABLE EDGES	0.60	1.0050	1.0305	1.1917	1.2708	1.4182	1.0250	1.0515	1.2116	1.2909	1.4383
	0.80	0.9922	1.0227	1.1699	1.2436	1.3784	1.0124	1.0427	1.1900	1.2636	1.3989
	1.00	0.9762	0.9923	1.1416	1.2145	1.3336	0.9962	1.0126	1.1619	1.2346	1.3539

	$\bar{\beta}=\frac{A_0}{h}$	HEPTAGON $k(\frac{E}{G_c})$					OCTAGON $k(\frac{E}{G_c})$				
		0	2.5	20	30	50	0	2.5	20	30	50
IMMOVABLE EDGES.	0.60	0.9784	0.9994	1.1204	1.1808	1.2791	0.9954	1.0194	1.1414	1.2008	1.3001
	0.80	0.9201	0.9417	1.0417	1.0891	1.1677	0.9429	0.9627	1.0627	1.1101	1.1887
	1.00	0.8632	0.8821	0.9640	0.9987	1.0594	0.8830	0.9031	0.9843	1.0197	1.0804
MOVABLE EDGES	0.60	1.0451	1.0716	1.2317	1.3110	1.4584	1.0652	1.0917	1.2518	1.3311	1.4785
	0.80	1.0352	1.0627	1.2103	1.2837	1.4190	1.0567	1.0827	1.2303	1.3041	1.4391
	1.00	1.0177	1.0329	1.1822	1.2549	1.3742	1.0391	1.0532	1.2025	1.2752	1.3945

Observations :

Numerical results obtained from different tables of this chapter show that the new approach presented in the present study can be conveniently applied to study the static as well as dynamic behaviours of different thick plates of different shapes under different edge conditions with ease and accuracy.

CONCLUSION OF THE THESIS :

The present project is an attempt to offer a new set of uncoupled differential equations in the theory of non-linear analysis of moderately thick isotropic plates. It is observed that numerical results showing the effects of shear deformation and rotatory inertia obtained from different tables for plates of different shapes are in excellent agreement with other known results. Moreover, results of movable as well as immovable edge conditions can be obtained from a single differential equation. This is also an additional advantage. Thus the proposed differential equations presented in the thesis are able to supply void in the literature of non-linear theory of moderately thick plates.

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INFLUENCES OF LARGE AMPLITUDES, TRANSVERSE SHEAR DEFORMATION AND ROTATORY INERTIA ON FREE LATERAL VIBRATIONS OF TRANSVERSELY ISOTROPIC PLATES—A NEW APPROACH

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Abstract—In this paper, using an improved Reissner's variational theorem [E. Reissner, *J. math. Phys.*, 90–95 (1950)] along with Banerjee's hypothesis, [B. Banerjee and S. Dutta, *Int. J. Non-Linear Mech.* **16**, 47–52 (1981)] a new set of governing equations whhich include the effects of transverse shear deformation and rotatory inertia is derived for the large amplitude free vibrations of isotropic plates. The case of a simply supported square plate has been discussed in detail. Numerical results have been computed showing the effect of the transverse shear deformation and compared with other known results.

INTRODUCTION

With the advent of modern plate and shell constructions subjected to severe operational conditions, the classical linear theory for small deflections is no longer applicable in many cases. Methods of analysis dealing with large deflections, therefore, are of increasingly practical importance. It is well-known that the classical plate equations for studying the non-linear behaviour of thin plates are due to Von Karman [3]. Many works have been done on Von Karman equations among which the works of Chu and Herrmann [4] and Yamaki [5] need special mention. It is also well-known that Berger [6] offered a simplified approach to study the non-linear behaviour of thin plates. The outstanding research workers who utilised Berger's equations in their respective investigations and obtained very satisfactory results are Nowinski [7], Nash and Modeer [8] and Wah [9]. Banerjee [10] offered a modified strain–energy expression for the investigation of non-linear behaviour of thin elastic plates and obtained satisfactory results both for movable as well as immovable edge conditions.

All these works mentioned above do not take into account the effects of transverse shear deformation and rotatory inertia which are important for moderately thick plates. The study of the non-linear behaviour of moderately thick plates is gaining momentum day by day due to its wide application in modern design. Important works in this field are: Wu and Vinson [11] and Kanaka Raju and Venkateswara Rao [12]. Wu and Vinson have applied Berger type equations whereas Kanaka Raju and Venkateswara have applied the finite element method to obtain their solutions. A discussion on various non-linear theories applicable for moderately thick plates can be found in papers by Sathyamoorthy and Chia [13] where it has been shown that the effects of transverse shear and rotatory inertia play a significant role in the large amplitude vibrations of moderately thick plates of various geometries.

The analytical work so far carried out is based mainly on single mode approximations and is often done with the aid of either Von Karman-type non-linear equations or Berger's approximation. Finite element methods have recently been used by Reddy *et al.* [14, 15] in the investigations concerning fundamental modes for moderately thick plates. Berger's equation is a purely approximate method. It yields accurate results for clamped edges. It yields fairly accurate results for simply supported edges and but fails for movable edges [16].

The present paper deals with the use of Reissner's variational theorem along with Banerjee's modified strain–energy expression for studying the non-linear behaviour of moderately thick isotropic plates. A new set of decoupled differential equations has been formed and solved with the help of Galerkin's procedure. The main advantage of this method is that it is simple and a single differential equation for the governing time function gives sufficiently accurate results both for movable as well as immovable edge conditions. The case of a simply supported square plate has been discussed in detail. Numerical results have been computed showing the effect of shear deformation and compared with other known results.

ANALYSIS

Let us consider the free vibrations of a square plate of thickness h and edge length $2a$. The material is transversely isotropic (such as pyrolytic graphite, for example). The origin of the coordinates is located at the centre of the plate. The deflections are considered to be of the same order of magnitude as the plate thickness.

Reissner's strain–energy expression after integrating with respect to z takes the following form [1]

$$\psi = \int_{-a}^{+a}\int_{-a}^{+a}\left\{\frac{Eh}{2(1-\nu^2)}[\bar{I}_e^2 - 2(1-\nu)\overline{II}_e] + M_x\frac{\partial\alpha}{\partial x} + M_y\frac{\partial\beta}{\partial y} + M_{xy}\left(\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\right)\right.$$
$$+ Qx\left(\frac{\partial\omega}{\partial x}+\alpha\right) + Qy\left(\frac{\partial\omega}{\partial y}+\beta\right) - \frac{1}{2E}\left[\frac{12}{h^3}(M_x^2+M_y^2) - 24\frac{M_xM_y\nu}{h^3}\right]$$
$$\left. - \frac{3}{5G_c h}(Q_x^2+Q_y^2)\right\}\mathrm{d}x\,\mathrm{d}y \ldots \quad (1)$$

where $\bar{I}_e$, $\overline{II}_e$ are the first and second invariants of the middle surface strains. These are,

$$\bar{I}_e = \varepsilon x_0 + \varepsilon y_0;$$
$$\overline{II}_e = \varepsilon x_0 \varepsilon y_0 - \varepsilon^2 x_0 y_0.$$

The kinetic energy equation after integrating through the thickness is [11]

$$T = \int_{-a}^{+a}\int_{-a}^{+a}\left\{\frac{\rho h}{2}\left[\left(\frac{\partial u_0}{\partial t}\right)^2 + \left(\frac{\partial v_0}{\partial t}\right)^2 + \left(\frac{\partial\omega}{\partial t}\right)^2\right]\right.$$
$$\left. + \frac{\rho h^3}{24}\left[\left(\frac{\partial\alpha}{\partial t}\right)^2 + \left(\frac{\partial\beta}{\partial t}\right)^2\right]\right\}\mathrm{d}x\,\mathrm{d}y \ldots . \quad (2)$$

In order to derive the equation of motion we now apply Hamilton's principle in conjunction with the strain–energy as well as the kinetic energy given by ψ and T. Therefore, we have to minimise the integral,

$$\phi = \int_{t_1}^{t_2}(\psi - T)\,\mathrm{d}t \ldots . \quad (3)$$

Using now Banerjee's hypothesis, taking the variation of ϕ, equating it to zero and finally eliminating M_x, M_y, M_{xy}, etc., we get the following set of decoupled differential equations governing the vibrations of the plates

$$\nabla^4 W + \frac{6k}{5(1-\nu^2)}\left(\frac{E}{G_c}\right)\frac{\bar{\alpha}^2h^2}{12}\tau^2(t)\nabla^2\left(\frac{\partial^2\omega}{\partial x^2} + \nu\frac{\partial^2\omega}{\partial y^2}\right)$$
$$+ \frac{3}{5}\frac{\lambda}{(1-\nu^2)}K\left(\frac{E}{G_c}\right)\nabla^2\left[\nabla^2 W\left\{\left(\frac{\partial\omega}{\partial x}\right)^2 + \left(\frac{\partial\omega}{\partial y}\right)^2\right\}\right.$$
$$\left. + 2\left\{\frac{\partial^2\omega}{\partial x^2}\left(\frac{\partial\omega}{\partial x}\right)^2 + \frac{\partial^2\omega}{\partial y^2}\left(\frac{\partial\omega}{\partial y}\right)^2\right\} + 4\frac{\partial^2\omega}{\partial x\partial y}\frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y}\right] - \frac{6}{5}\frac{\rho}{G_c}\frac{\delta^2}{\partial t^2}(\nabla^2\omega)$$
$$- \bar{\alpha}^2\tau^2(t)\left[\frac{\partial^2\omega}{\partial x^2} + \nu\frac{\partial^2\omega}{\partial y^2}\right] - \frac{6\lambda}{h^2}\left[\nabla^2\omega\left\{\left(\frac{\partial\omega}{\partial x}\right)^2 + \left(\frac{\partial\omega}{\partial y}\right)^2\right\} + 2\left\{\frac{\partial^2\omega}{\partial x^2}\left(\frac{\partial\omega}{\partial x^2}\right)^2\right.\right.$$
$$\left.\left. + \frac{\partial^2\omega}{\partial y^2}\left(\frac{\partial\omega}{\partial y}\right)^2\right\} + 4\frac{\partial^2\omega}{\partial x\partial y}\frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y}\right] + \frac{12}{h^2C_p^2}\frac{\partial^2\omega}{\partial t^2} = 0 \ldots \quad (4)$$

where

$$\frac{\bar{\alpha}^2 h^2}{12}\tau^2(t) = \frac{\partial u_0}{\partial x} + \nu\frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial \omega}{\partial y}\right)^2 \ldots. \tag{5}$$

We are primarily interested in the fundamental mode of vibrations of the plate. For a square plate of side $2a$, let us choose the deflection function in the following form:

$$\omega = A_{00}\tau(t)\,\mathrm{Cos}\frac{\pi x}{2a}\cos\frac{\pi y}{2a}\ldots. \tag{6}$$

Clearly this form of ω satisfies the following simply supported edge conditions

$$\omega = 0 \quad \text{at } x = \pm a.$$

$$\omega = 0 \quad \text{at } y = \pm a.$$

$$\frac{\partial^2 \omega}{\partial x^2} = 0 \quad \text{at } x = \pm a.$$

$$\frac{\partial^2 \omega}{\partial y^2} = 0 \quad \text{at } y = \pm a.$$

Putting (6) in (5) and integrating over the area of the plate one gets

$$\bar{\alpha}^2 = \frac{3}{8}\frac{A_{00}^2\pi^2(1+\nu)}{a^2h^2}\ldots. \tag{7}$$

For transverse vibration the normal displacement $\omega(x, y, t)$ is our primary interest. So the in-plane displacements have been eliminated through integration by choosing suitable expressions for them. Putting (6) in (4), considering (7) and applying Galerkin's error minimising technique one gets the following differential equation for the time function $\tau(t)$

$$\left[\frac{12}{h^2C_p^2} + \frac{3}{5}\frac{\pi^2\rho}{G_c a^2}\right]\ddot{\tau}(t) + \frac{\pi^4}{4a^4}\tau(t) + \left[\frac{15}{32}\frac{\pi^4\lambda}{a^4}\left(\frac{A_{00}}{h}\right)^2 + \frac{3}{32}\frac{\pi^4(1+\nu)^2}{a^4}\left(\frac{A_{00}}{h}\right)^2\right.$$

$$\left. + \frac{3}{640}\frac{\pi^6h^2(1+\nu)^2}{a^6(1-\nu^2)}K\left(\frac{E}{G_c}\right)\left(\frac{A_{00}}{h}\right)^2 + \frac{3}{128}\frac{\lambda\pi^6h^2}{a^6(1-\nu^2)}K\left(\frac{E}{G_c}\right)\left(\frac{A_{00}}{h}\right)^2\right]$$

$$\tau^3(t) = 0\ldots. \tag{8}$$

The solution of the above equation subject to the boundary conditions

$$\tau(0) = 1$$

$$\dot{\tau}(0) = 0$$

is well-known and is obtained in terms of Jacobic's elliptic function. The ratio of the non-linear and linear time period is

$$\frac{T^*}{T} = \frac{2k}{\pi}$$

$$\left[\frac{1 + \dfrac{\pi^2}{20(1-\nu^2)}\left(\dfrac{E}{G_c}\right)h^2/a^2}{1 + \dfrac{15}{8}\lambda\bar{\beta}^2 + \dfrac{3}{8}(1+\nu)^2\bar{\beta}^2 + \dfrac{3}{32}\dfrac{\lambda}{(1-\nu^2)}K\left(\dfrac{E}{G_c}\right)\dfrac{\pi^2h^2}{a^2}\bar{\beta}^2 + \dfrac{3}{160}K\left(\dfrac{E}{G_c}\right)\dfrac{\pi^2h^2(1+\nu)^2}{a^2(1-\nu^2)}\bar{\beta}^2}\right]^{1/2} \ldots \tag{9}$$

where $\bar{\beta} = A_{00}/h$ is the ratio of the static deflection to the thickness of the place.

NUMERICAL RESULTS

Numerical results are presented here in the tabular forms (both for movable as well as immovable edges) for moderately thick isotropic square plates and compared with other known results.

The ratios of the non-linear period T^* of vibration including the effects of transverse shear deformation and rotatory inertia, to the corresponding linear period T of the classical plate (not including transverse shear and rotatory inertia) are computed for various thickness parameters $\left(\frac{h}{2a} = \frac{1}{10}, \frac{1}{20}, \frac{1}{30}\right)$ and material constants ($\nu = 0.3$, $\lambda = \nu^2$[2], $K\left(\frac{E}{G_c}\right) = 0, 2.5, 20, 30, 50$) at different non-dimensional amplitudes vibration $\left(\bar{\beta} = \frac{A_{00}}{h}\right)$. For moderately thick plates, the non-linear periods are dependent on the thickness parameter whereas they are independent of the same for thin plates.

Table 1. Present findings for immovable edges [11]

		T^*/T E/G_c				T^*/T E/G_c			
	$\bar{\beta}$	2.5	20	30	50	2.5	20	30	50
	0	1.0268	1.1976	1.2850	1.4440	1.0268	1.1976	1.2850	1.4440
	0.2	1.0140	1.1774	1.2602	1.4092	1.0037	1.1606	1.2397	1.3806
$\frac{h}{2a} = \frac{1}{10}$	0.4	0.9785	1.1228	1.1940	1.3187	0.9418	1.0683	1.1290	1.2290
	0.6	0.9270	1.0469	1.1066	1.2012	0.8606	0.9577	0.9978	1.0656
	0.8	0.8624	0.9636	1.0113	1.0819	0.7758	0.8422	0.8710	0.9159
	1.0	0.8055	0.8809	0.9123	0.9678	0.6976	0.7449	0.7648	0.7948

Table 2. Present findings [11]

		T^*/T E/G_c				T^*/T E/G_c			
	$\bar{\beta}$	2.5	20	30	50	2.5	20	30	50
	0	1.0067	1.0529	1.0785	1.1274	1.0067	1.0529	1.0785	1.1274
	0.2	0.9947	1.0391	1.0635	1.1100	0.9846	1.0173	1.0511	1.0966
$\frac{h}{2a} = \frac{1}{20}$	0.4	0.9610	1.0009	1.0227	1.0644	0.9270	0.9617	0.9810	1.0176
	0.6	0.9121	0.9460	0.9643	0.9905	0.8487	0.8757	0.8903	0.9175
	0.8	0.8548	0.8825	0.8968	0.9251	0.7670	0.7869	0.7973	0.8166
	1.0	0.7948	0.8170	0.8273	0.8503	0.6900	0.7049	0.7119	0.7255

Table 3. Present findings [11]

		T^*/T E/G_c				T^*/T E/G_c			
	$\bar{\beta}$	2.5	20	30	50	2.5	20	30	50
	0	1.0030	1.0239	1.0355	1.0585	1.0030	1.0239	1.0355	1.0585
	0.2	0.9912	1.0111	1.0225	1.0445	0.9811	1.0005	1.0113	1.0221
$\frac{h}{2a} = \frac{1}{30}$	0.4	0.9578	0.9759	0.9860	1.0058	0.9172	0.9393	0.9482	0.9656
	0.6	0.9093	0.9247	0.9334	0.9501	0.8464	0.8586	0.8653	0.8784
	0.8	0.8525	0.8641	0.8722	0.8855	0.7630	0.7742	0.7791	0.7885
	1.0	0.7930	0.8031	0.8087	0.8197	0.6889	0.6952	0.6986	0.7052

Table 4. Present study [11]—classical thin plate theory

$\bar{\beta}$	T^*/T $E/G_c = 0$	T^*/T $E/G_c = 0$
0	1	1
0.2	0.9882	0.9782
0.4	0.9552	0.9210
0.6	0.9072	0.8446
0.8	0.8507	0.7640
1.0	0.7917	0.6878

Table 5. Present findings for movable edges

	$\bar{\beta}$	T^*/T E/G_c 2.5	20	30	50
$\frac{h}{2a}=\frac{1}{10}$	0	1.02680	1.19760	1.2850	1.4440
	0.2	1.02406	1.19325	1.2796	1.4366
	0.4	1.01601	1.18056	1.2641	1.4147
	0.6	1.00298	1.1604	1.2394	1.3802
	0.8	0.9857	1.13376	1.2071	1.3362
	1.0	0.9647	1.1022	1.1693	1.2852

Table 6

	$\bar{\beta}$	T^*/T E/G_c 2.5	20	30	50
$\frac{h}{2a}=\frac{1}{20}$	0	1.0067	1.0529	1.07850	1.12754
	0.2	1.0042	1.0499	1.0752	1.1239
	0.4	0.9966	1.0412	1.0659	1.1134
	0.6	0.9844	1.0291	1.0509	1.0963
	0.8	0.9678	1.0089	1.0310	1.0738
	1.0	0.9480	0.9859	1.0070	1.0468

Table 7

	$\bar{\beta}$	T^*/T E/G_c 2.5	20	30	50
$\frac{h}{2a}=\frac{1}{30}$	0	1.0030	1.0239	1.0355	1.0585
	0.2	1.0005	1.0213	1.0327	1.0555
	0.4	0.9930	1.0132	1.0245	1.0467
	0.6	0.9808	1.0003	1.0110	1.0326
	0.8	0.9647	0.9831	0.9934	1.0139
	1.0	0.9449	0.9622	0.9720	0.9910

Table 8

	$\bar{\beta}$	T^*/T	
$E/G_c=0$	0	1	Note that absurd results are obtained by Berger's method for movable edge conditions.
	0.2	0.9975	
	0.4	0.9900	
	0.6	0.9779	
	0.8	0.9616	
	1.0	0.9416	

OBSERVATIONS

(1) It has been observed from the present study that

(a) for the same value of $\frac{E}{G_c}$, $\frac{T^*}{T}$ decreases as $\frac{h}{2a}$ decreases, both for immovable as well as movable edges.

(b) for the same $\frac{h}{2a}$, as $\frac{E}{G_c}$ increases $\frac{T^*}{T}$ increases, both for immovable as well as movable edges.

(c) when $\frac{E}{G_c} = 0$, $\frac{T^*}{T}$ decreases as $\bar{\beta}$ increases.

(2) The results obtained in the present study for immovable edges are in good agreement with those obtained by Wu and Vinson [11].

(3) It appears from the tables that the effects of rotatory inertia and shear deformation are more prominent in the present study than those obtained by Vinson and Wu. This is due to the fact that Berger's equations used by them involve the neglect of the membrane shear deformation.

(4) Results for movable edge conditions have also been computed but cannot be compared because of the absence of any known results.

(5) A single differential equation obtained in the present study is able to predict the effect of transverse shear deformation and rotatory inertia on large amplitude free vibrations of moderately thick plates of movable as well as immovable edges with ease and accuracy while Berger's approximation yields fairly good results for similar problems with immovable edges only. This is certainly an advantage of the present study.

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INFLUENCES OF LARGE AMPLITUDES, SHEAR DEFORMATION AND ROTATORY INERTIA ON AXISYMMETRIC VIBRATIONS OF MODERATELY THICK CIRCULAR PLATES: A NEW APPROACH

1. INTRODUCTION

A new approach is presented for the determination of the axisymmetric vibrations of moderately thick circular plates by using Reissner's variational theorem [1] along with Banerjee's hypothesis [2, 3]. A set of governing equations including the effect of shear deformation and rotatory inertia is derived. The case of a clamped circular plate is studied in detail. Numerical results have been computed showing the effects of shear deformation and rotatory inertia and these are compared with other known results. The case of a simply supported square plate has been discussed in a separate paper [4] and the results obtained there have been found to be in excellent agreement with other known results.

2. ANALYSIS

Consider a circular plate of radius a. It is assumed that the origin is located at the centre of the plate and the deflection is of the same order of magnitude as the plate thickness.

Using Banerjee's hypothesis [2], taking the variation of ϕ as given in reference [1], equating it to zero, eliminating M_x, M_y, M_{xy} etc., as given in reference [6] and finally transforming the set of equations into polar co-ordinates, one obtains the following set of differential equations governing the vibrations of the circular plate:

$$\begin{aligned}&\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right]+\frac{6}{5(1-\nu^2)}K\left(\frac{E}{G_c}\right)\frac{\bar{\alpha}^2h^2}{12}r^{\nu-1}\left[\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\frac{\partial w}{\partial r}\right]\tau^2(t)\\
&+\frac{3\lambda}{5(1-\nu^2)}K\left(\frac{E}{G_c}\right)\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)\left(\frac{\partial w}{\partial r}\right)^2+2\frac{\partial^2 w}{\partial r^2}\left(\frac{\partial w}{\partial r}\right)^2\right]\\
&-\frac{6}{5}\frac{\rho}{G_c}\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right]-\bar{\alpha}^2\tau^2(t)r^{\nu-1}\left[\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\cdot\frac{\partial w}{\partial r}\right]\\
&-\frac{6\lambda}{h^2}\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)\left(\frac{\partial w}{\partial r}\right)^2+2\frac{\partial^2 w}{\partial r^2}\left(\frac{\partial w}{\partial r}\right)^2\right]+\frac{12}{h^2C_p^2}\frac{\partial^2 w}{\partial t^2}=0,\end{aligned}\tag{1}$$

$$r^{\nu-1}\frac{\bar{\alpha}^2h^2}{12}\tau^2(t)=\frac{\partial u_0}{\partial r}+\nu\frac{v_0}{r}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2.\tag{2a}$$

For movable edge conditions

$$\bar{\alpha}=0.\tag{2b}$$

Of primary interest here is the fundamental mode of vibration of the plate. For a circular plate of radius a, the deflection function is taken to be

$$w=A_0\tau(t)(1-r^2/a^2)^2.\tag{3}$$

Clearly this form satisfies the clamped edge conditions $(w)_{r=a}=0$ and $(\partial w/\partial r)_{r=a}=0$. Putting expression (3) in equation (2a) and integrating over the area of the plate gives

$$\bar{\alpha}^2=\{1536\nu/a^{\nu+1}(\nu+3)(\nu+5)(\nu+7)\}(A_0^2/h^2).\tag{4}$$

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TABLE 1

Ratio of non-linear to linear period for the fundamental mode of vibration of a circular plate

	T^*/T, clamped immovable edges								T^*/T, movable edges			
	$h/a=0\cdot20$, $K(E/G_c)=8\cdot1971$		$h/a=0\cdot15$, $K(E/G_c)=8\cdot8133$		$h/a=0\cdot10$, $K(E/G_c)=10\cdot4869$		$h/a=0\cdot05$, $K(E/G_c)=19\cdot3165$		$h/a=0\cdot20$, $K(E/G_c)=8\cdot1971$	$h/a=0\cdot15$, $K(E/G_c)=8\cdot8133$	$h/a=0\cdot10$, $K(E/G_c)=10\cdot4869$	$h/a=0\cdot05$, $K(E/G_c)=19\cdot3165$
$\bar{\beta}=A_0/h$	Present study	[5]	Present study	[5]	Present study	[5]	Present study	[5]				
0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
0·2	0·9821	0·9921	0·9907	0·9924	0·9919	0·9927	0·9926	0·9928	0·9972	0·9976	0·9979	0·9980
0·4	0·9584	0·9699	0·9661	0·9710	0·9689	0·9718	0·9716	0·9722	0·9891	0·9905	0·9915	0·9921
0·6	0·9138	0·9366	0·9251	0·9388	0·9339	0·9402	0·9394	0·9410	0·9759	0·9790	0·9812	0·9826
0·8	0·8576	0·8965	0·8774	0·8995	0·8909	0·9015	0·8993	0·9026	0·9583	0·9635	0·9674	0·9697
1·0	0·8029	0·8533	0·8258	0·8568	0·8435	0·8591	0·8548	0·8603	0·9371	0·9448	0·9504	0·9538

Putting expression (3) in equation (1), considering expression (4) and applying Galerkin's error minimizing technique then gives the following differential equation for the time function $\tau(t)$:

$$\left[\frac{6}{5}\frac{a^2}{h^2C_p^2}+\frac{4}{5}\frac{\rho}{G_c}\right]\ddot{\tau}(t)+\frac{32}{3a^2}\tau(t)+\left[864\cdot79872K\left(\frac{E}{G_c}\right)\frac{A_0^2}{a^4}\frac{\nu}{(1-\nu^2)(\nu+5)(\nu+7)}\right.$$
$$+1541\cdot2224\frac{A_0^2}{a^2h^2}\frac{\nu}{(\nu+3)(\nu+5)(\nu+7)}+10\cdot24\lambda K\left(\frac{E}{G_c}\right)\frac{1}{(1-\nu^2)}\frac{A_0^2}{a^4}$$
$$\left.+7\cdot3142\frac{\lambda}{h^2}\frac{A_0}{a^2}\right]\tau^3(t)=0. \quad (5)$$

The solution of this equation subject to the boundary conditions $\tau(0)=1$ and $\dot{\tau}(0)=0$ is well known and is obtained in terms of Jacobi's elliptic function. The ratio of the non-linear and linear time periods is

$$\frac{T^*}{T}=\frac{2K}{\pi}\left[1\Big/\left\{1+81\cdot1255K\left(\frac{E}{G_c}\right)\frac{h^2}{a^2}\frac{\nu}{(1-\nu^2)(\nu+5)(\nu+7)}\bar{\beta}^2\right.\right.$$
$$+\frac{0\cdot9606\lambda}{(1-\nu^2)}K\left(\frac{E}{G_c}\right)\frac{h^2}{a^2}\bar{\beta}^2+\frac{144\cdot4986\nu}{(\nu+3)(\nu+5)(\nu+7)}\bar{\beta}^2$$
$$\left.\left.+0\cdot6857062\lambda\bar{\beta}^2\right\}\right]^{1/2}. \quad (6)$$

3. NUMERICAL RESULTS

Numerical results have been computed both for movable as well as for immovable edge conditions for clamped circular plates, and are presented in Table 1. The ratios of the non-linear time period T^* of vibration, including the effects of shear deformation and rotatory inertia, to the corresponding linear period T of the classical plate (not including shear deformation and rotatory inertia) are shown for various values of the thickness parameter ($h/a=0\cdot20, 0\cdot15, 0\cdot10, 0\cdot05$) and material constants ($\nu=0\cdot3$, $\lambda=0\cdot18$ [2], $K(E/G_c)=8\cdot1971$, $8\cdot813339$, $10\cdot4869$ and $19\cdot3165$) at different non-dimensional amplitudes of vibration ($\bar{\beta}=A_0/h$).

4. CONCLUSION

A single differential equation obtained in the present study is able to predict the effect of transverse shear deformation and rotatory inertia on large amplitude free vibrations of moderately thick circular plates for immovable as well as for movable edges with ease and accuracy. This is certainly an advantage of the present approach. Results for movable edges cannot be compared in absence of any known results.

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June 12, 1990

Professor Rekha Bhattacharya
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Dear Professor Bhattacharya:

I have today forwarded your MS (#90122) with Banerjee, "Influence of Large Amplitudes ... " to Professor Keer for publication as a Brief Note in the *Journal of Applied Mechanics*. Congratulations.

Sincerely yours,

J. G. Simmonds, Associate Editor
Journal of Applied Mechanics

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Please quote in all correspondence:

P/38/90(L) 31st July 1990

Professor B. Banerjee
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Dear Professor Banerjee,

Thank you for your letter of 19th May 1990. I am pleased to say that the manuscript is acceptable in its present "Letter to the Editor" form. I have edited it and sent it to the publishers. Proofs should be ready in approximately two months.

A copy of your covering letter has been sent to the second referee, for his information.

Yours sincerely,

P.E. Doak